Preconditioning and Locality in Algorithm Design

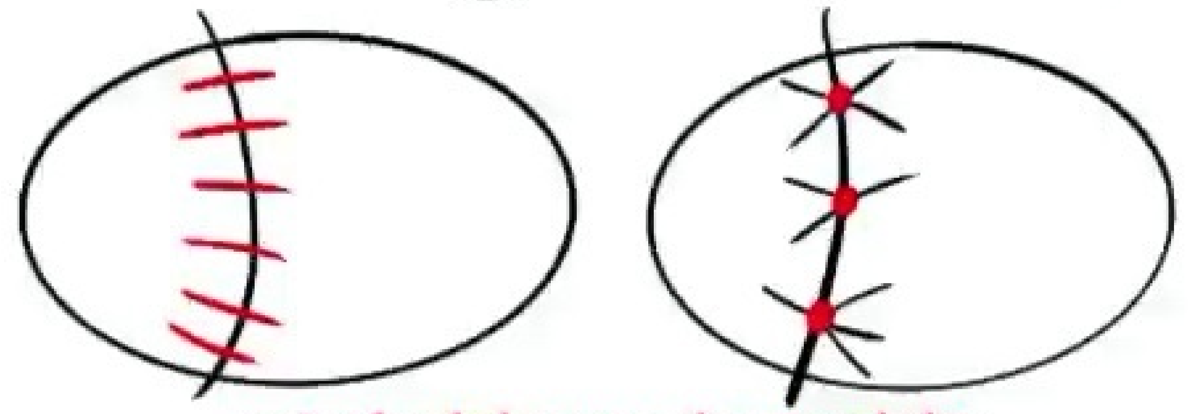
Jason Li

PhD Thesis

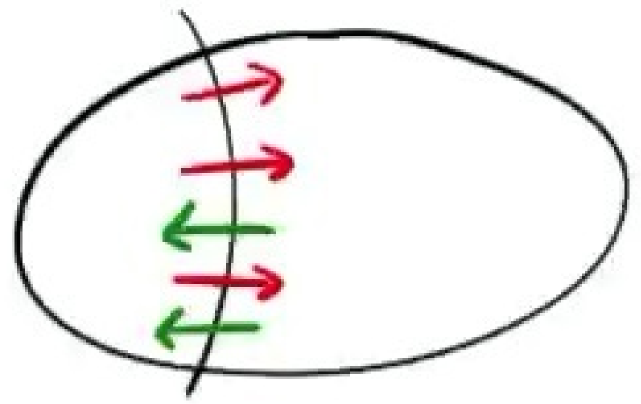
Problems Studied

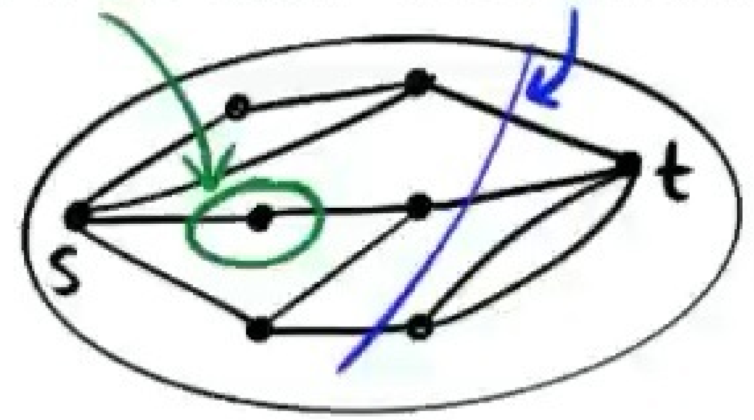
Graph cut problems

- Mincut: edge/vertex, undirected/directed, global/terminal/all-pairs



cut of minimum size/weight





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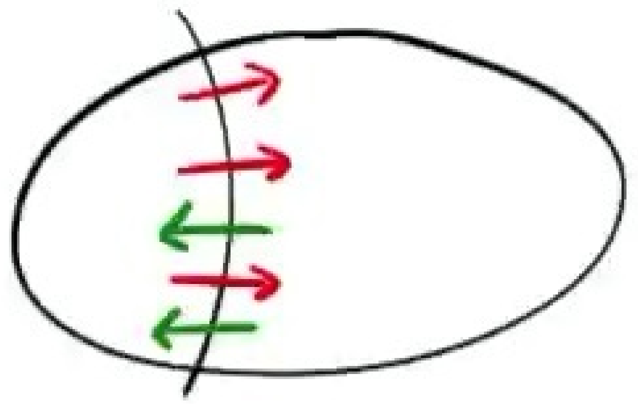
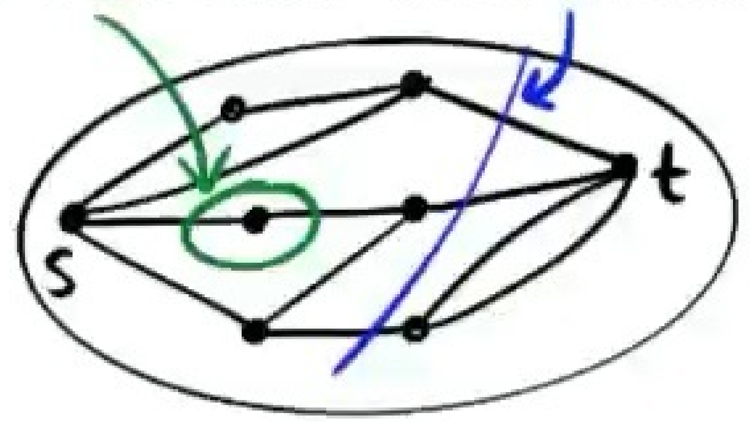
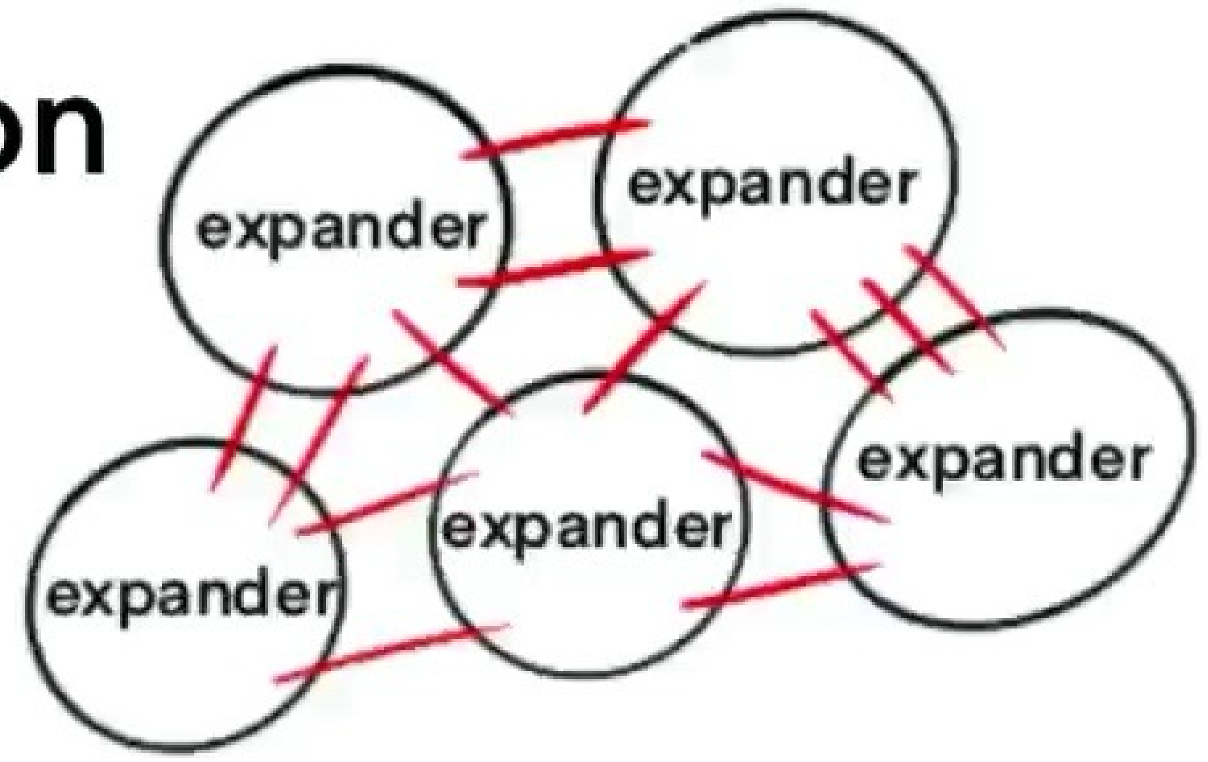




- Conductance and expander decomposition



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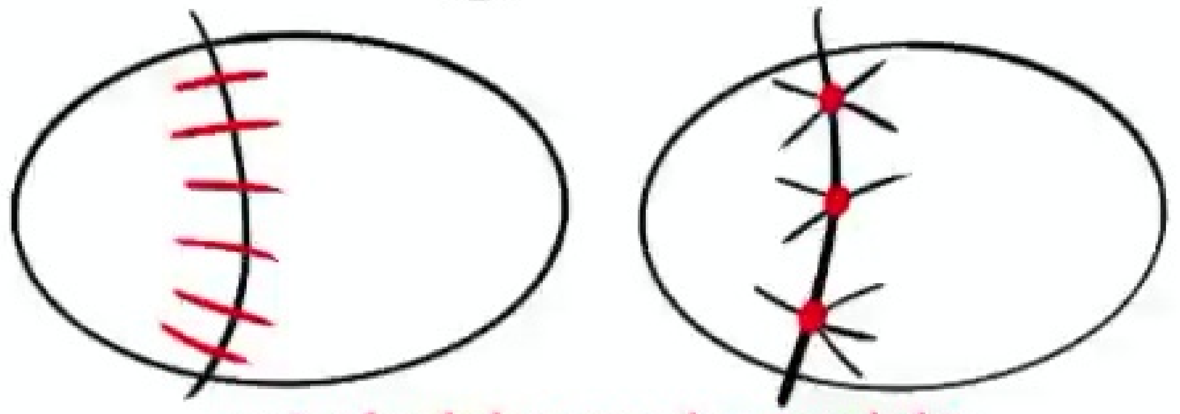

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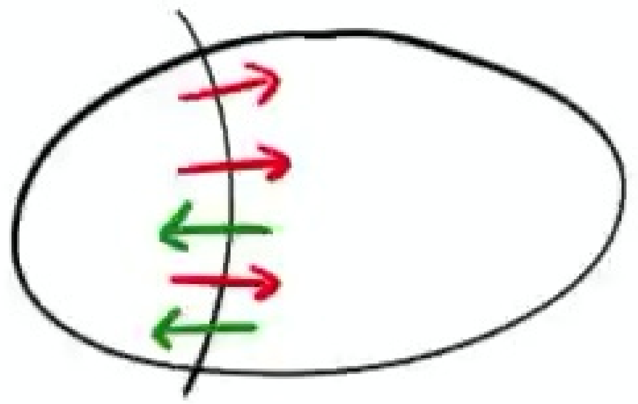

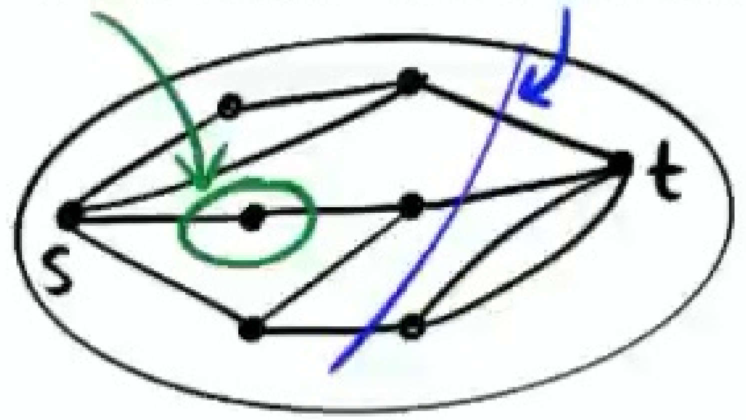

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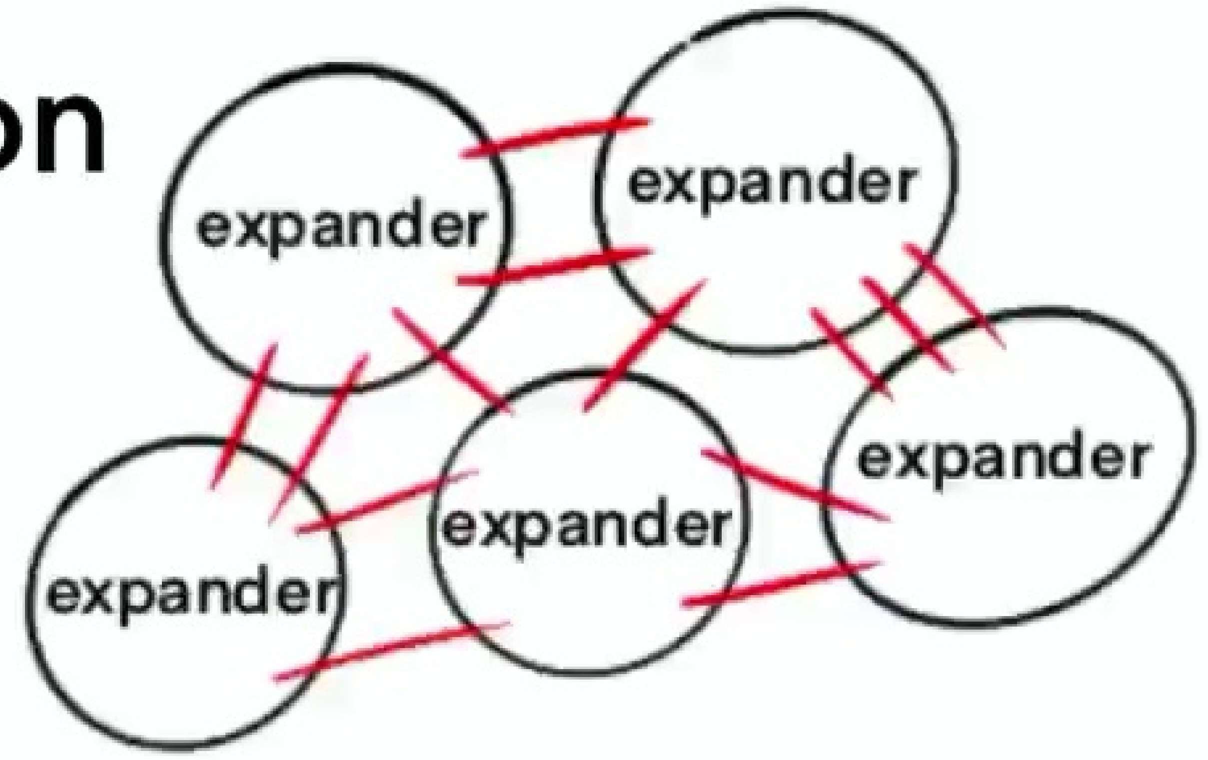

Graph distance problems

- Approximate shortest path, transshipment, L_1 embedding (PRAM model)

Preconditioning and Locality

Preconditioning: worst case vs. average case

- Assume that the input is random
  - expander (graph cut problems), low aspect ratio (distance)

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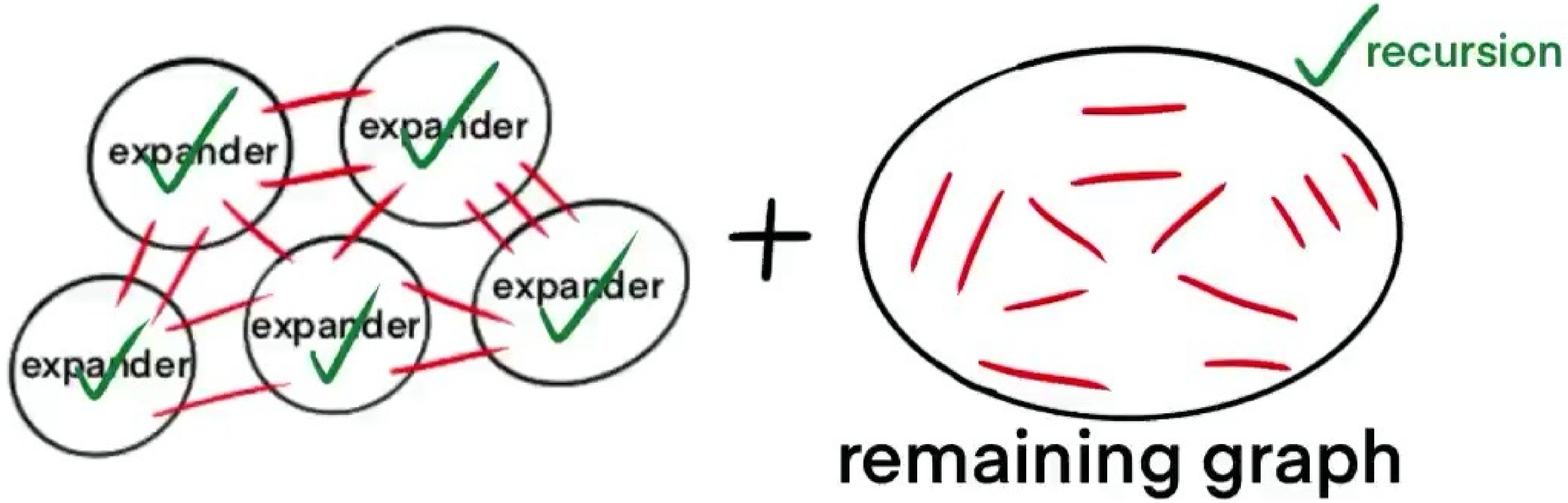

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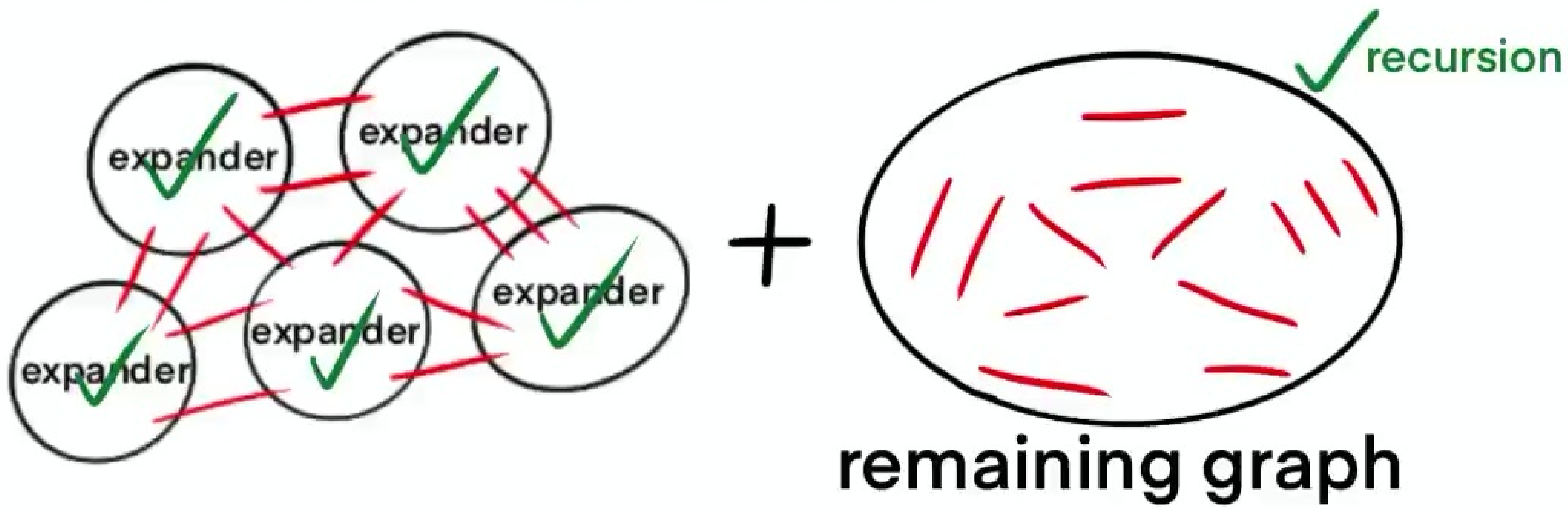
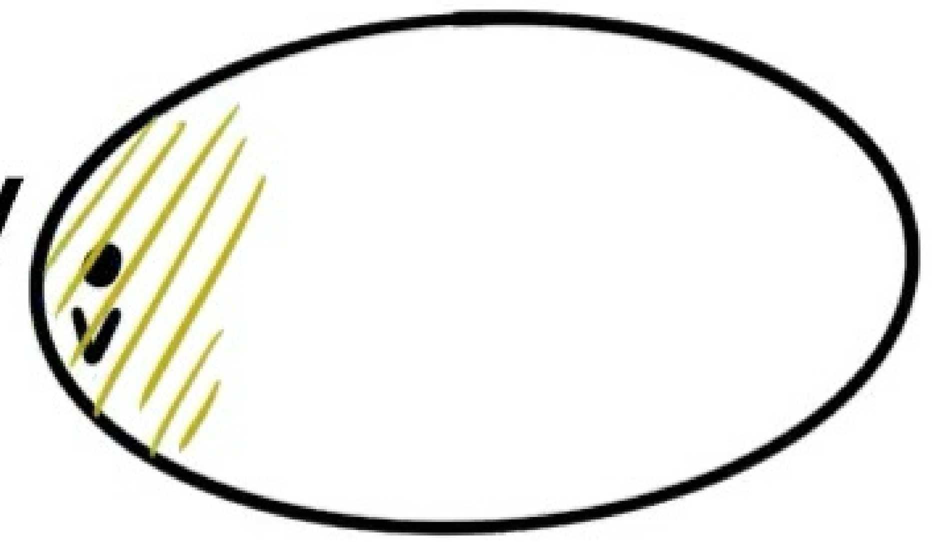


- Popularized by Spielman and Teng [ST'04] on Laplacian system solvers

Preconditioning and Locality

- Local algorithms: explore a small neighborhood around v



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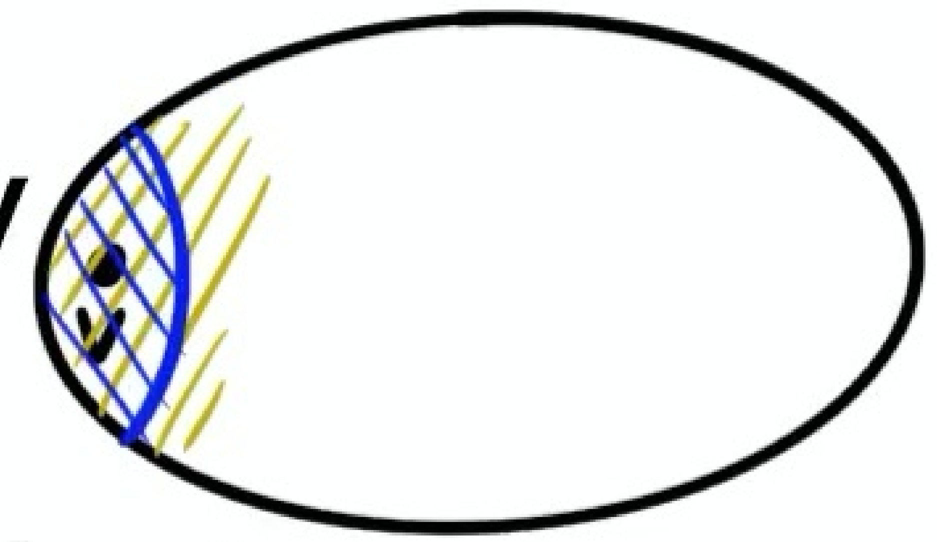

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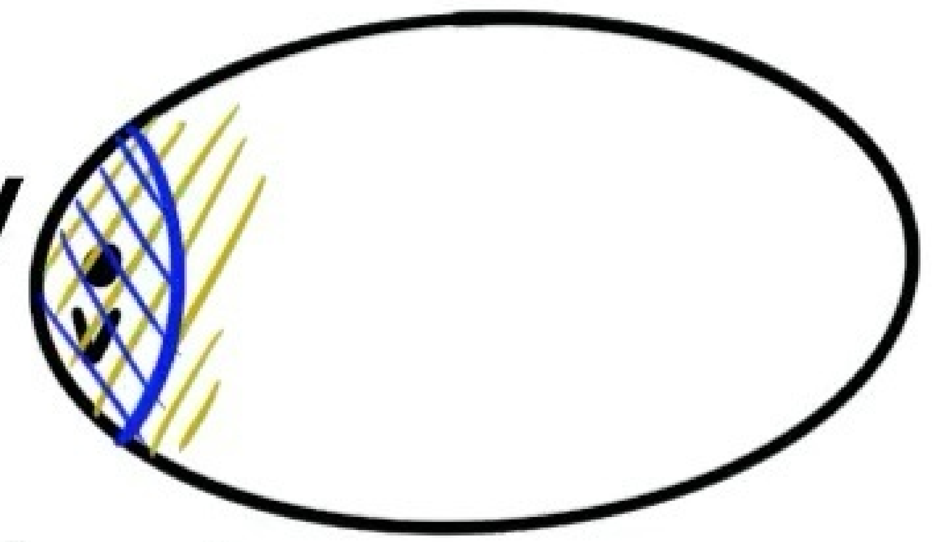

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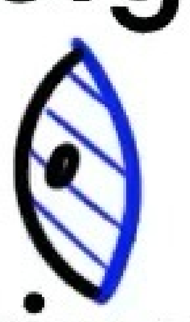

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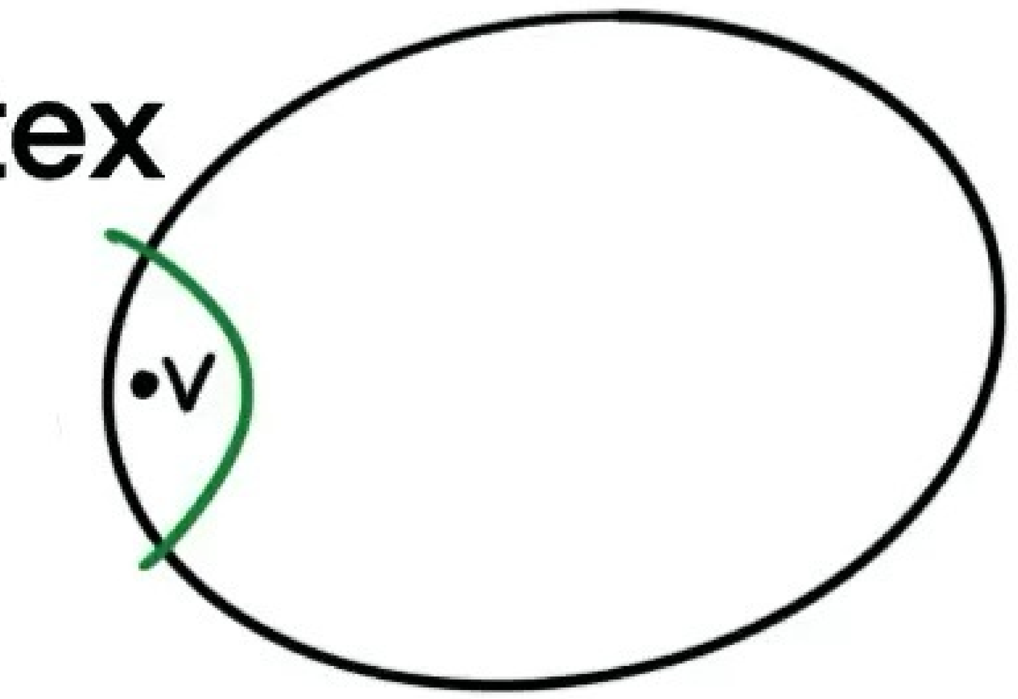
- Assume that the target solution is local to some vertex
  - e.g. mincut cuts a small neighborhood around v



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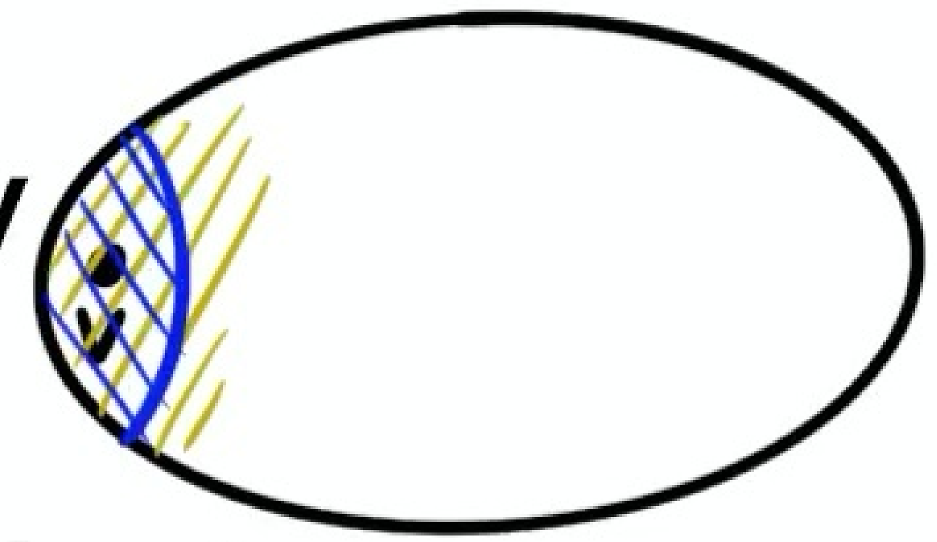

Locality: unbalanced vs. balanced

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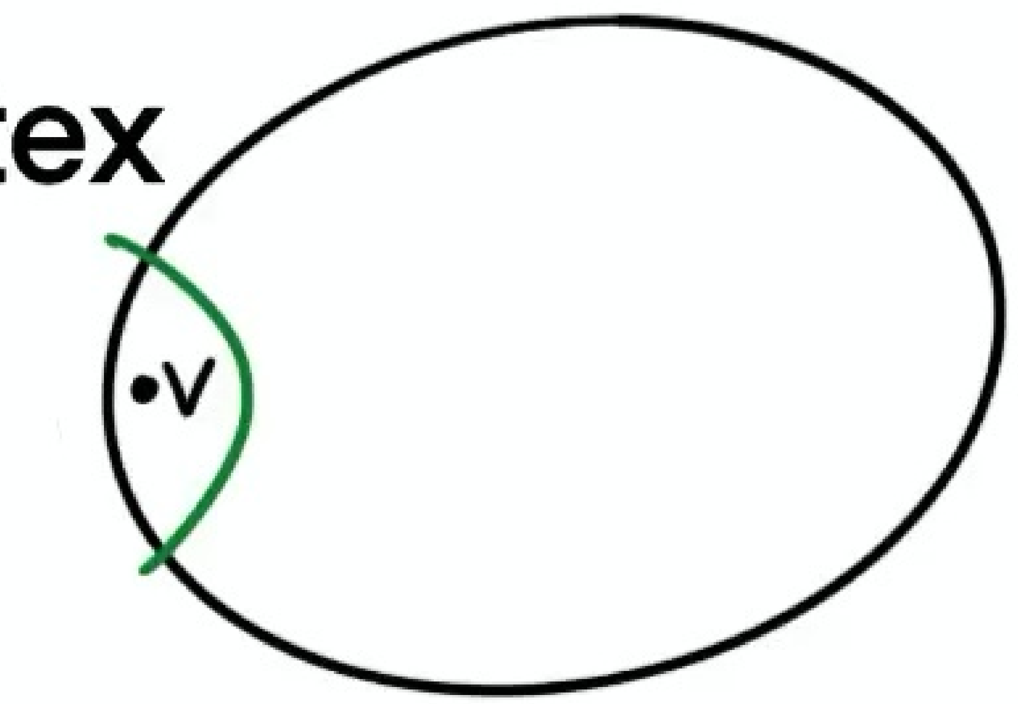
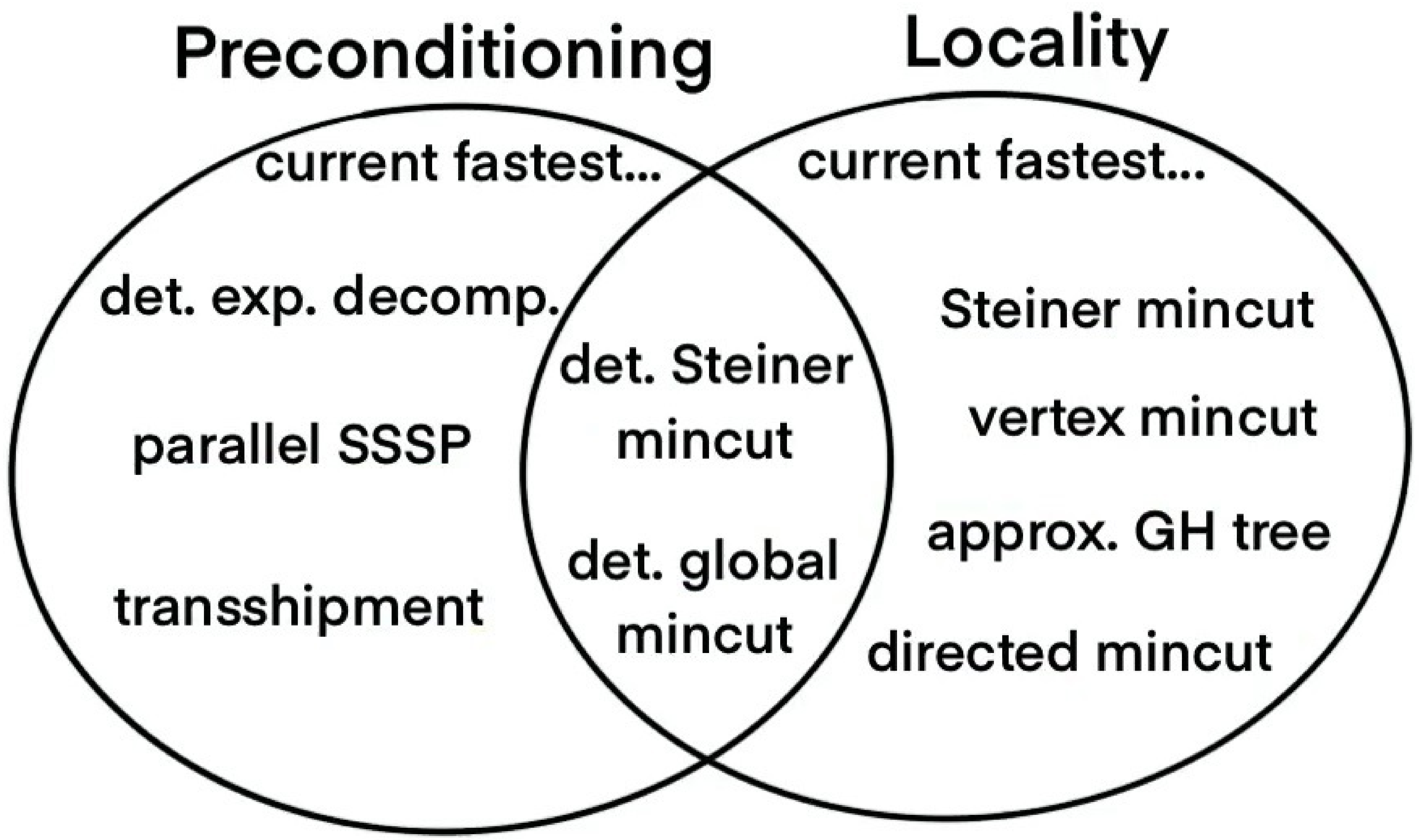
- Reduce to unbalanced instances
  - Straight reduction, or handle balanced case separately



The Case For Preconditioning and Locality

Powerful

- Resolves fundamental open problems



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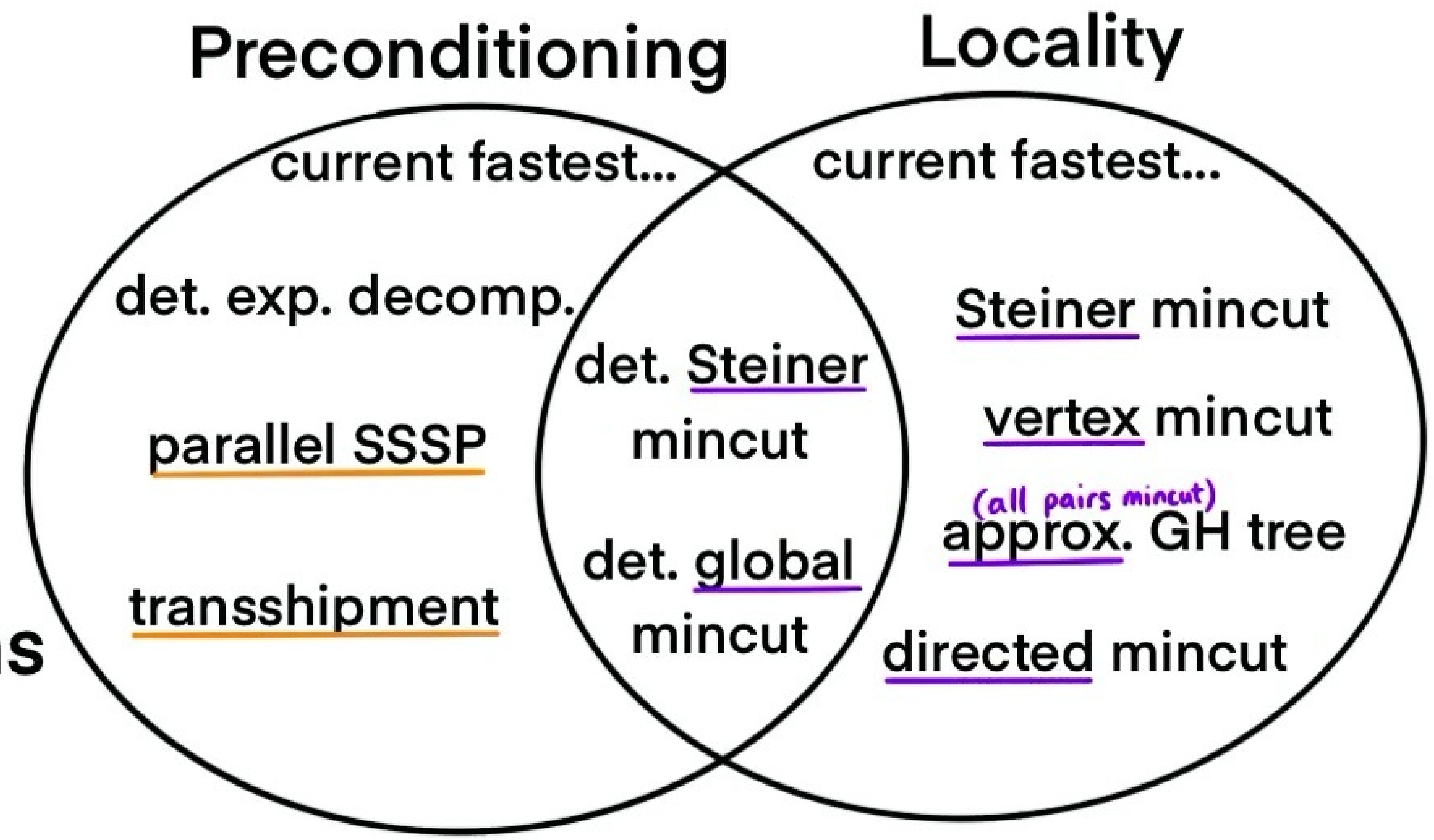

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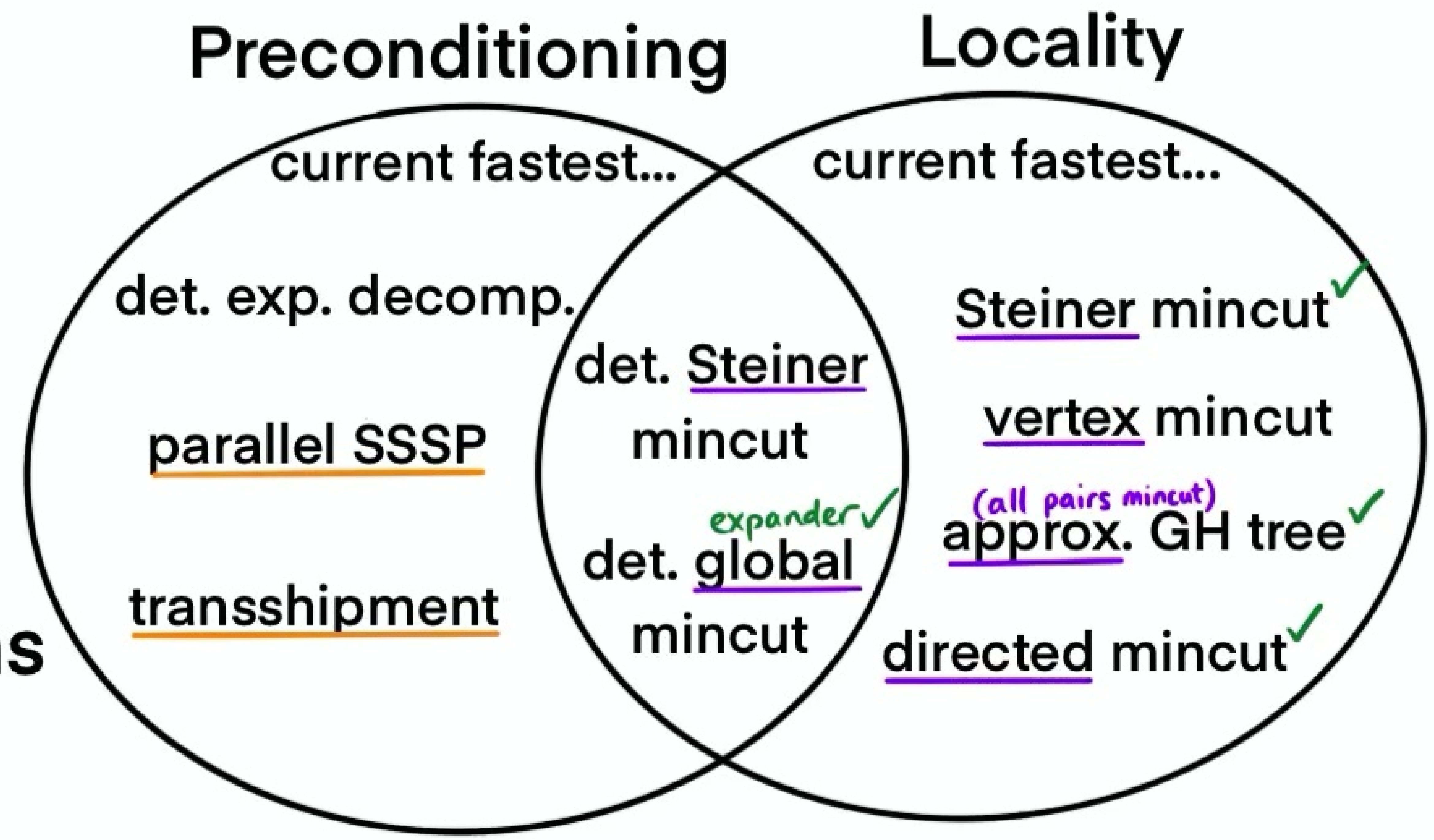

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Cutting-edge

- Mostly unexplored in the past => future potential
- Some results are remarkably simple
  - All tools were around 40+ years ago, was only missing perspective

Problems Studied in Talk

Locality:

- Minimum Isolating Cuts problem

  ⇒ simple, fastest Steiner mincut algorithm

  ⇒ simple, fastest single-source mincut algorithm

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- Minimum Isolating Cuts problem
  - ⇒ simple, fastest Steiner mincut algorithm
  - ⇒ simple, fastest single-source mincut algorithm
- Directed mincut: simple, fastest algorithm

Preconditioning:

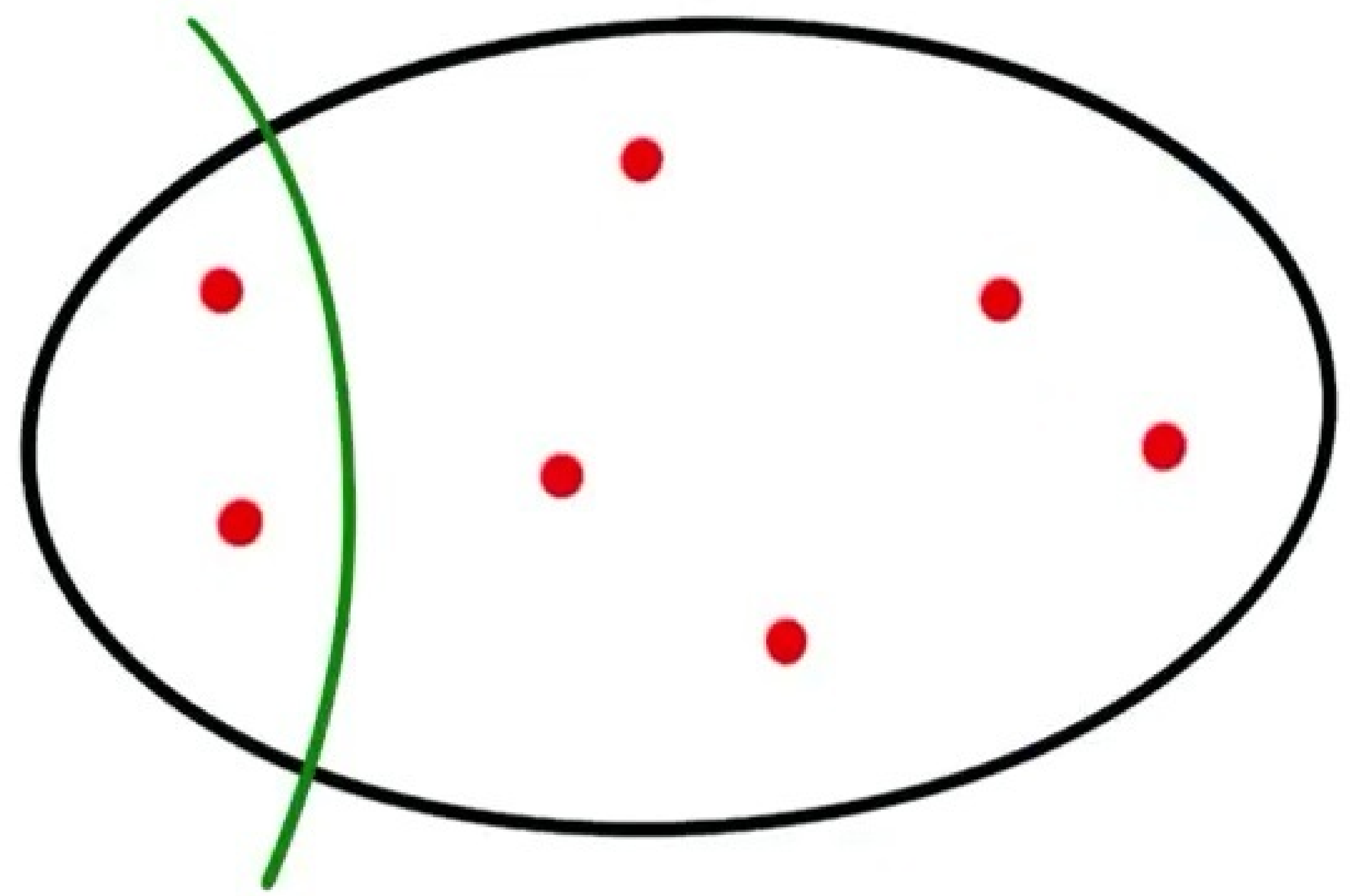
- Deterministic mincut: first almost-linear time algorithm
  - simple on expanders

Part I: Locality

1. Steiner mincut
2. Directed mincut

Steiner Mincut

Given a graph and a set R of terminals, find the mincut that separates at least two terminals



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- Generalizes s-t mincut: R = {s,t}
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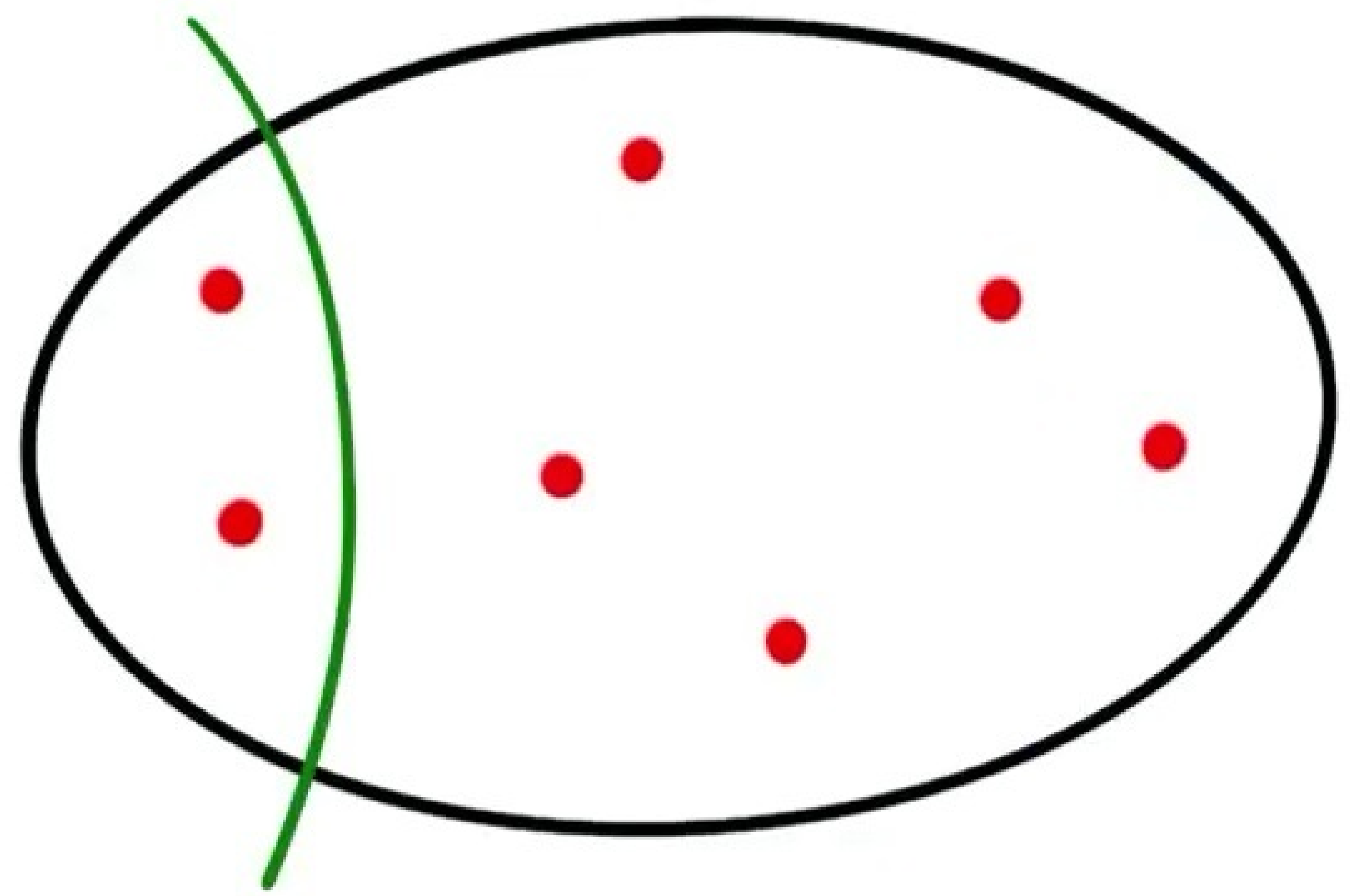

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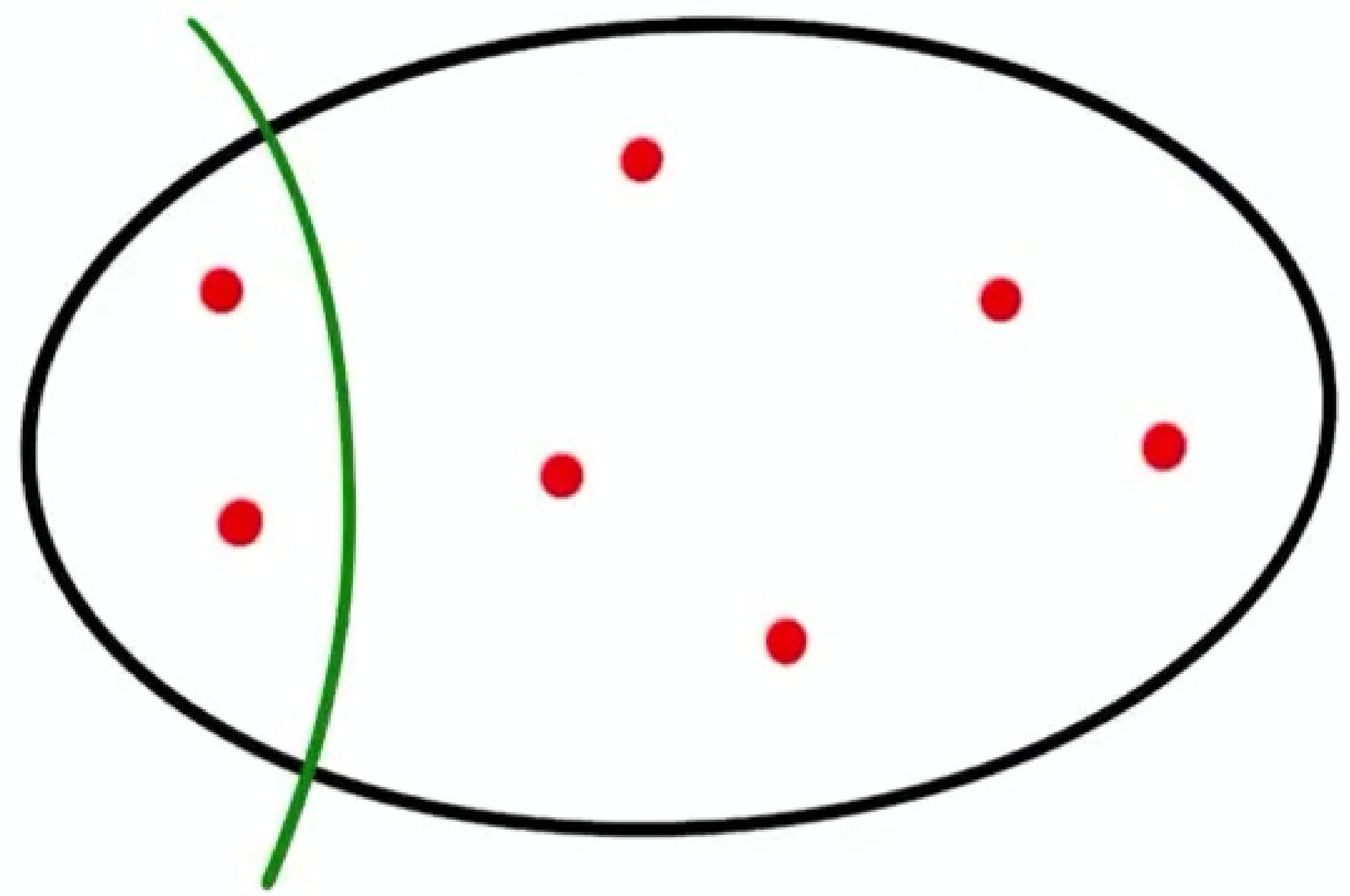

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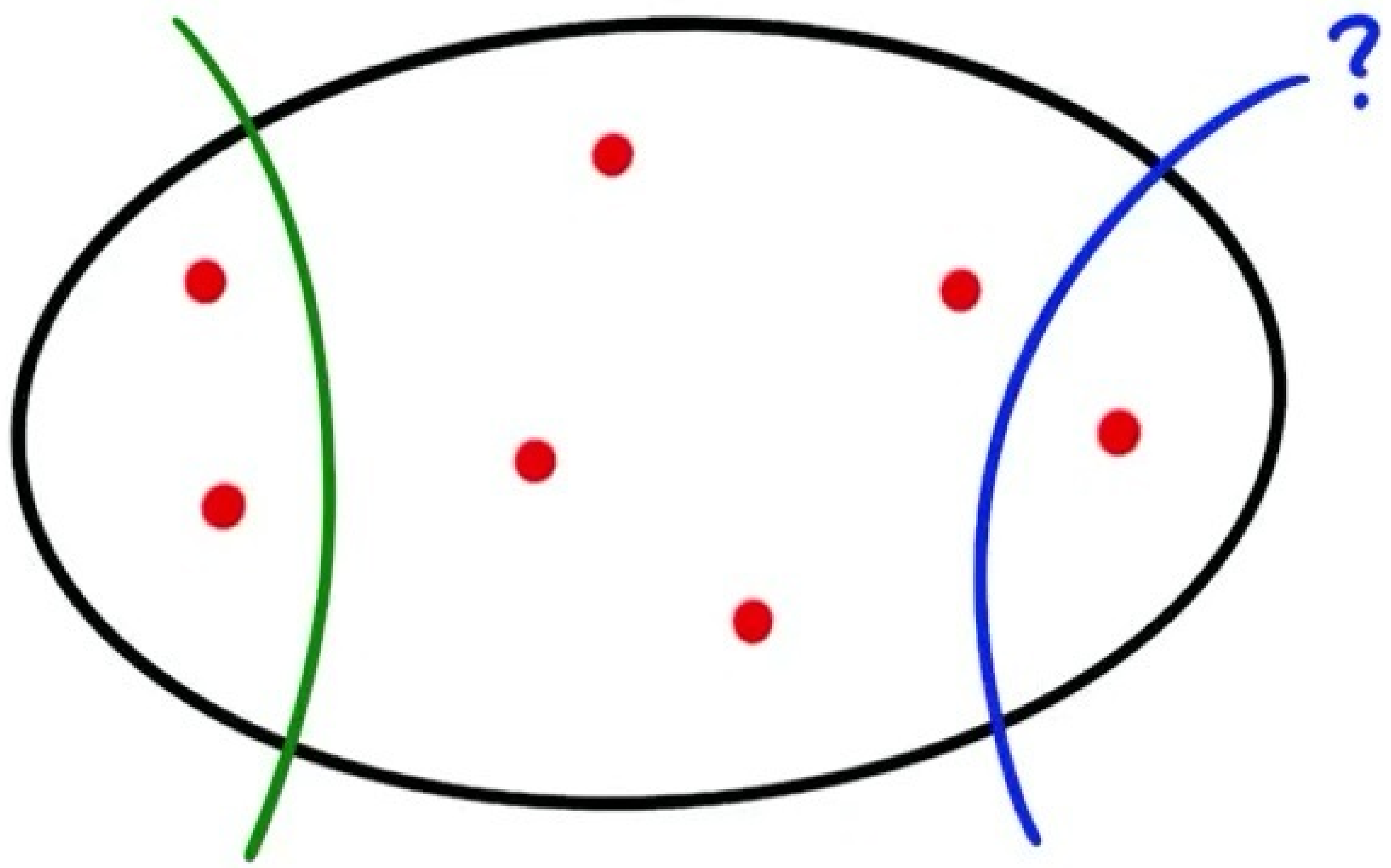

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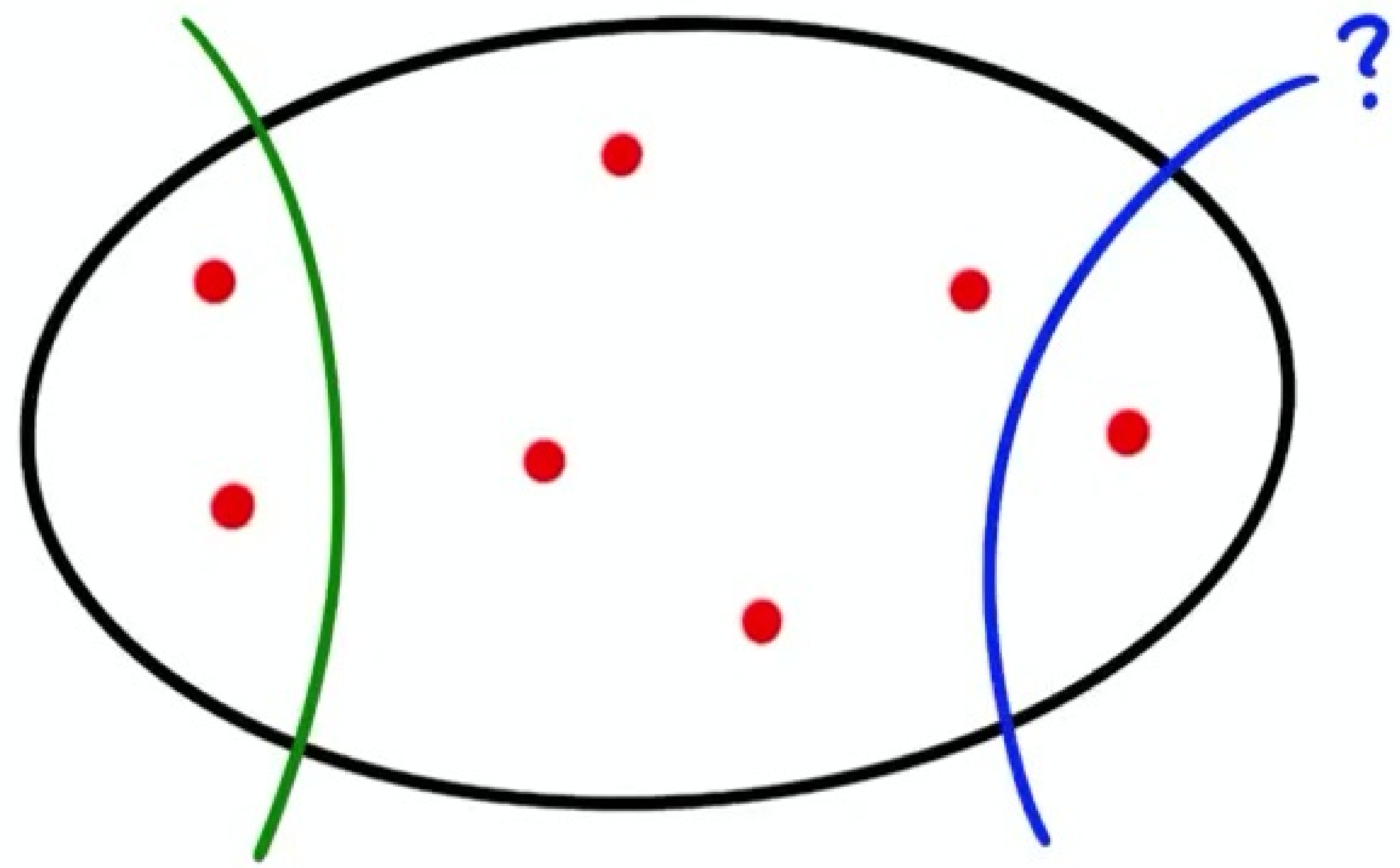

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Can be reduced to this case! (random sampling)

Steiner Mincut

Theorem: unbalanced Steiner mincut can be solved in polylog(n) max-flow calls

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- Simple algorithm in O(log n) max-flows

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- Minimum Isolating Cuts: new problem capturing the locality assumption
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Theorem: (general) Steiner mincut can be solved in polylog(n) max-flow calls

- Simple random sampling: reduce to unbalanced!

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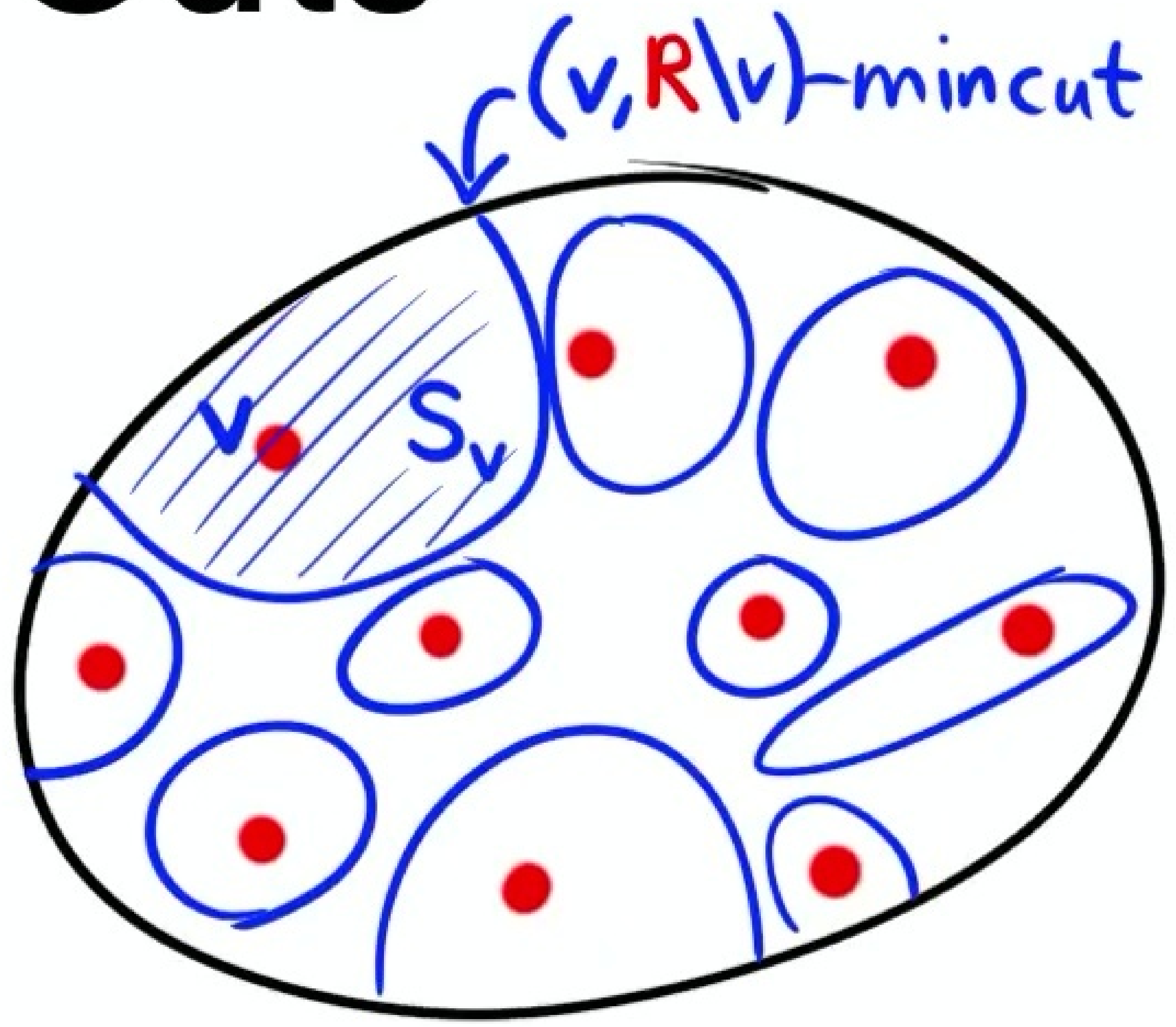
Given a graph and a set R of terminals, find, for each terminal v, the mincut S_v that isolates that terminal



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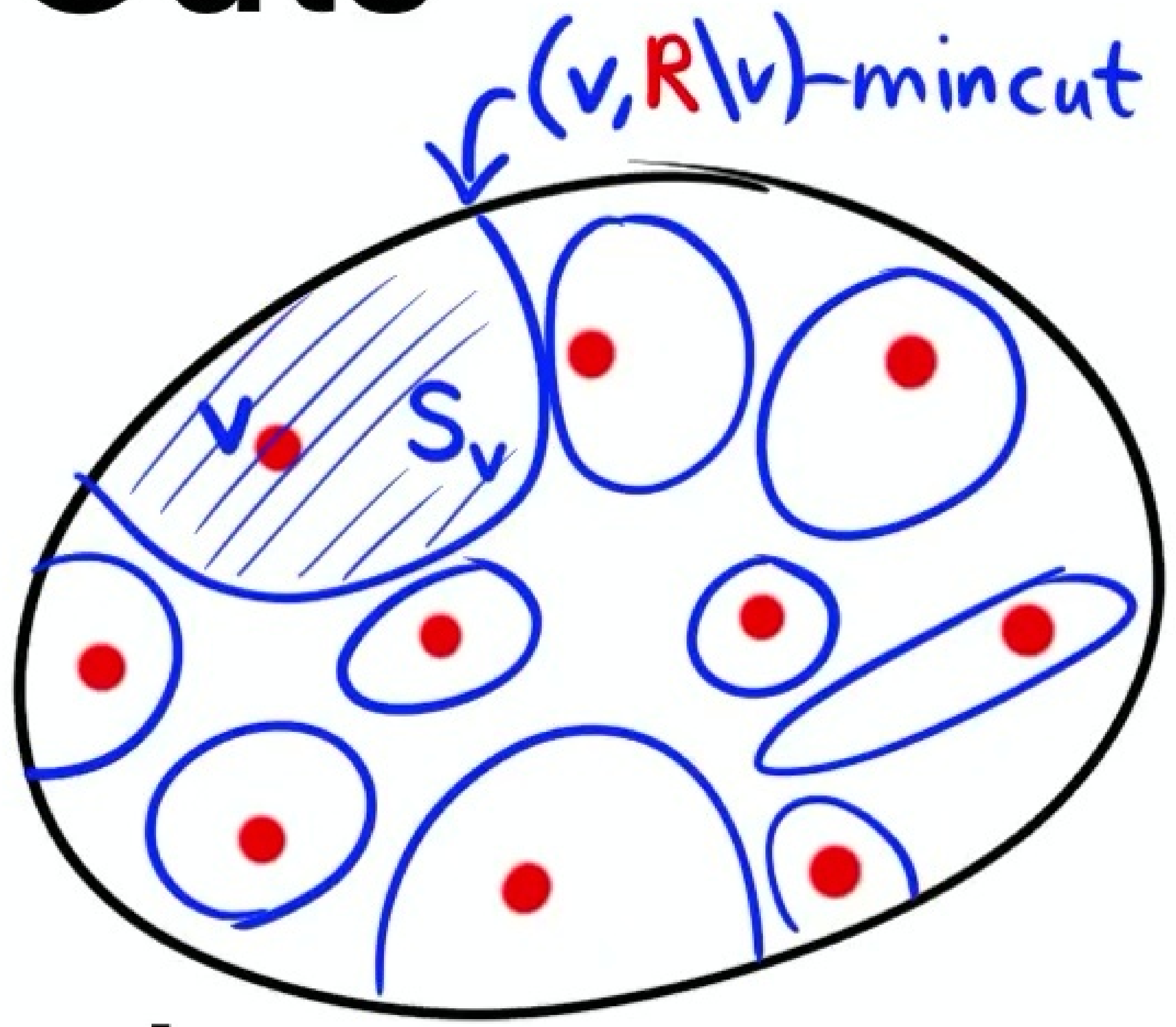

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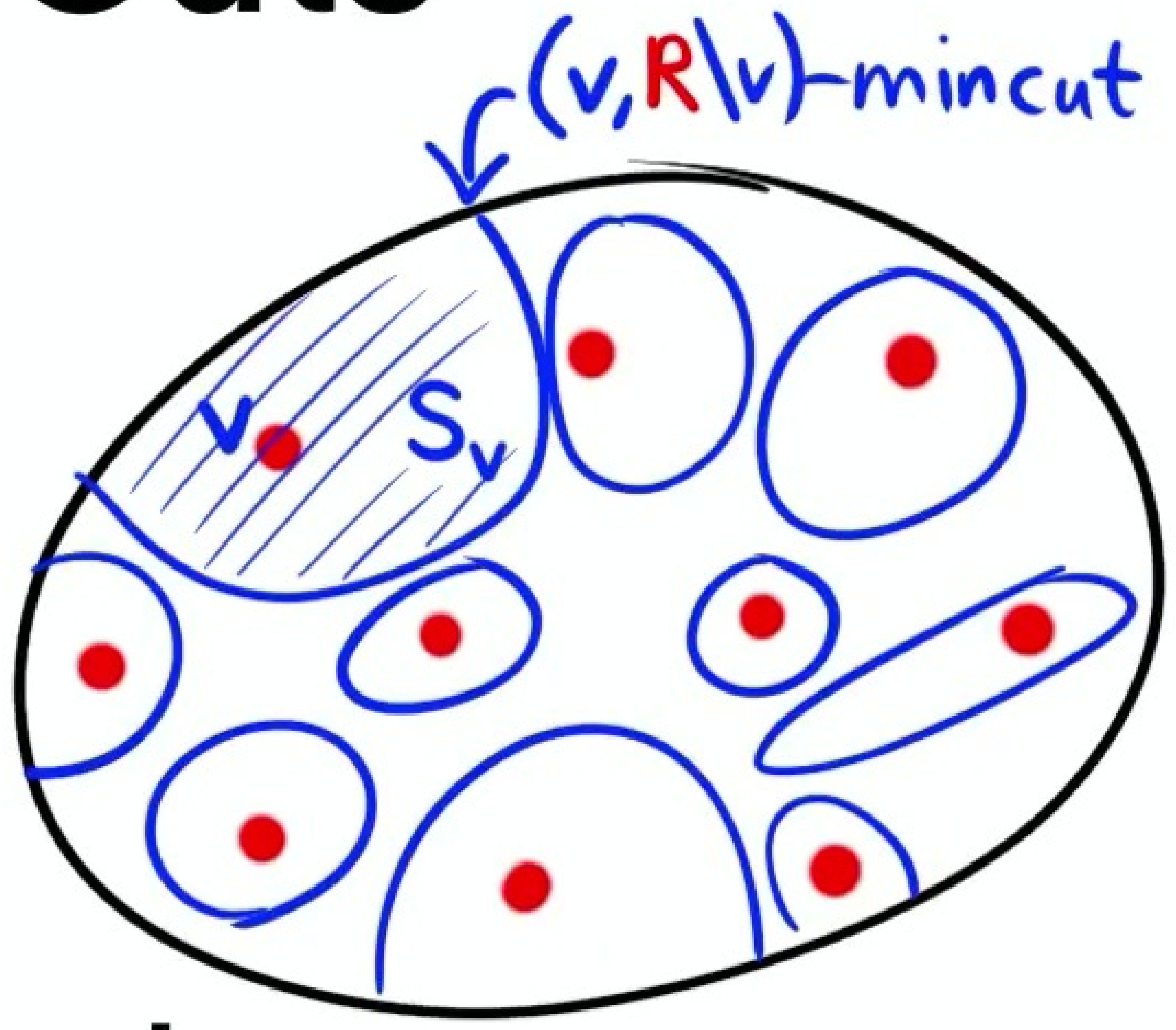

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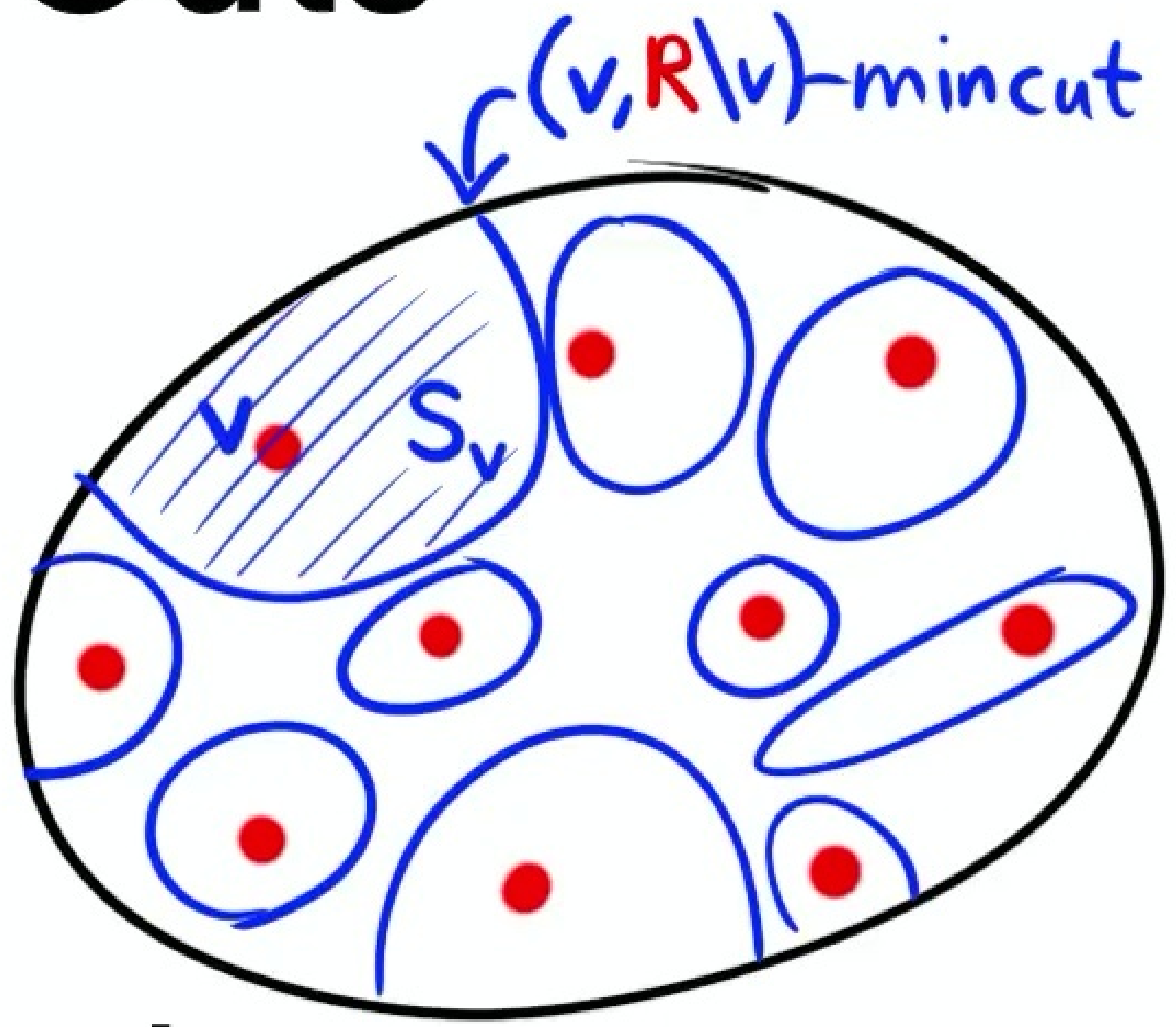

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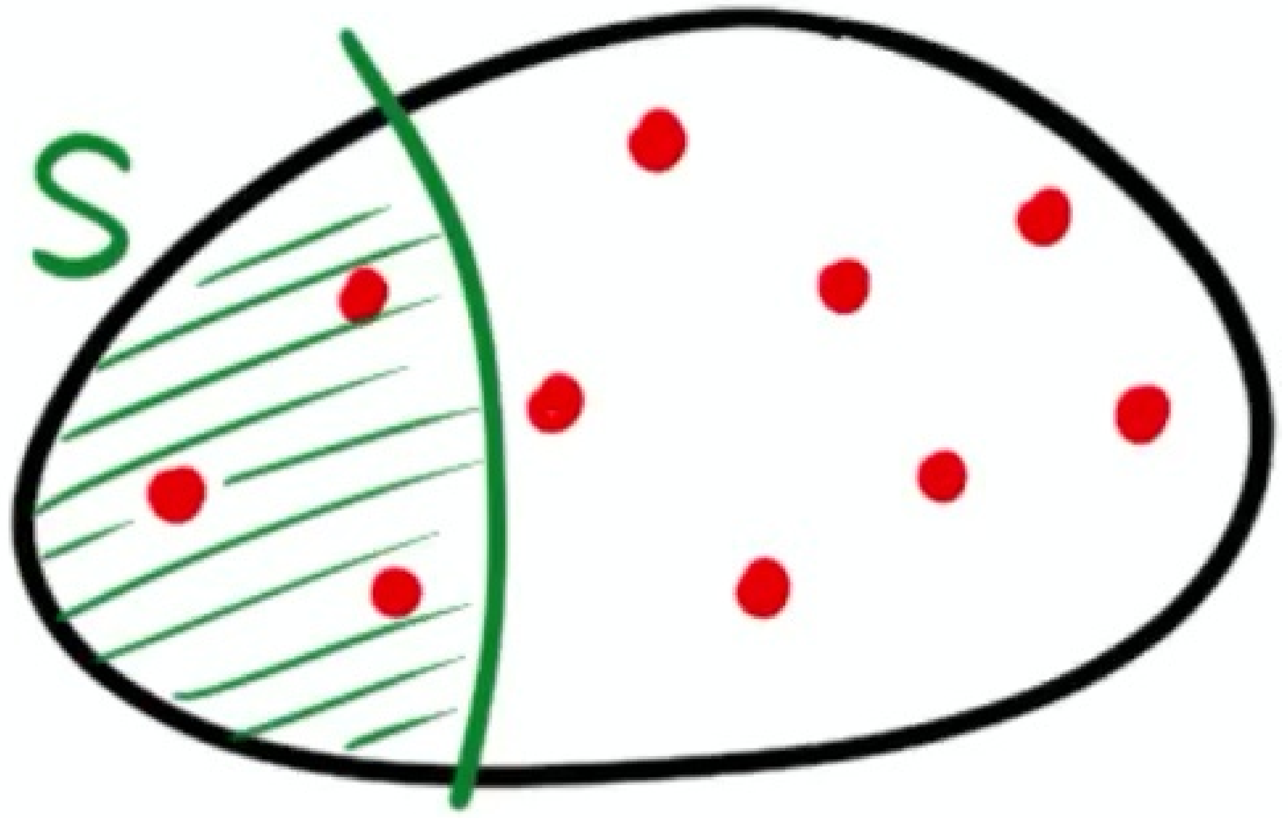

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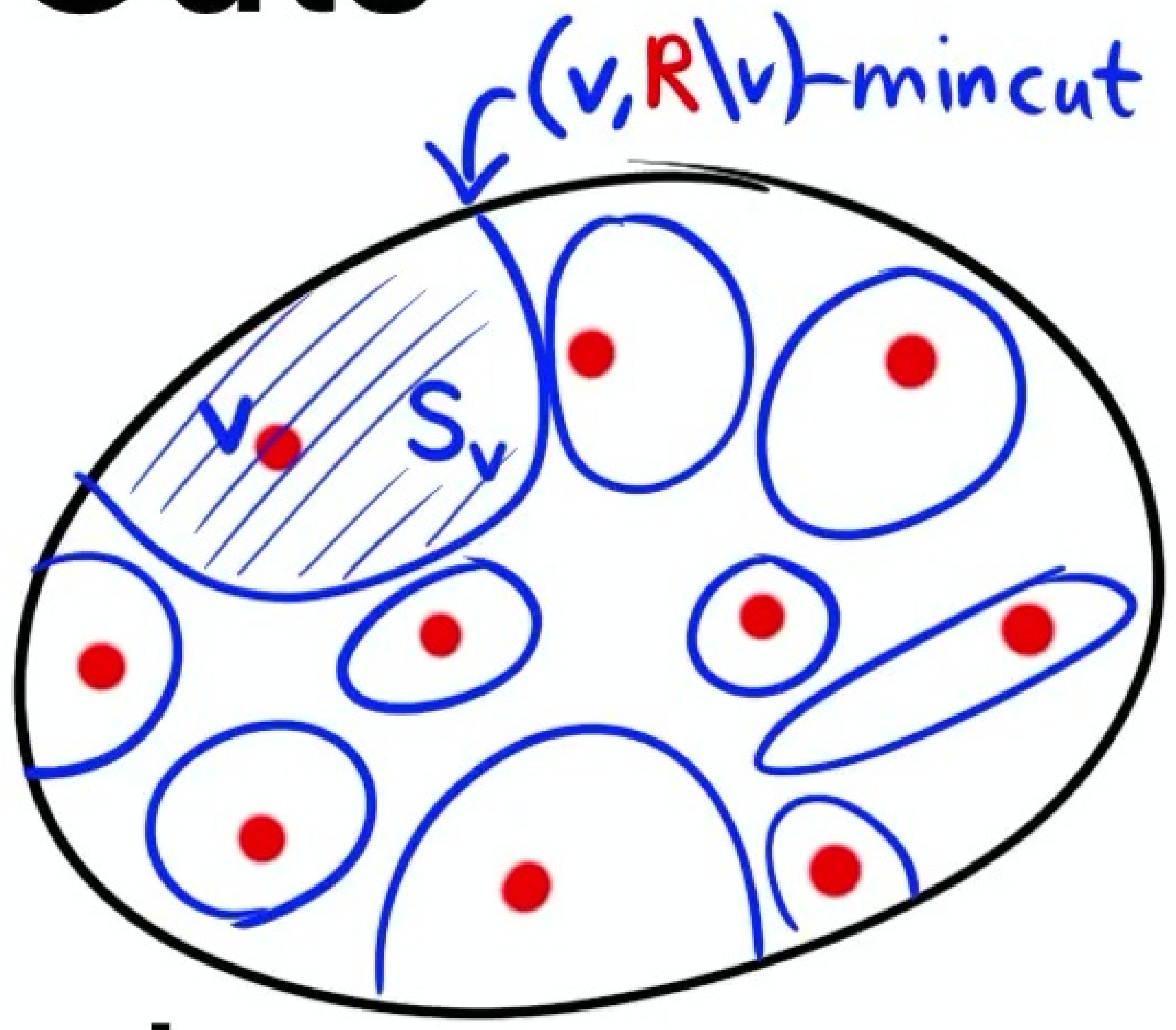

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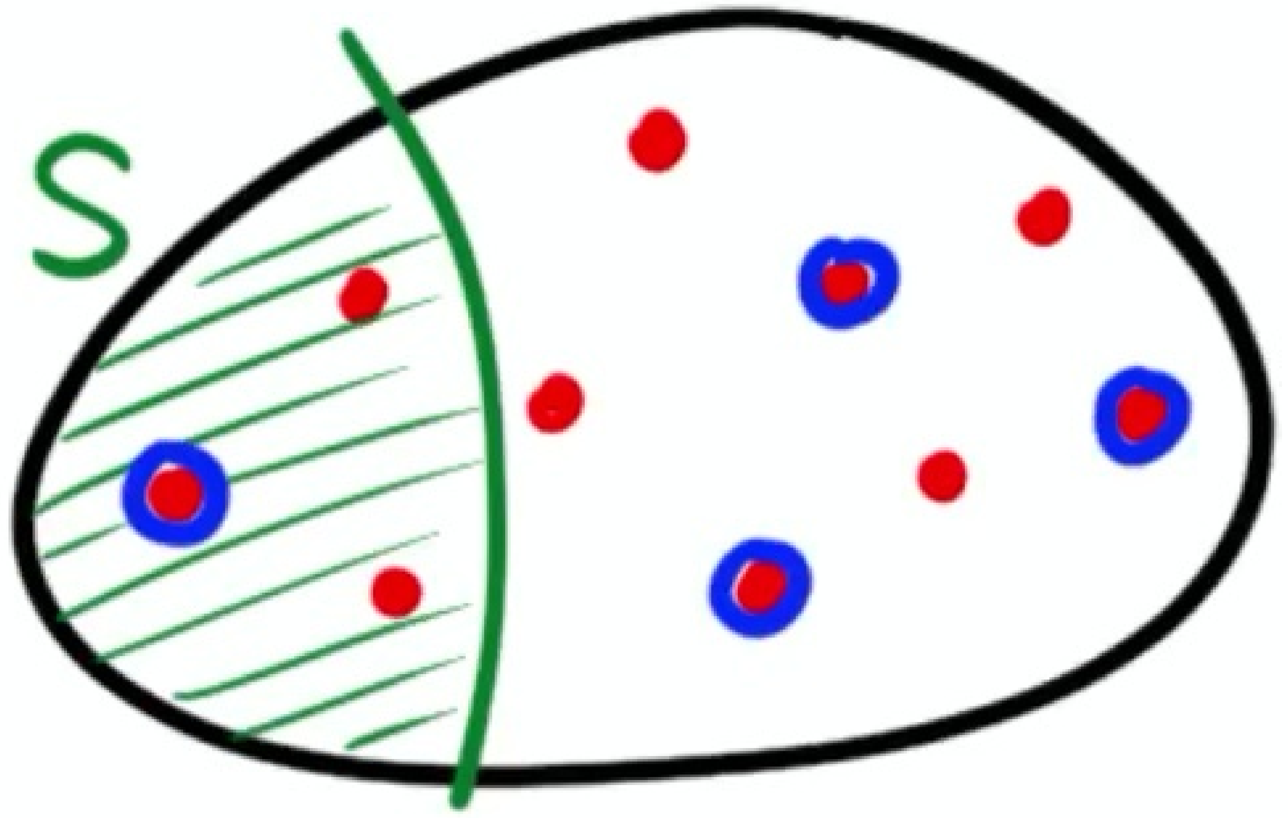

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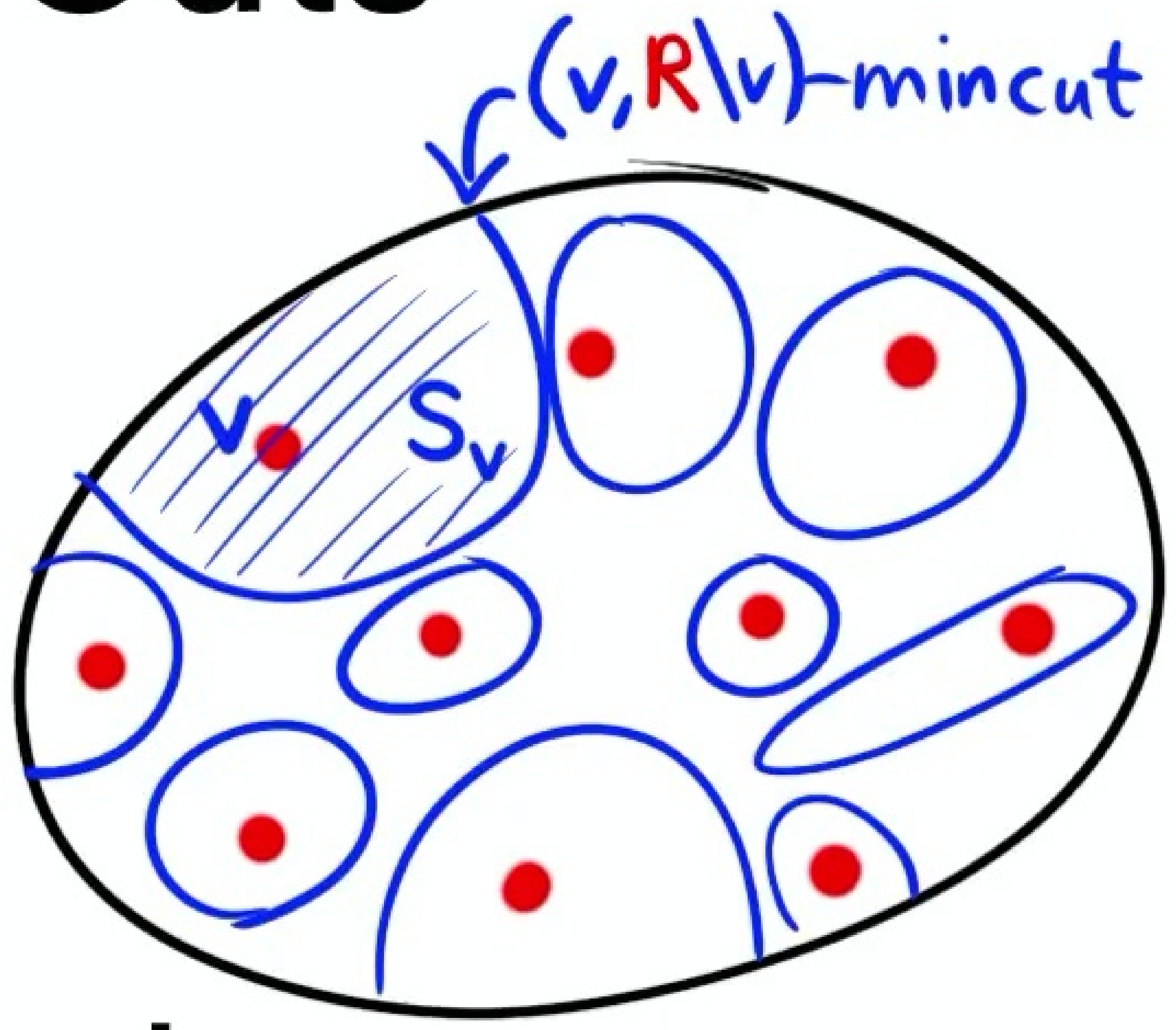

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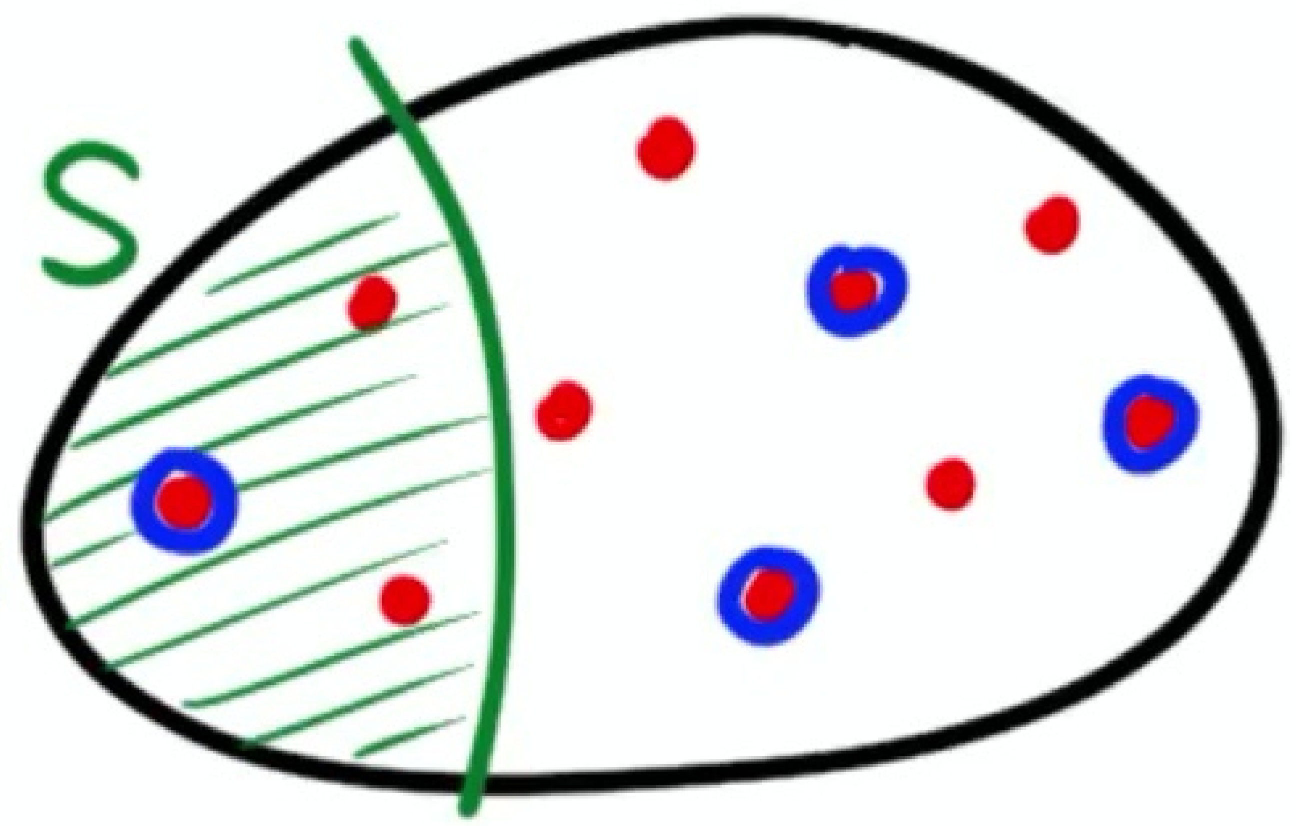

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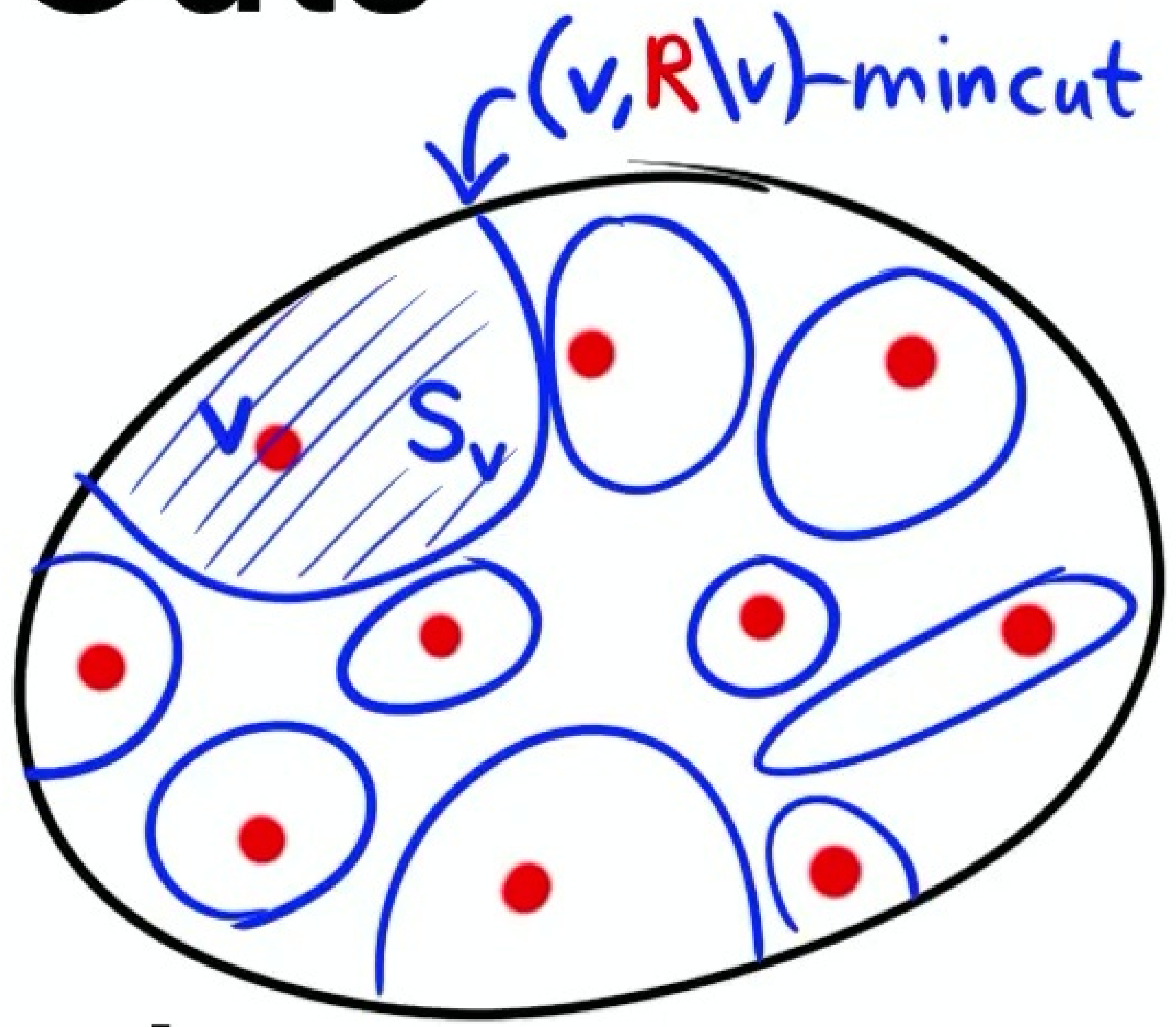

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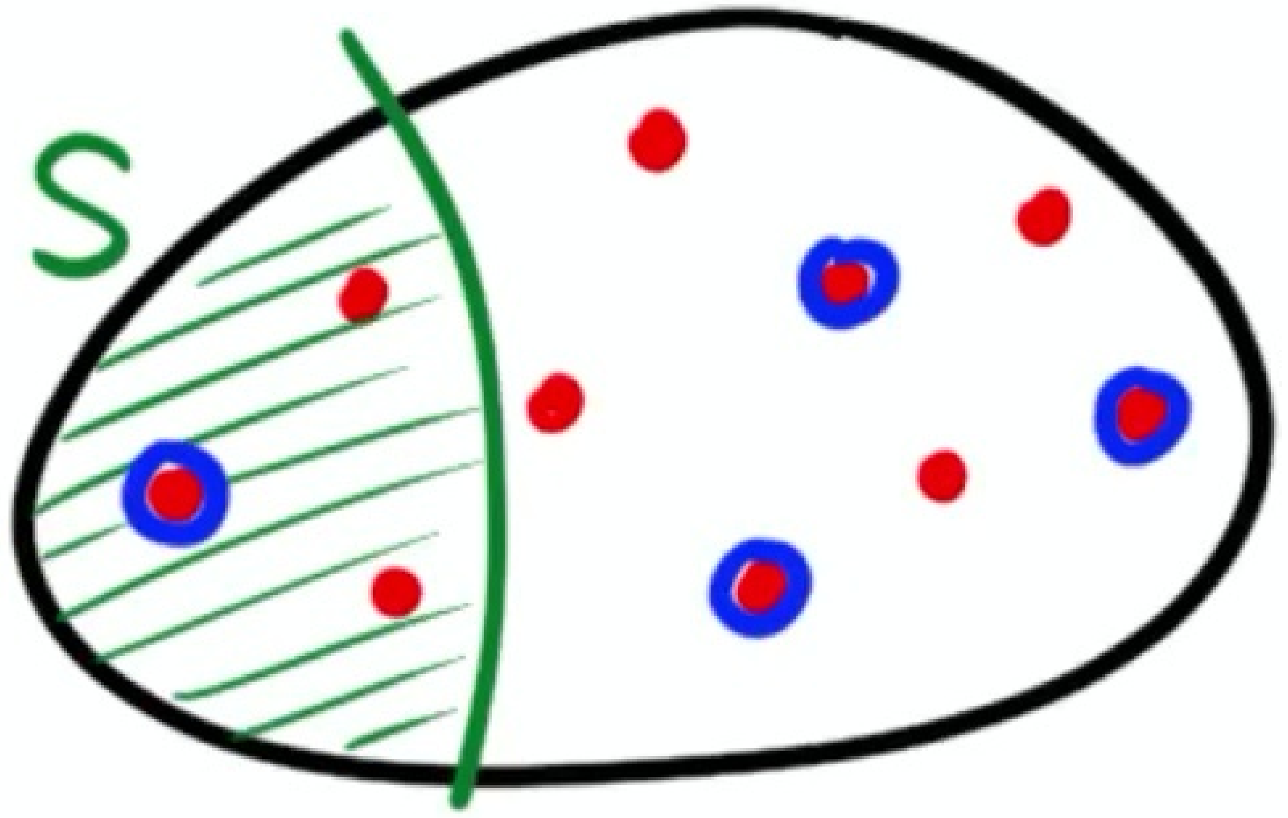
Reduce general Steiner mincut to unbalanced:

Sample at rate 1/2, 1/4, 1/8, ...

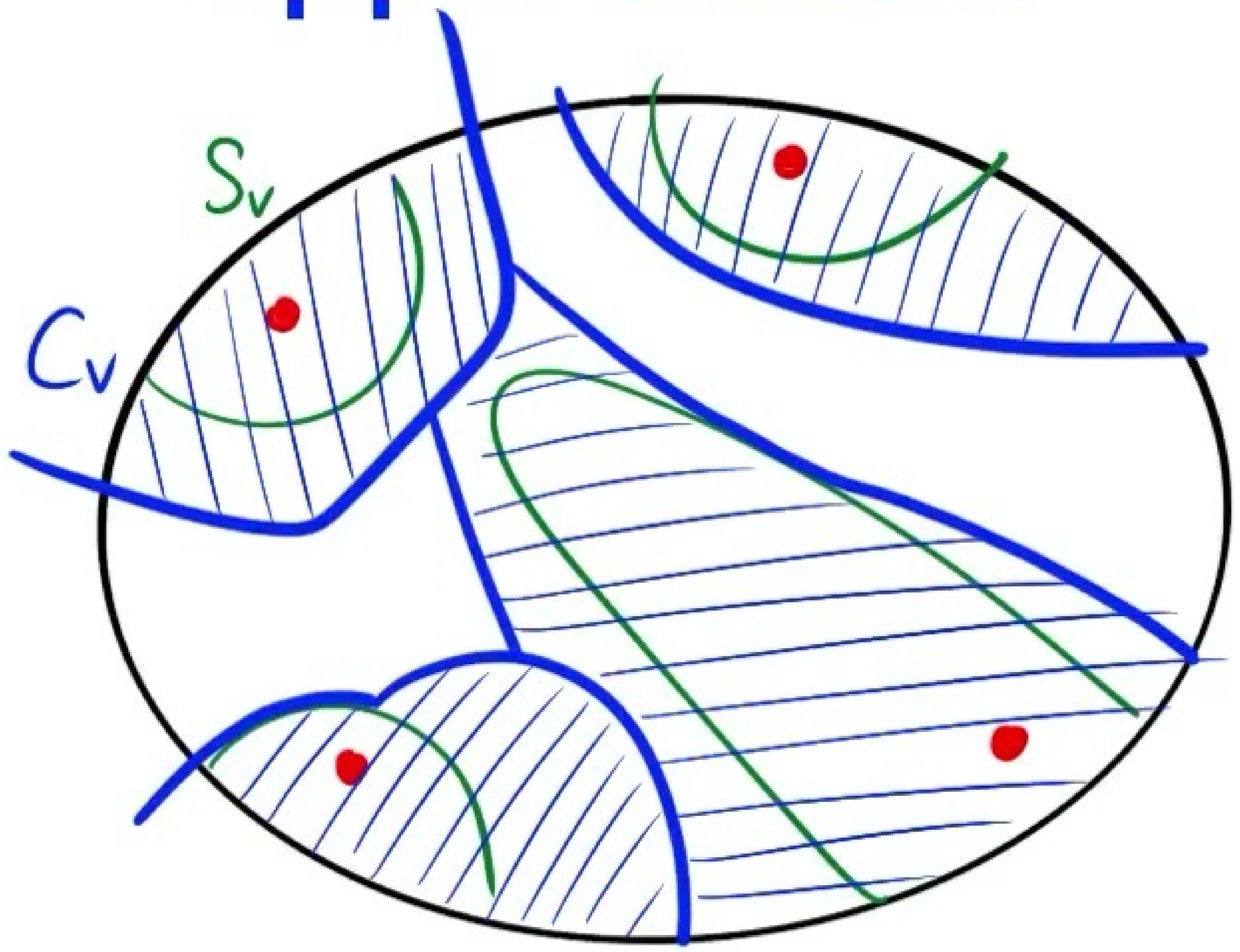
If sample at rate ~ $\frac{1}{|S \cap R|}$, then constant prob. success



Minimum Isolating Cuts

Idea: compute an "upper bound" for each isolating cut

- $C_v \supseteq S_v \quad \forall v \in R$
- C_v are disjoint



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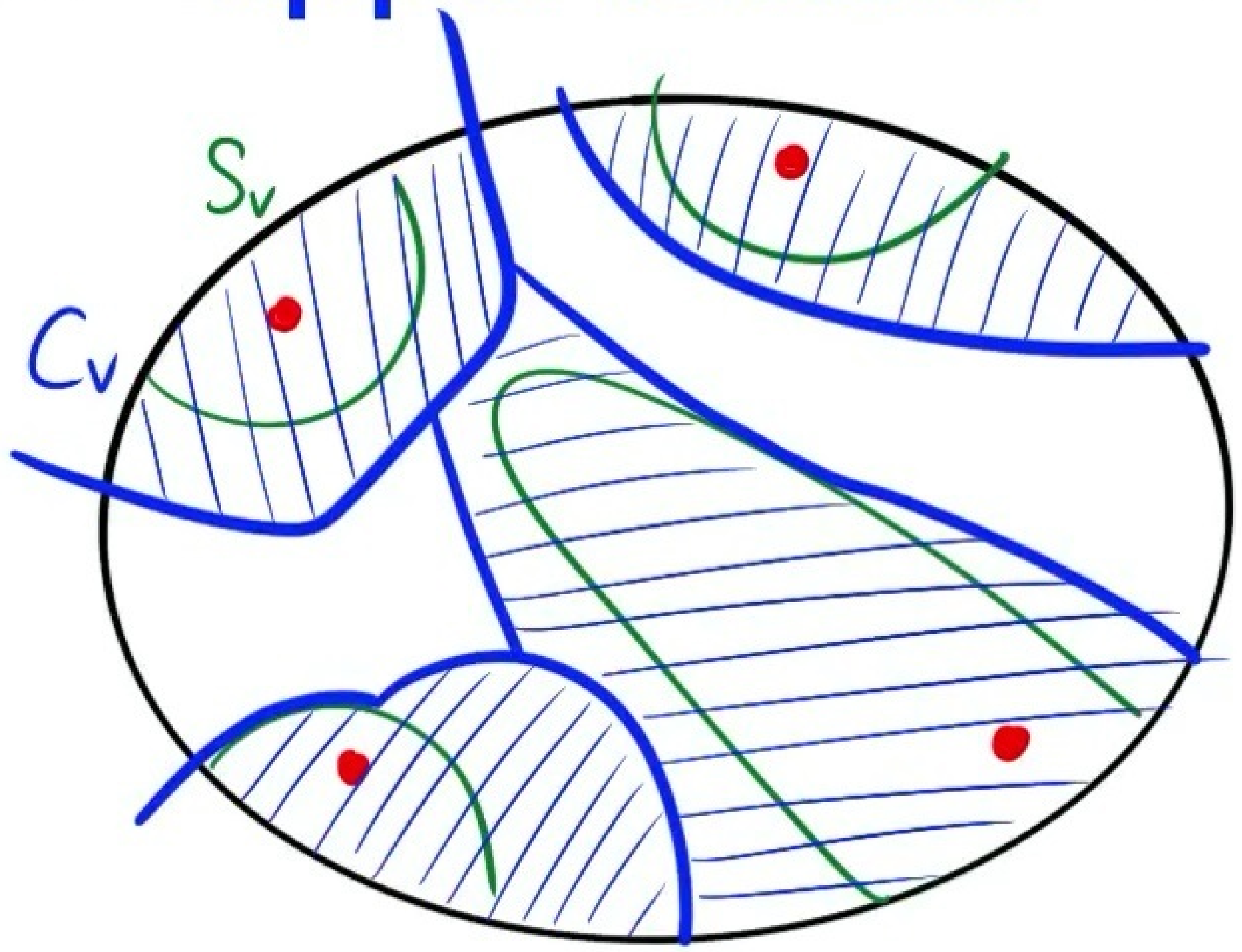

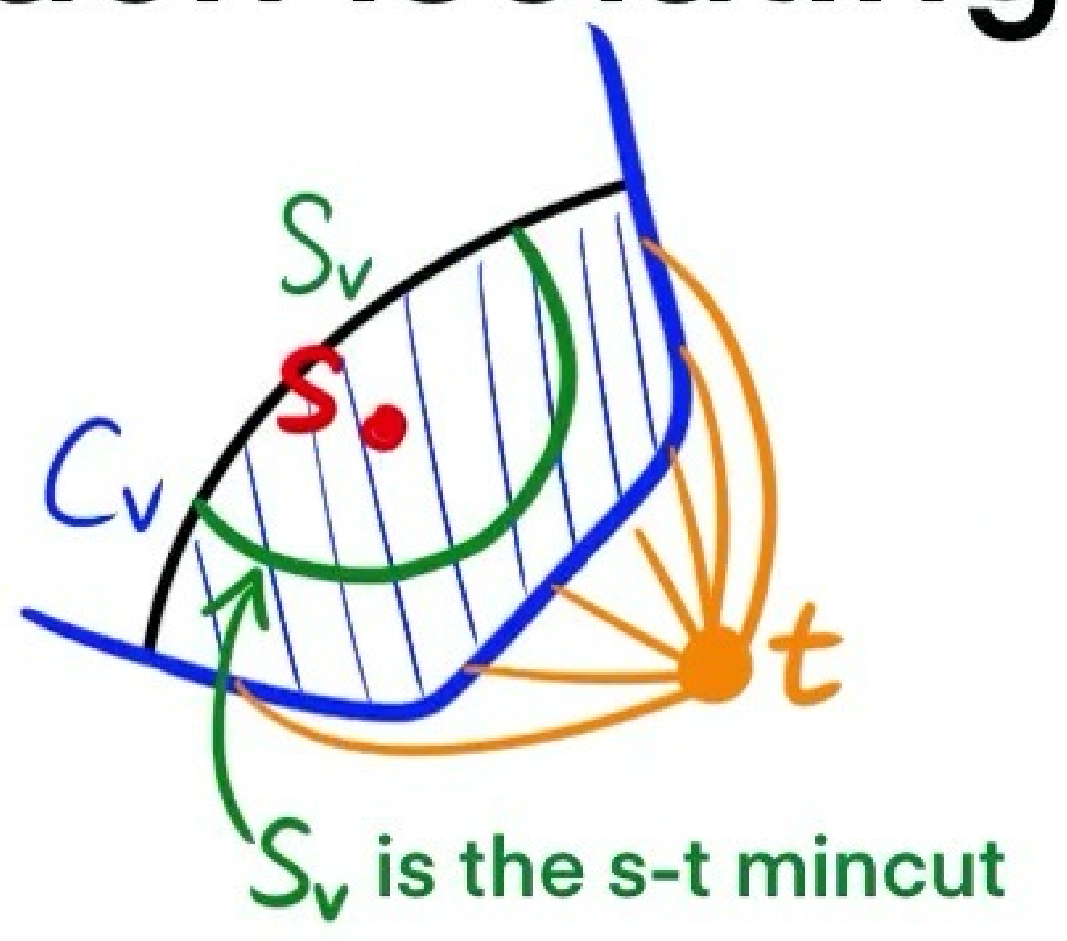

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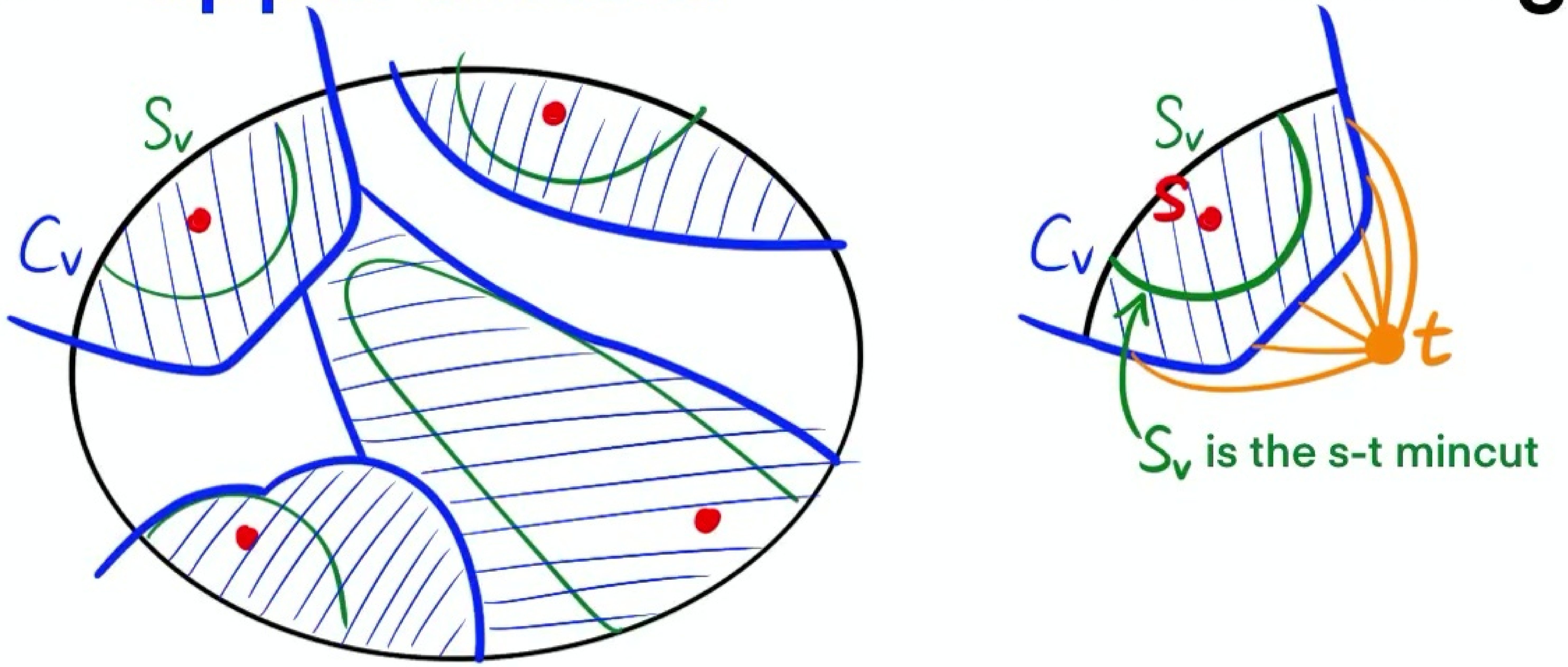

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Each edge in at most 2 such graphs => total size ≤ 2m

=> max-flow time on O(n) vertices, O(m) edges

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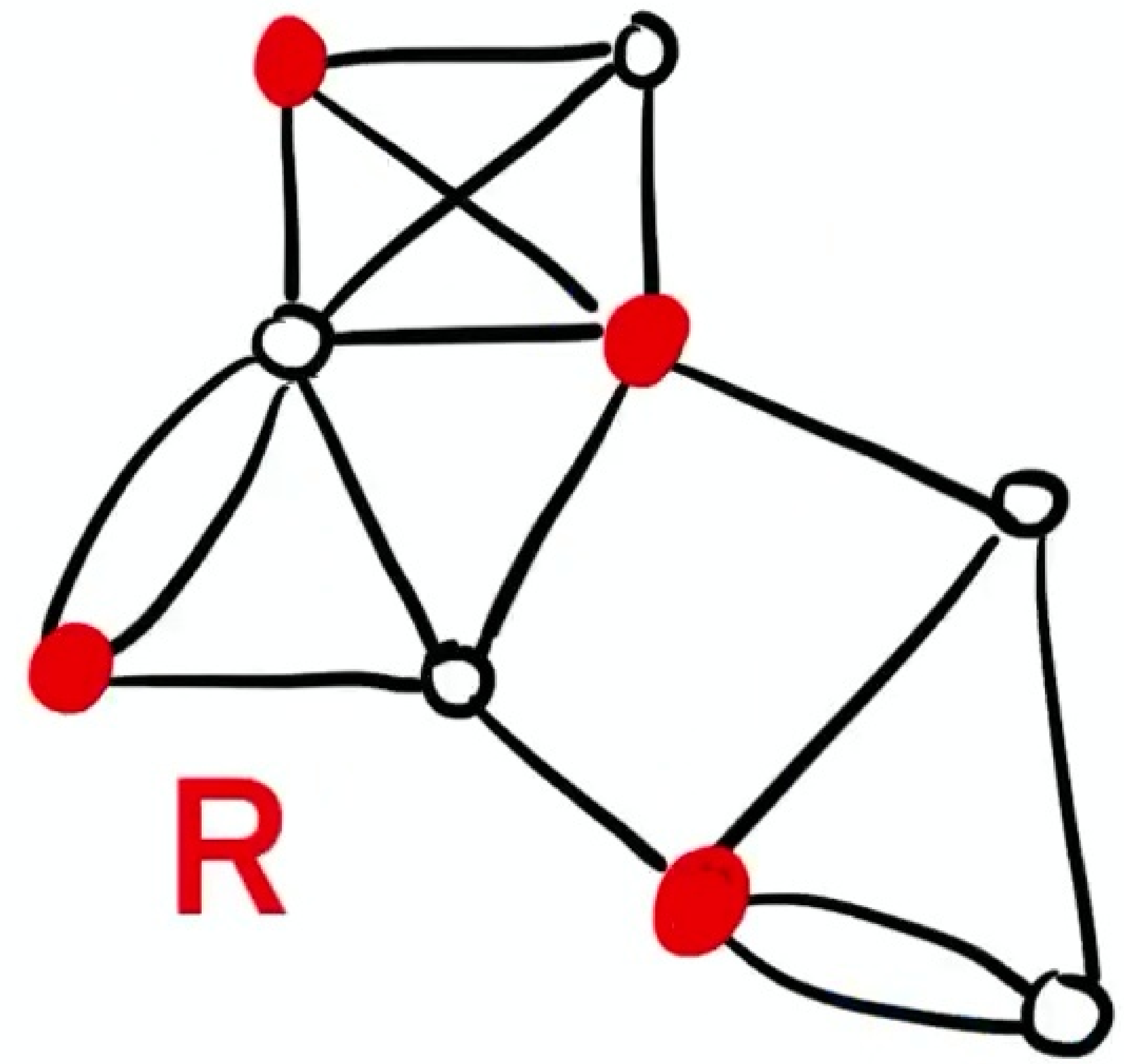
  - Want: each pair s,t in R is separated in at least one of them



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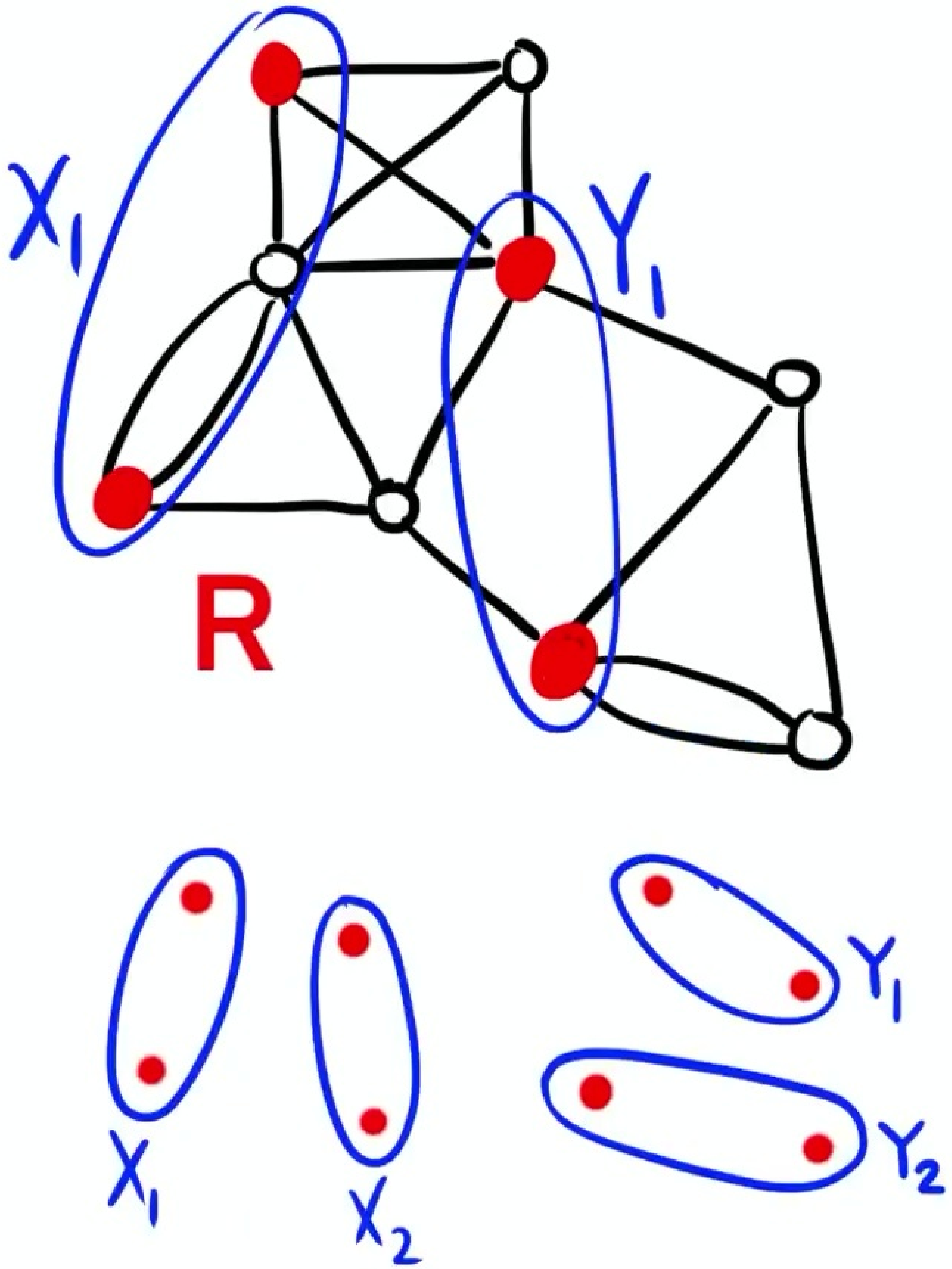

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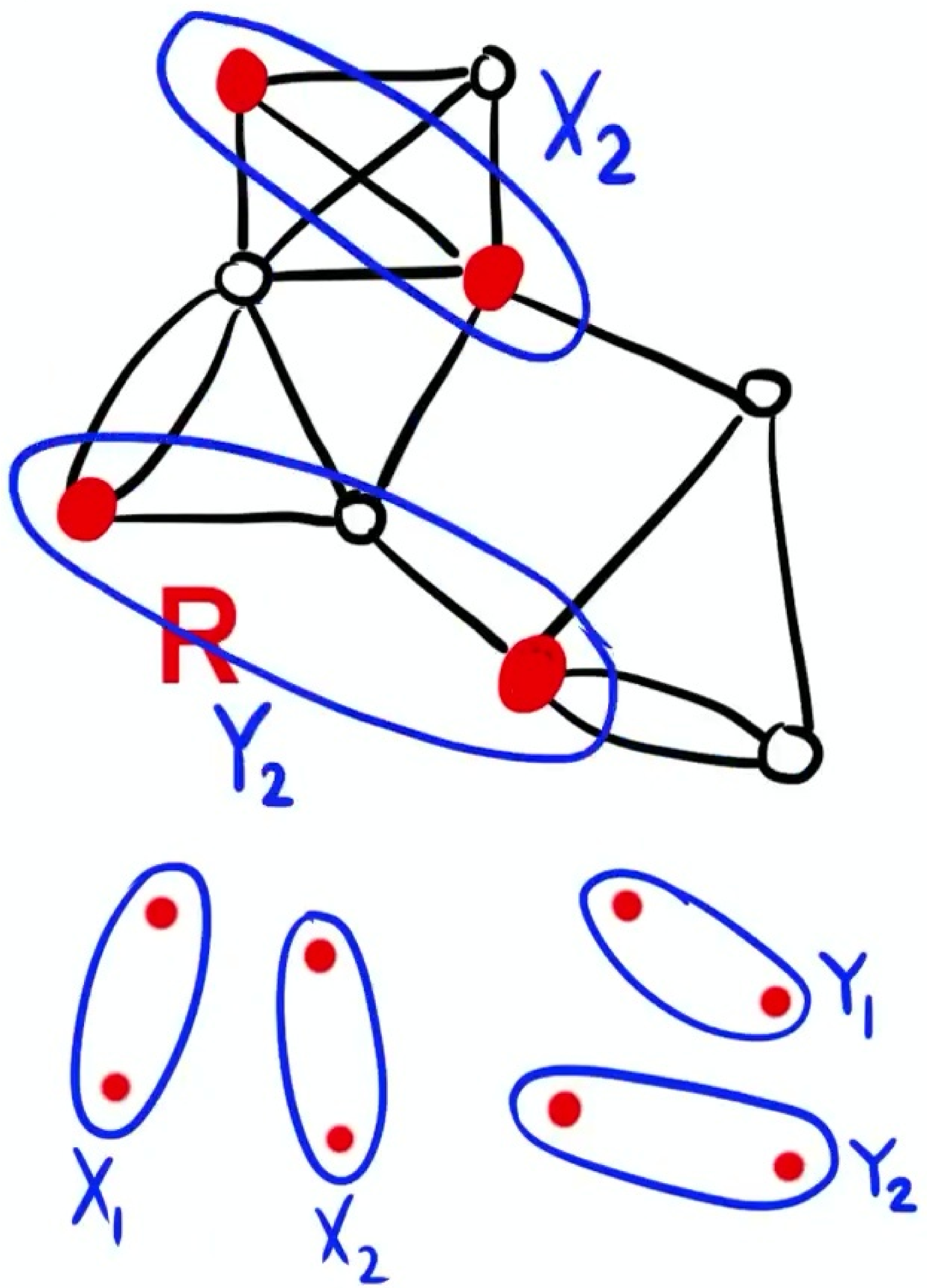

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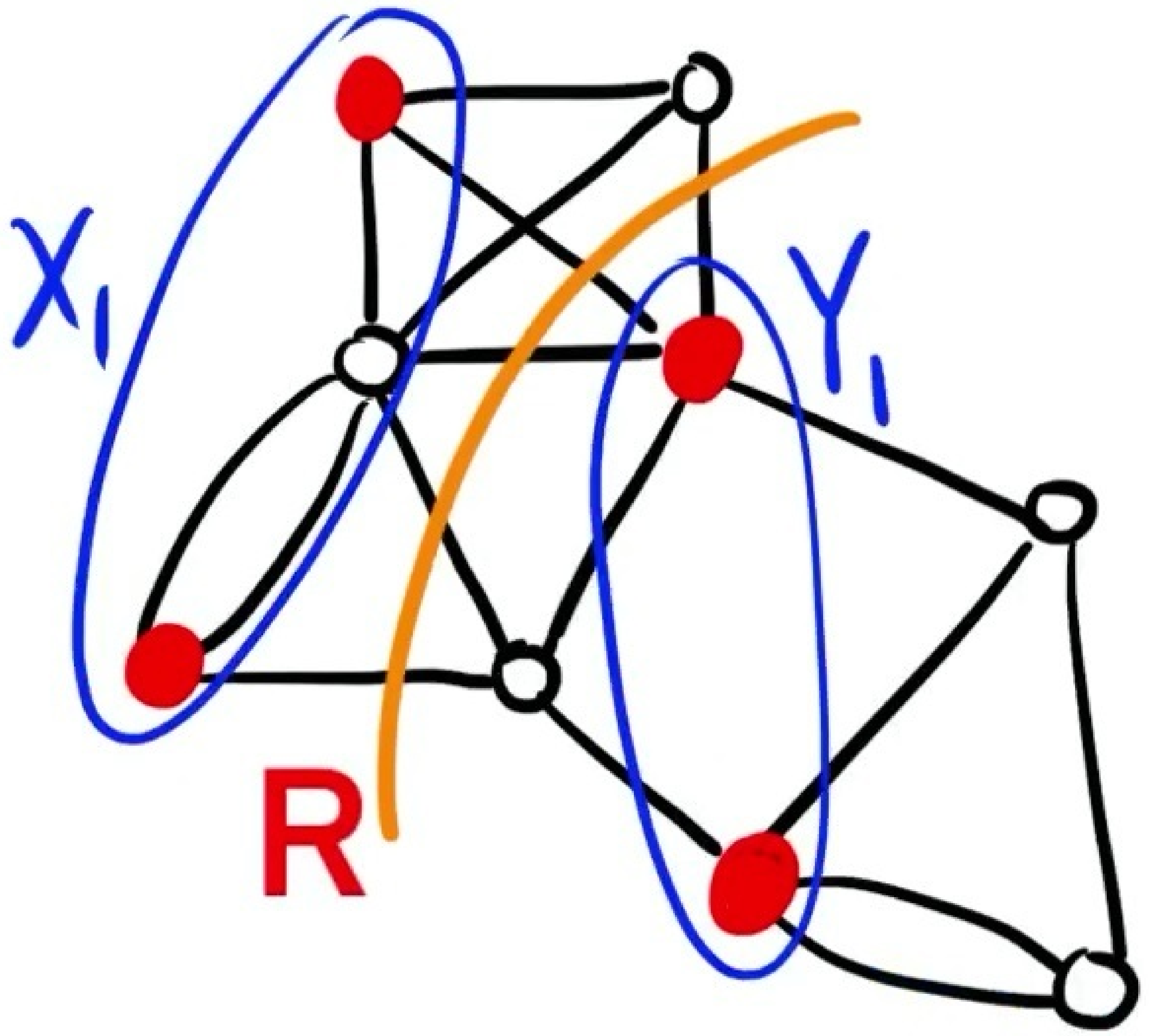

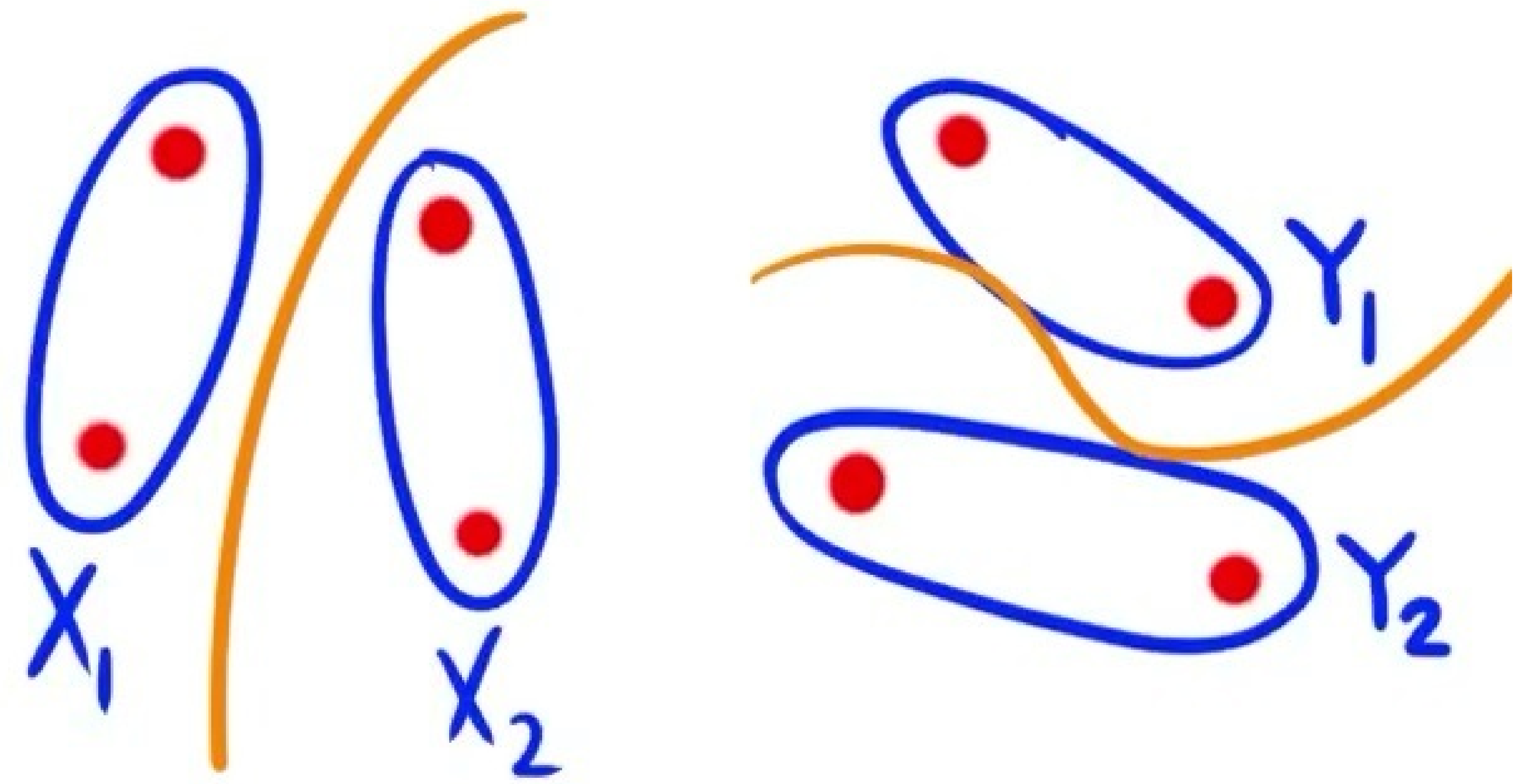

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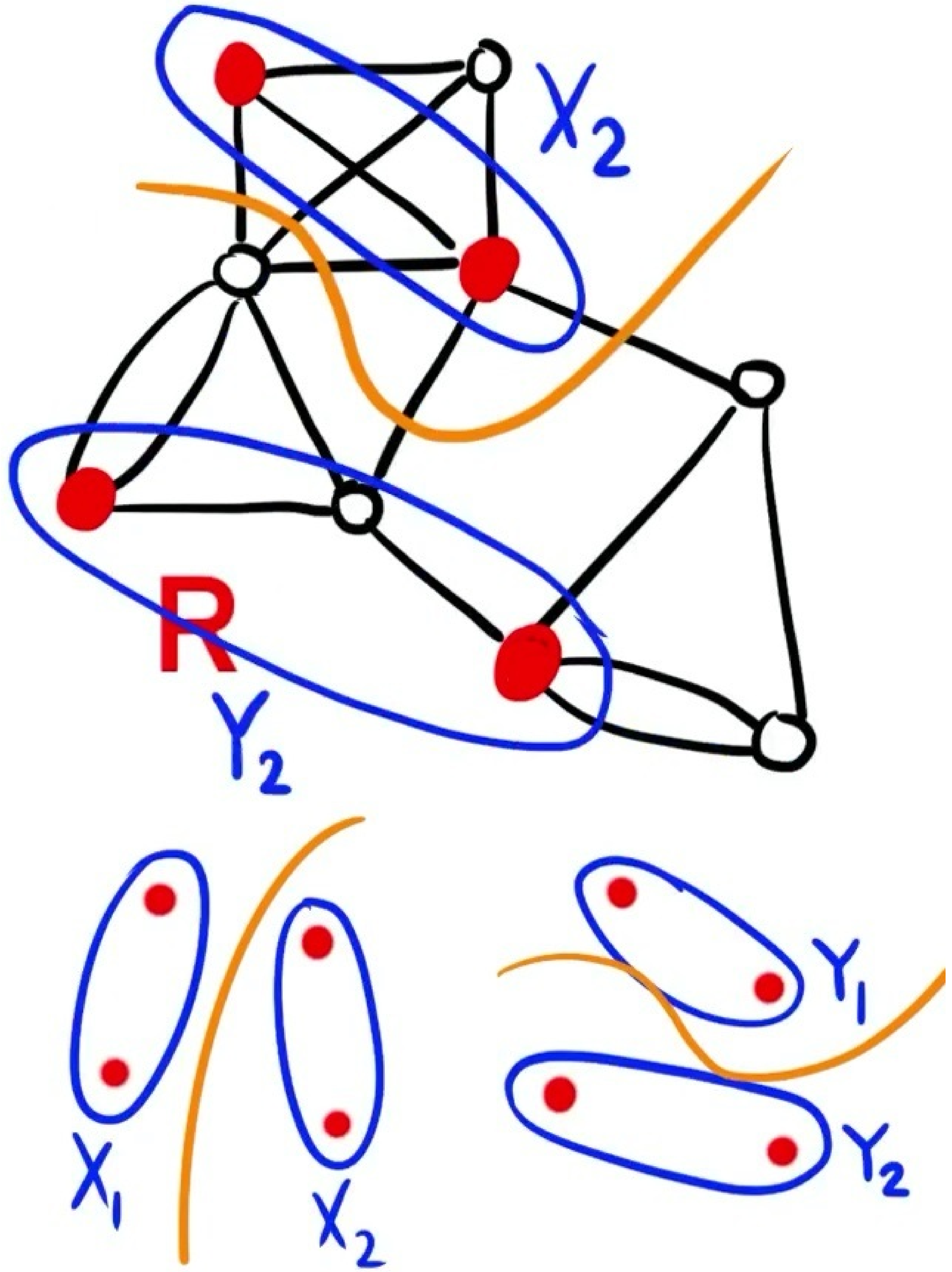

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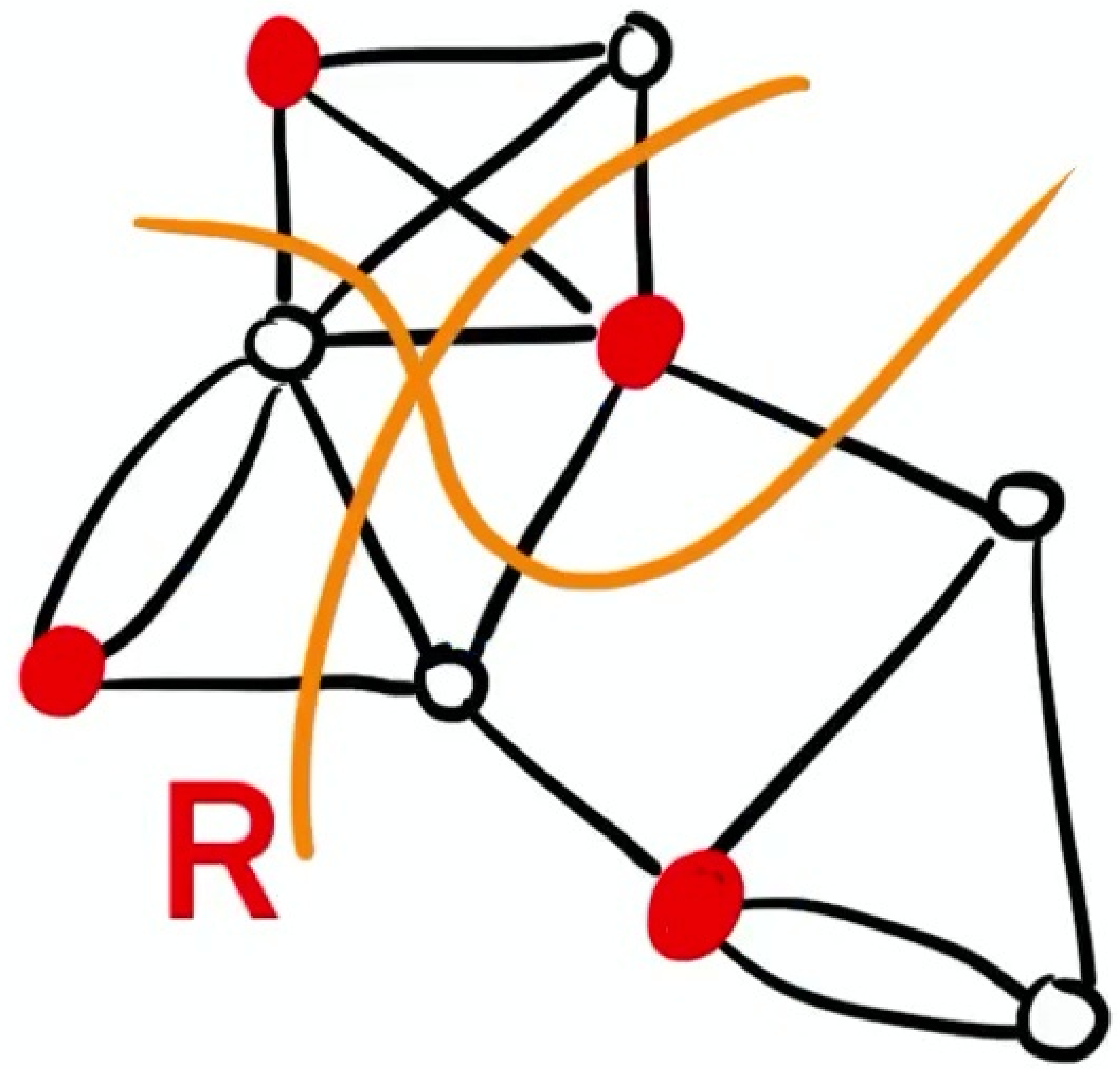
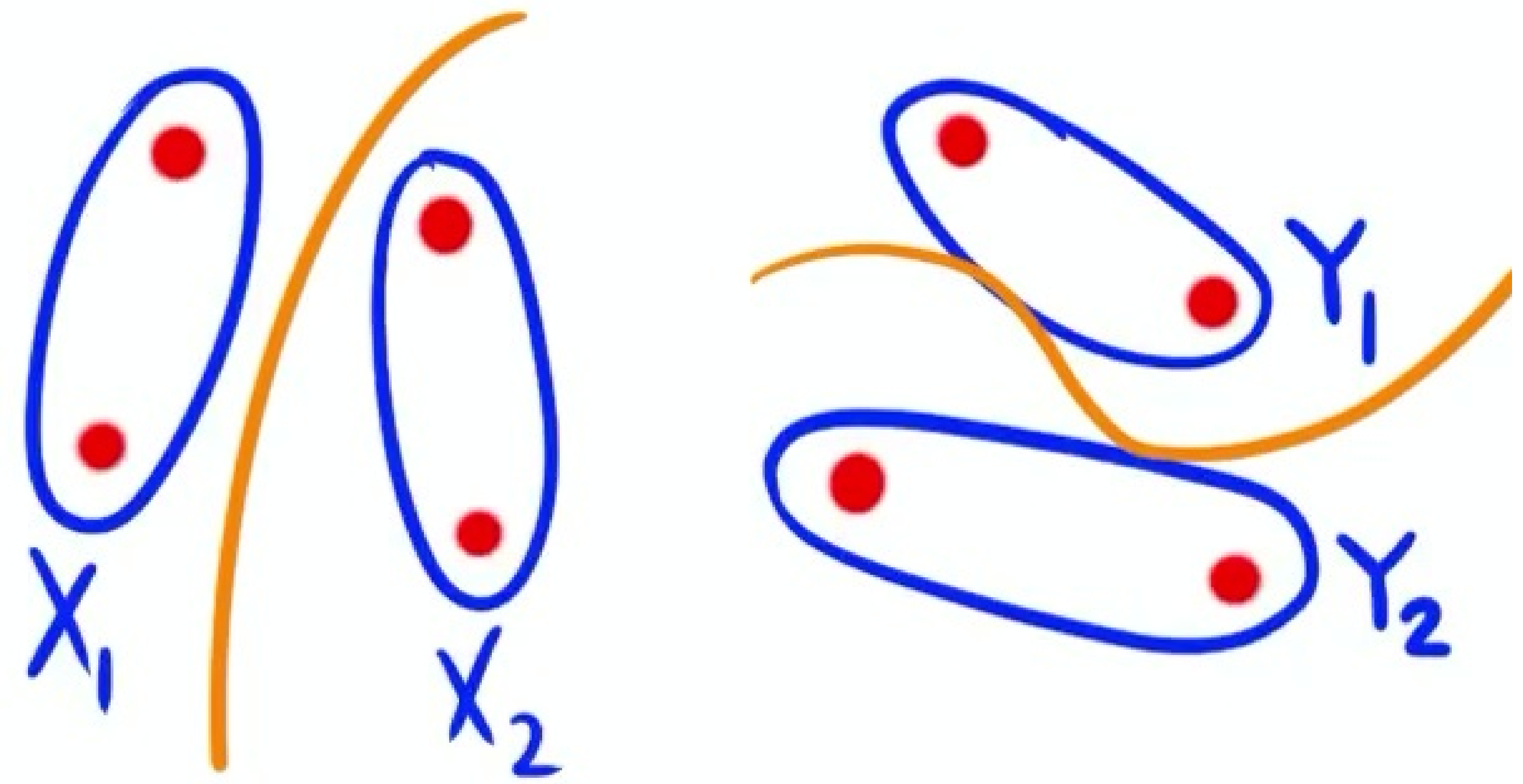
Claim: Union of min-cuts separates all of R





Minimum Isolating Cuts

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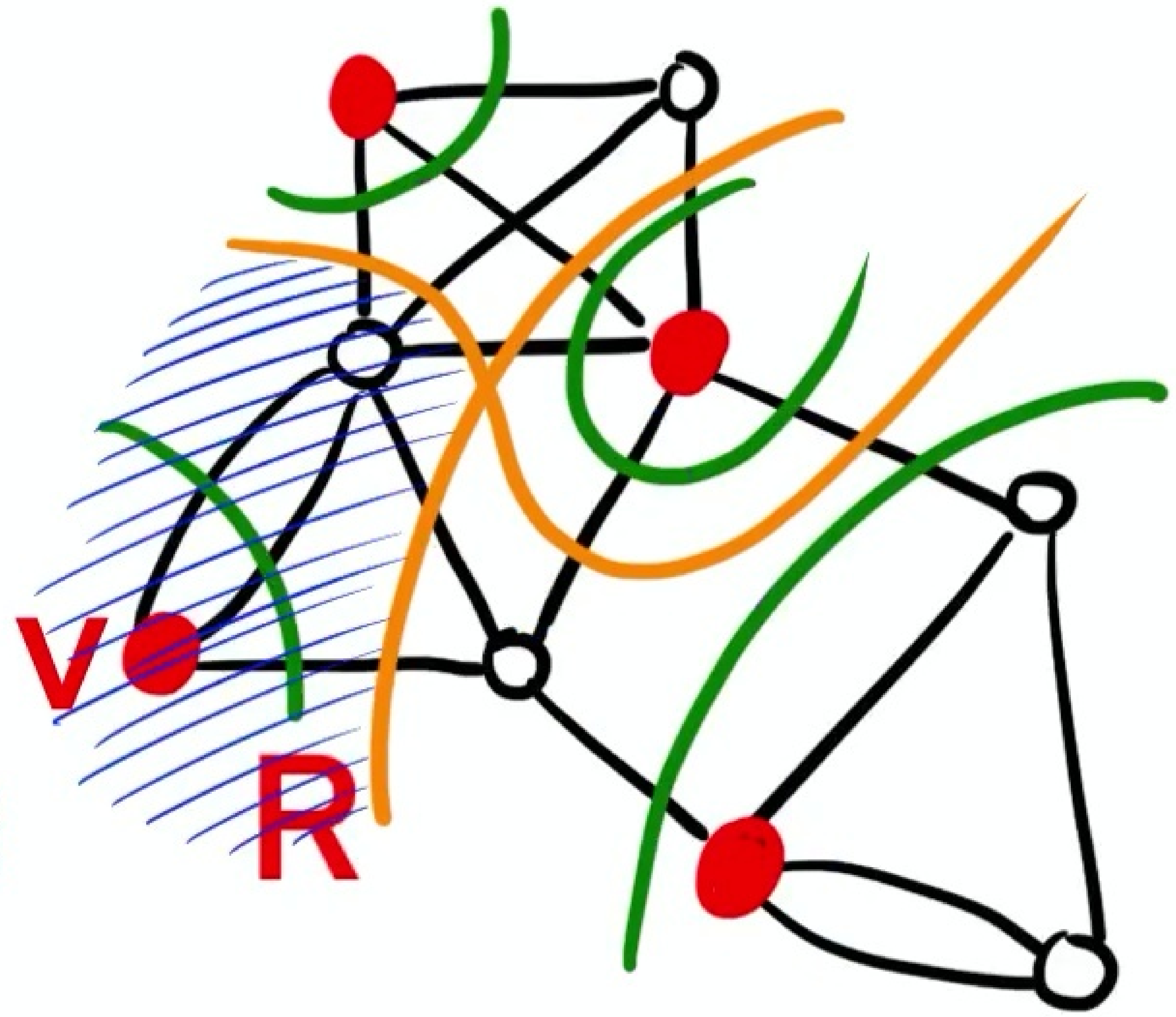
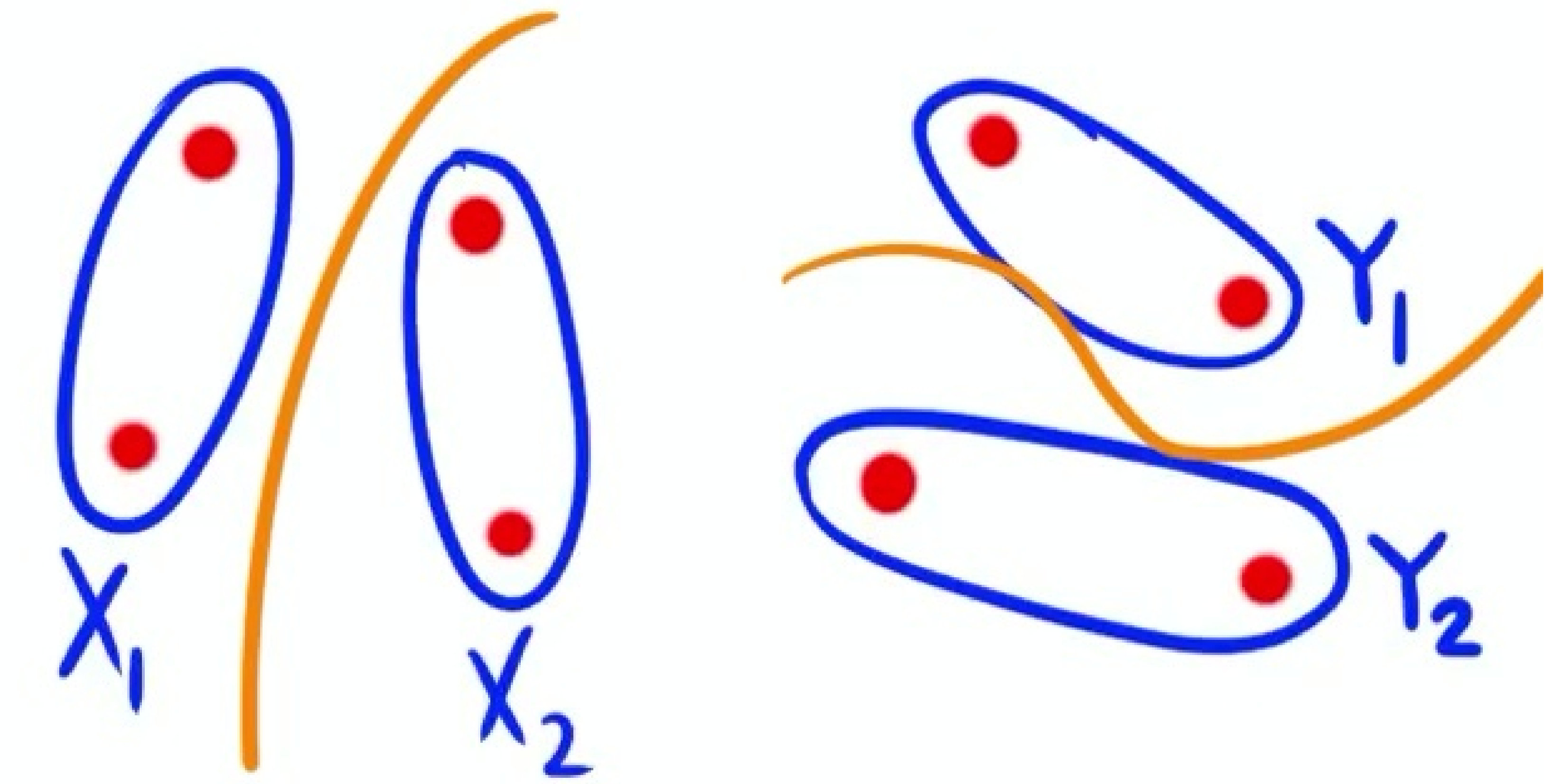
In G\(union of mincuts), v's connected component contains (v, R\v)-mincut





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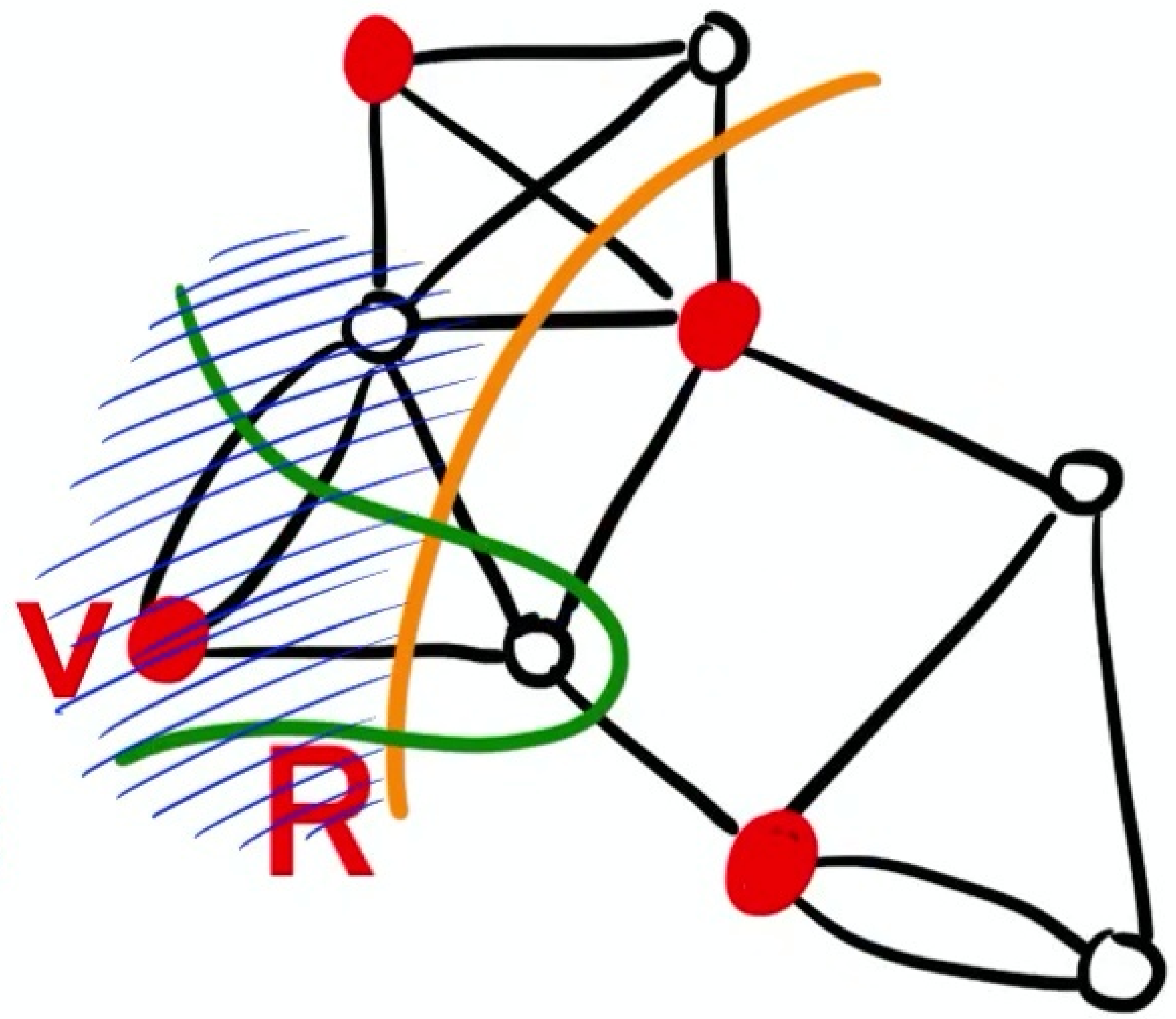

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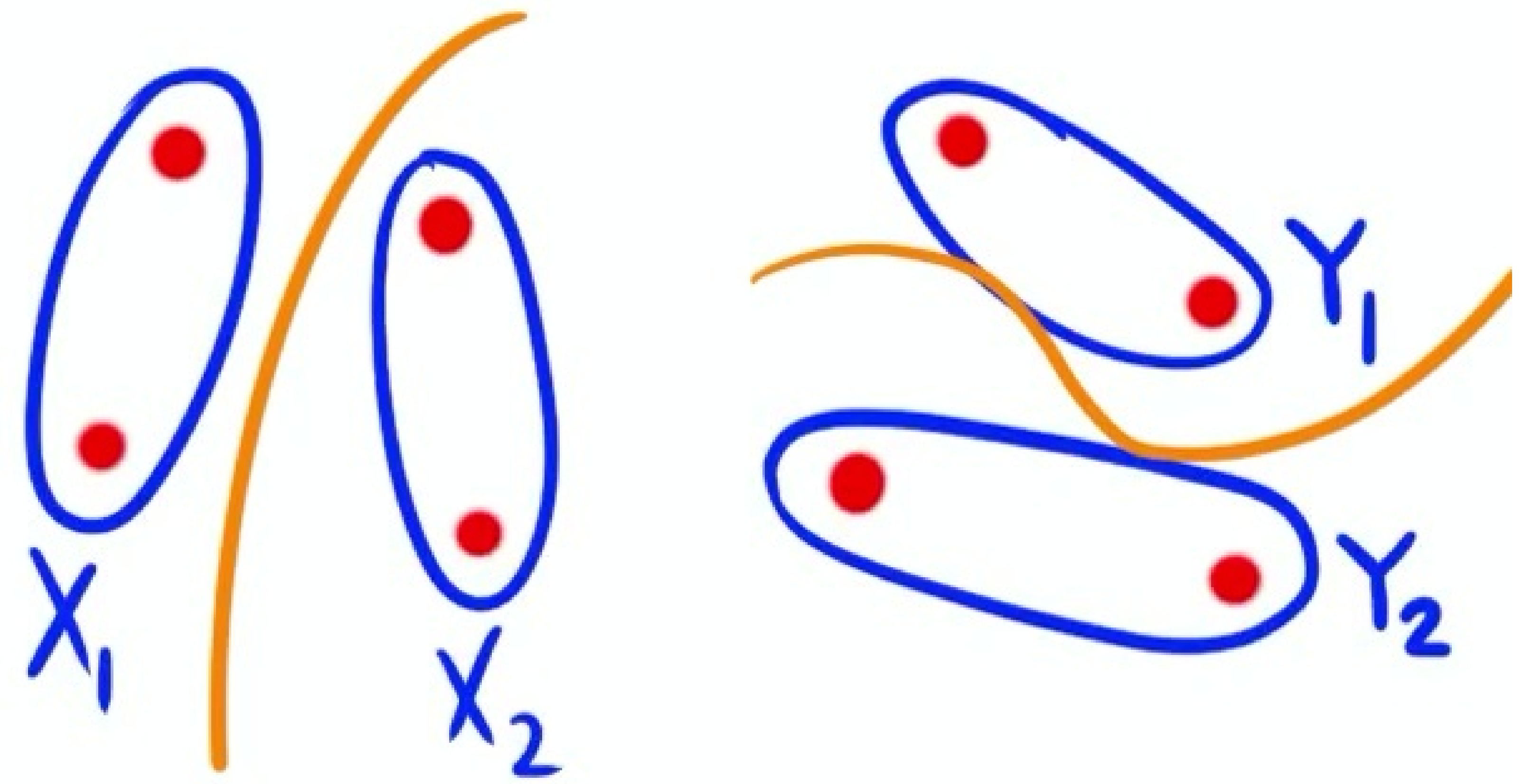

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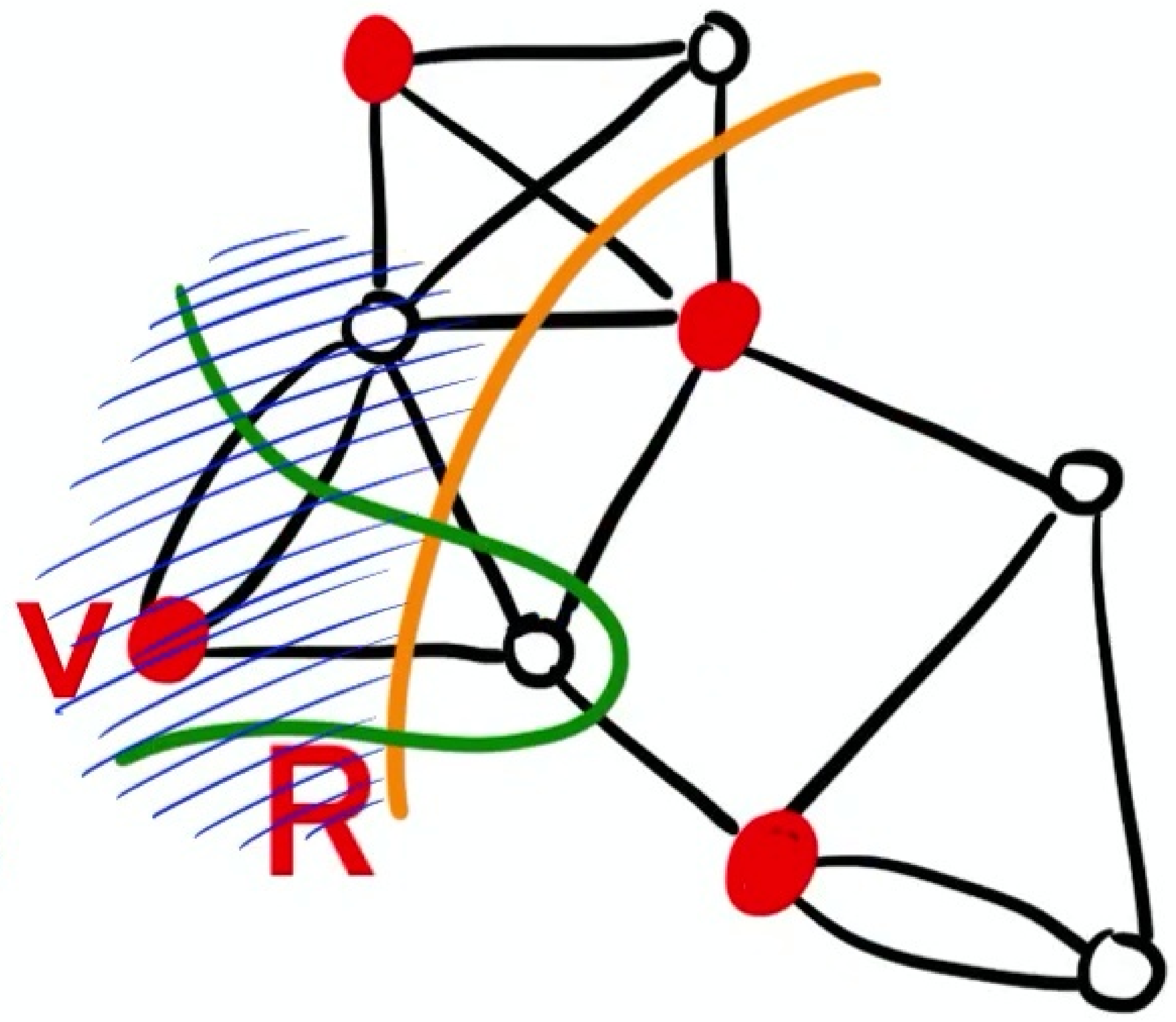

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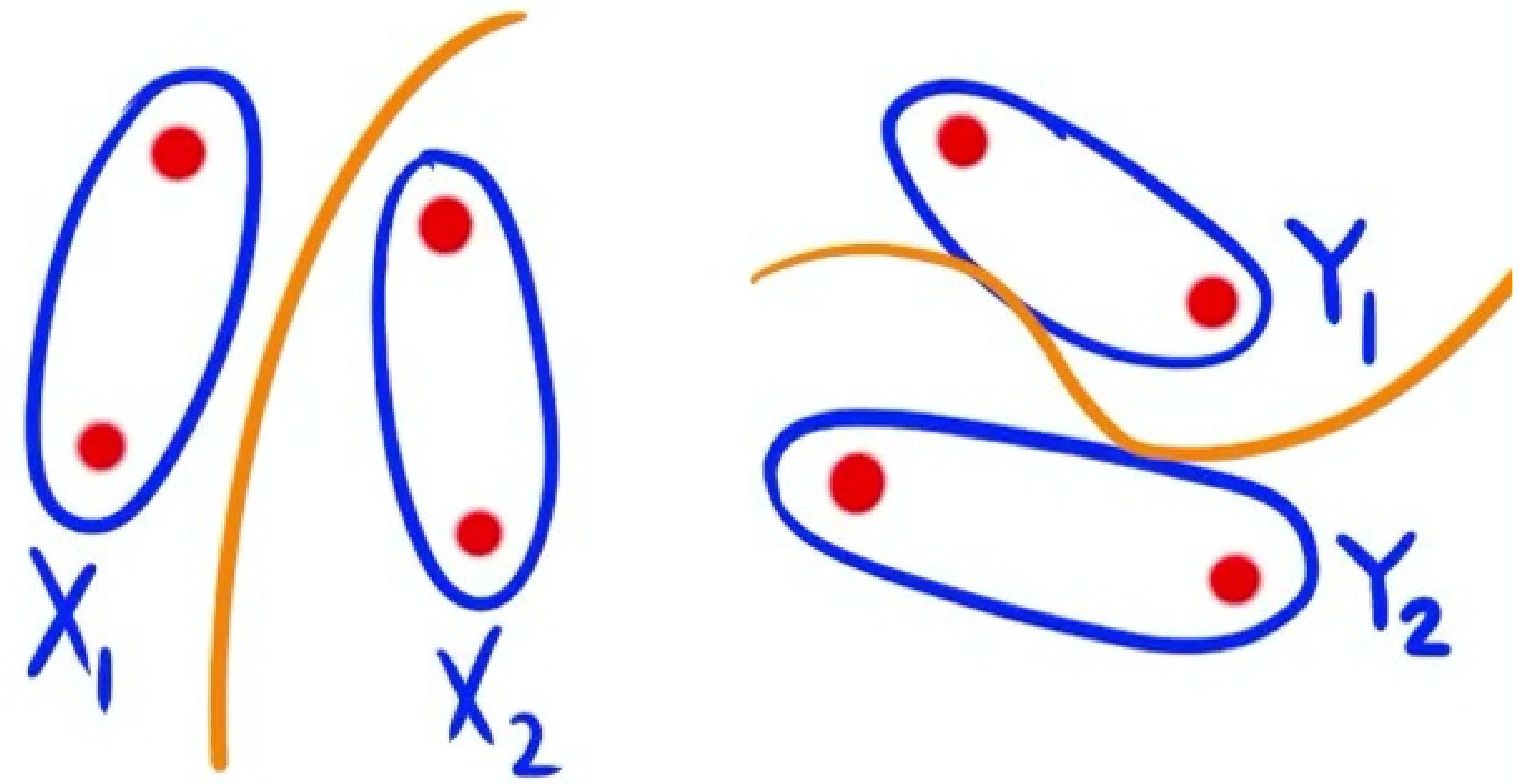

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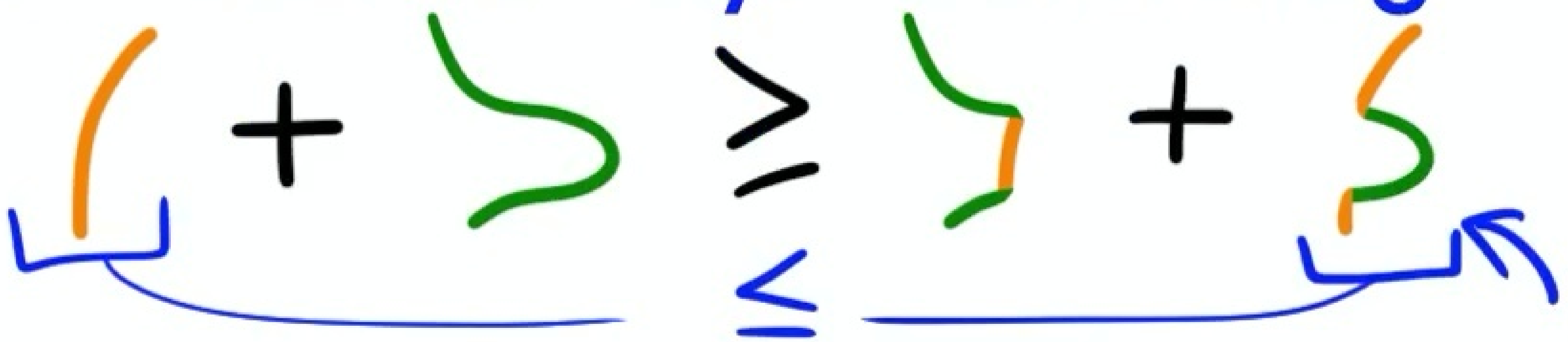

also (X_1,Y_1)-mincut

Minimum Isolating Cuts

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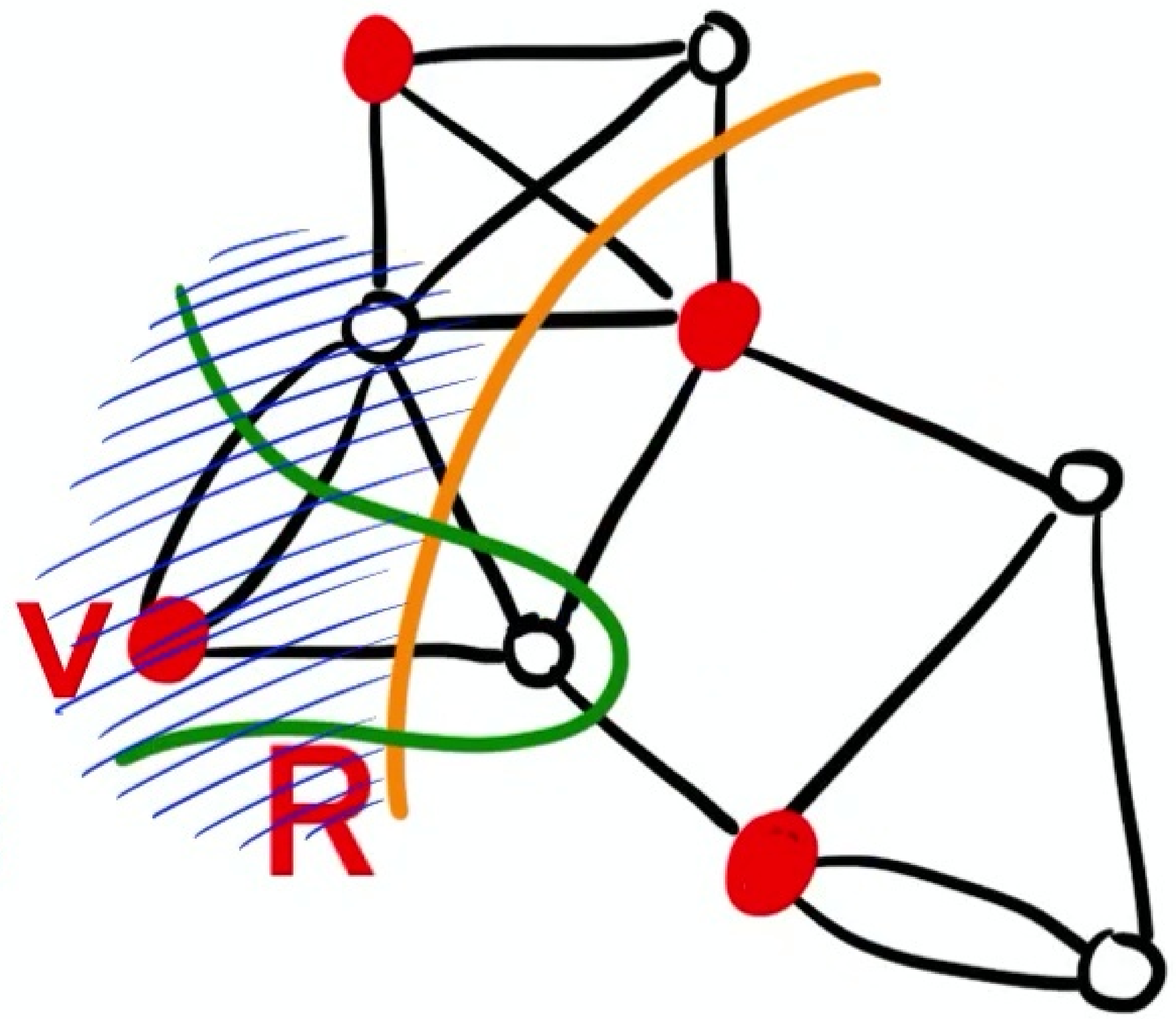

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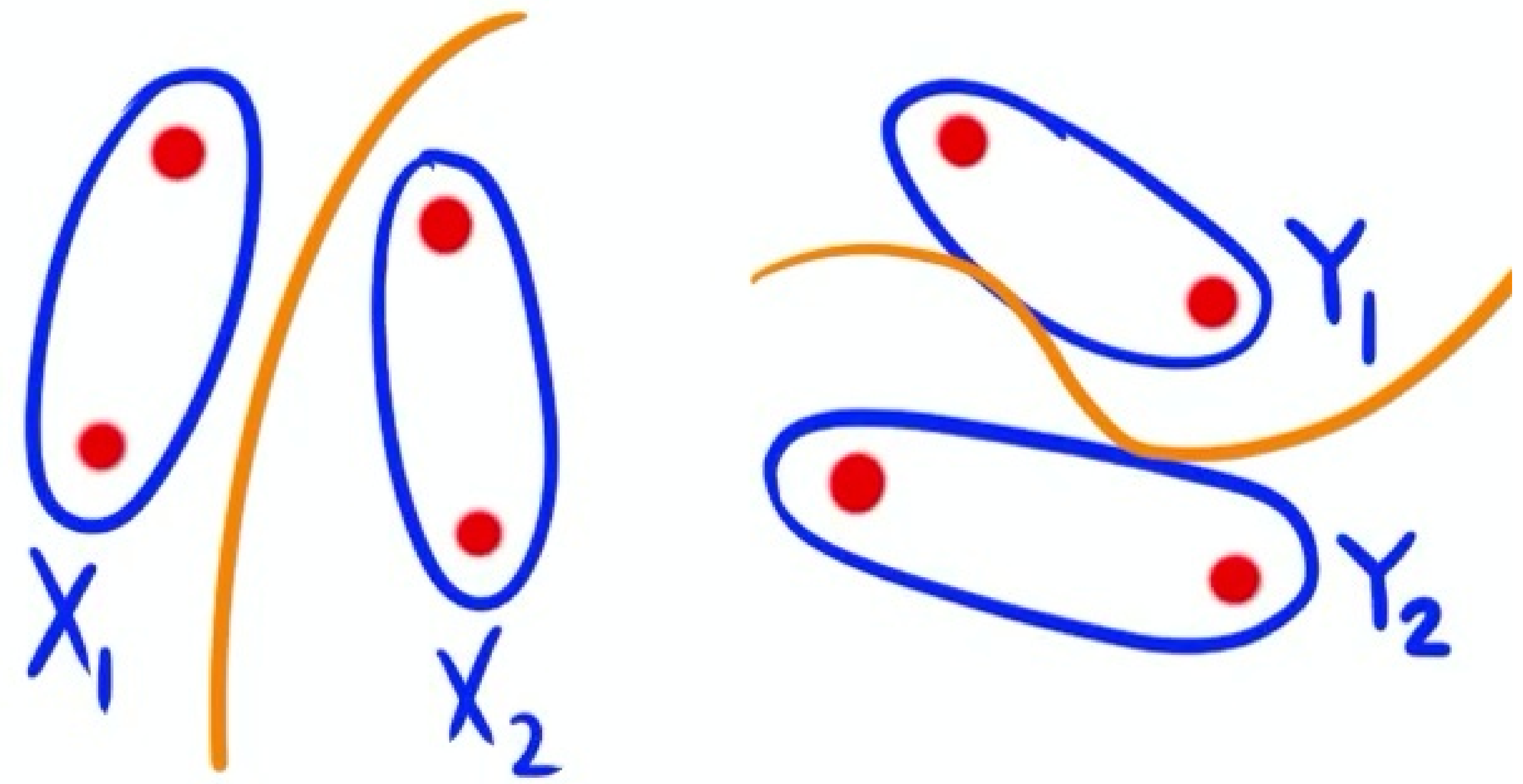

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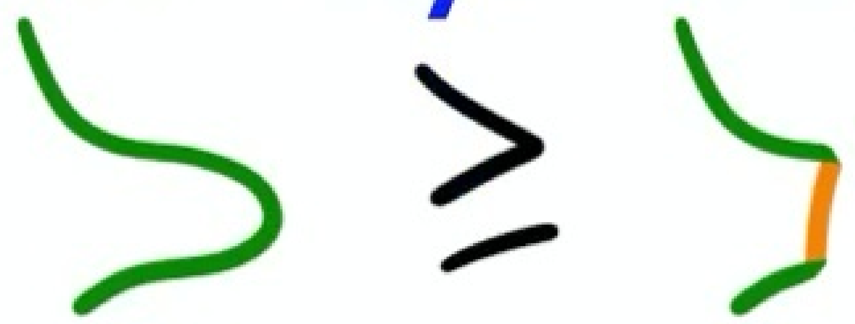

Recap: Steiner mincut

Thm: Steiner mincut in polylog(n) max-flows

Assumption inspired by locality: Steiner mincut is unbalanced (1 terminal on one side)

- Reduces to Minimum Isolating Cuts

Simple algorithm for Min. Iso. Cuts

Simple reduction from general Steiner mincut to unbalanced: random sampling

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Minimum Isolating Cuts: Applications

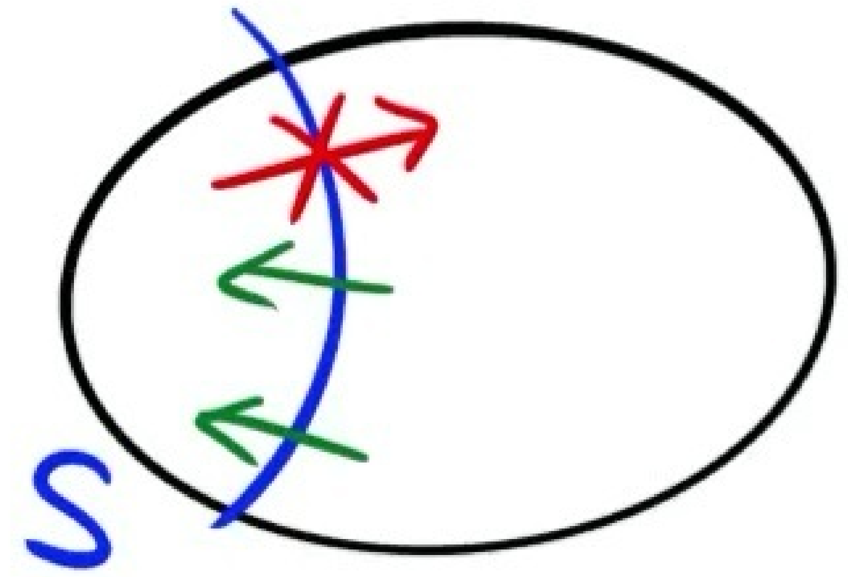
[L.-Panigrahi '21]: All-pairs mincut and Gomory-Hu tree:
(1+ε)-approximation in polylog(n) exact max-flows

[L.-Nanongkai-Panigrahi-Saranurak-Yingchareonthawornchai '21]
vertex connectivity in polylog(n) max-flows

[Chekuri-Quanrud, Mukhopadhyay-Nanongkai '21]
Symmetric bisubmodular function minimization,
hypergraph connectivity, element connectivity

Directed Mincut

Directed mincut: partition (S, V\S) s.t. no directed edge from S to V\S



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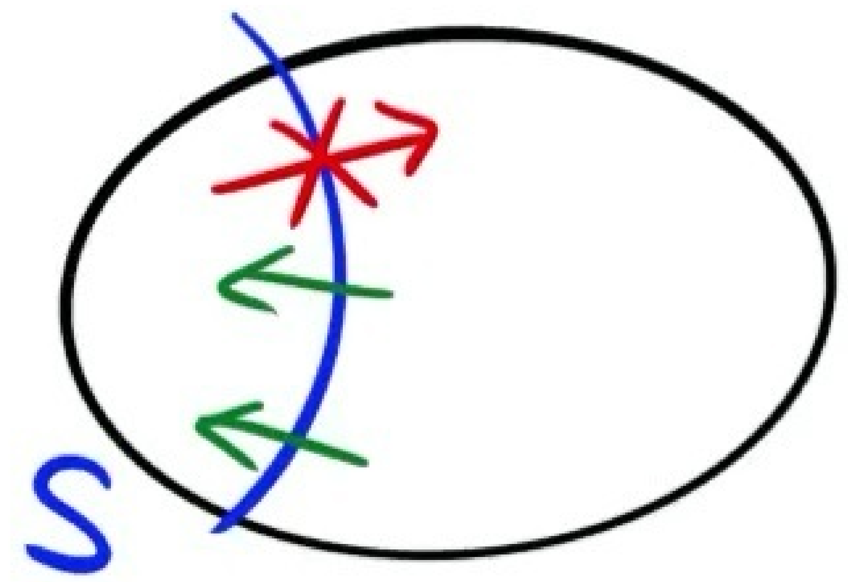

Much harder than undirected:

- Karger's randomized contraction doesn't work
- Sparsifiers for directed graphs are hard/impossible

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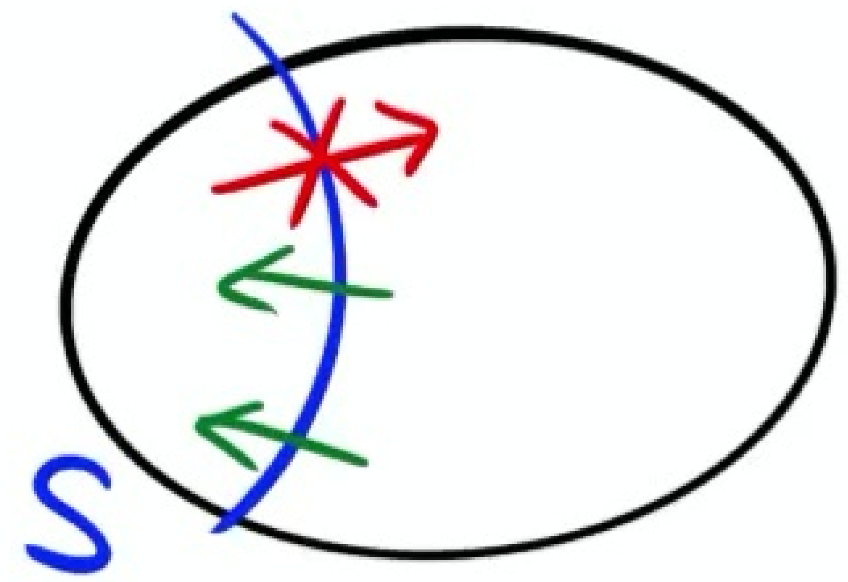

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Previous best: $\tilde{O}(mn)$ [Hao-Orlin'94]

[Cen-L.-Nanongkai-Panigrahi-Saranurak] $\sqrt{n}$ max-flows $\Rightarrow O(m\sqrt{n}+n^2)$

This talk: $(1+\varepsilon)$-approximation

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Why are directed sparsifiers hard?

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Thm [Karger]: For undirected graphs, if sample each edge w.p. $p \sim \frac{\log n}{\varepsilon^2 \lambda}$,

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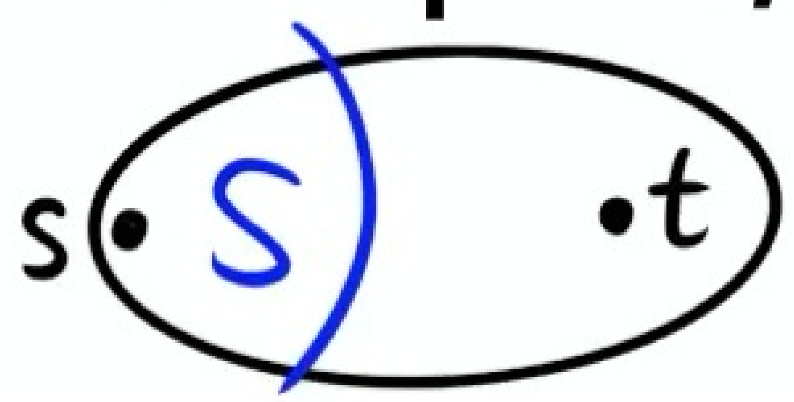
Balanced case: sample s,t at random and compute s-t mincut



occurs w.p. $\gtrsim k/n \implies$ repeat $\sim n/k$ times

Directed Mincut

Algorithm: compute partial sparsifier H, then find directed mincut $\partial_H S$ in sparsifier. Output $\partial_G S$

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Thm: suppose sampled graph H satisfies (for some p)

- all k-unbalanced cuts have $(1\pm\varepsilon)p$ fraction edges sampled
- all k-balanced cuts have size $\gg p\lambda$ (λ = mincut)

then mincut in H is $(1+\varepsilon)$-mincut in G

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by overlaying an expander: $|\partial S| \approx \frac{2\varepsilon\lambda}{k}|S|$ for $|S| \leq n/2$

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- $\frac{k}{\varepsilon}$-balanced cut increases by $\geq 2\lambda$
- k-unbalanced cut increases by $\leq 2\varepsilon\lambda$ (including mincut)

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Arborescence packing + minimum 1-respecting cut:

$\sim k$ max flows unbalanced, exact

$k = \sqrt{n}$: $\sim \sqrt{n}$ max flows

Recap: directed mincut

Thm: directed mincut in $\sqrt{n}$ max-flows

Directed sparsification is hard

Locality: partial sparsification of only unbalanced cuts

Balanced case: different strategy this time

$\Rightarrow$ simple $(1+\varepsilon)$-approximate directed mincut

few extra steps for exact

Part II: Preconditioning

1. Deterministic mincut

Deterministic Mincut

Global mincut: given a graph, find minimum # edges whose removal disconnects the graph

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- Karger '93, '96: $\tilde{O}(n^2)$ time, $\tilde{O}(m)$ time randomized
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- L.-Panigrahi '20: deterministic Steiner mincut in ~max-flow time
- L.: deterministic mincut in $m^{1+o(1)}$ time

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Preconditioning assumption: assume input is an expander

- Expander case: simple algorithm following [Karger '96]
- General case: expander decomposition (technical)

Mincut by Sparsification + Tree Packing

Thm [Karger '96]: Suppose given a skeleton graph H s.t.

- H has O(m) edges
- The mincut of H is $\lambda_H \geq \rho\lambda$
- For the mincut $\partial_G S^*$ in G, the $|\partial_H S^*| \leq (1+\varepsilon)\rho\lambda$

Then, can compute exact mincut in G in $m\lambda_H$ additional deterministic time

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This talk: deterministic skeleton for expander

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Sample each edge in G with prob $p := \frac{100 \log n}{\varepsilon^2 \lambda}$. Let H = sampled edges

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Locality assumption: $(1+\varepsilon)$-preserve only unbalanced cuts

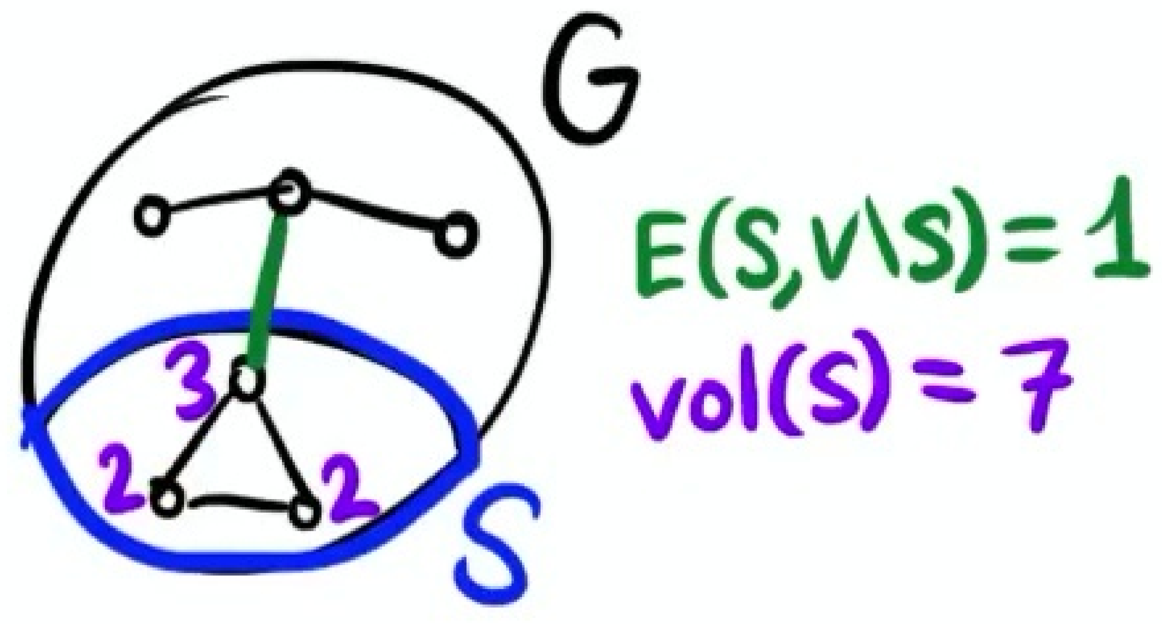
mincut is unbalanced for expander

Balanced cuts: overlay expander (same as before)

Expanders

Conductance of a graph: $\Phi(G) = \min_{\substack{S \subseteq V \\ vol(S) \leq vol(V \setminus S)}} \frac{|E(S, V \setminus S)|}{vol(S)}$

↑ "volume" of S: sum of degrees in S



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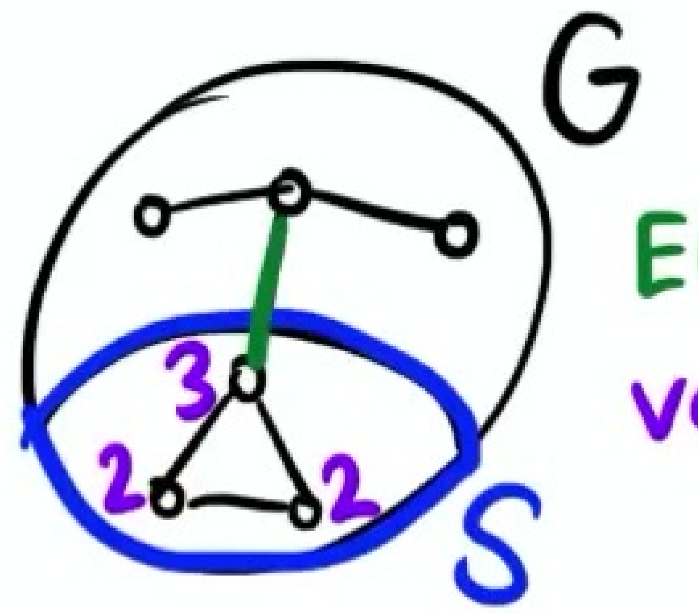

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Claim: in a ϕ-expander, any α-approx mincut ∂S $(|\partial S| \le \alpha\lambda)$

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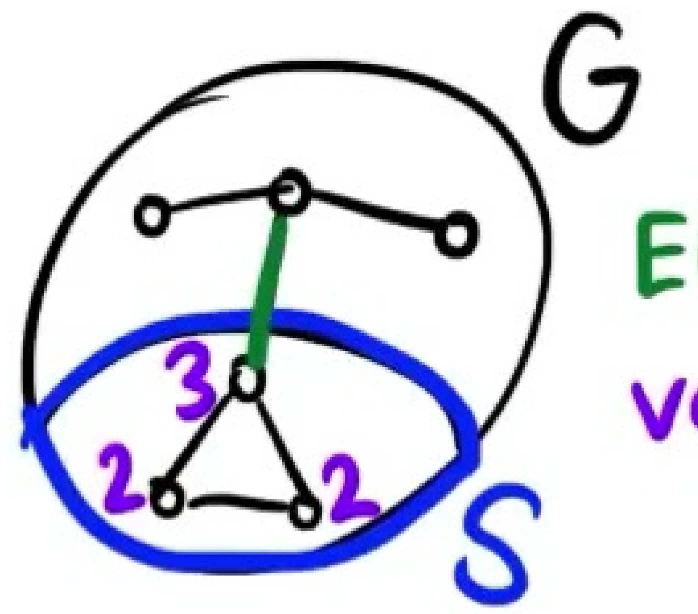

$E(S, V \setminus S) = 1$

$vol(S) = 7$

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must have $|S| \leq \alpha/\phi$

Proof: All degrees $\geq \lambda$ [λ = mincut]

so $vol(S) \geq \lambda |S|$

ϕ-expander: $|\partial S| \geq \phi \, vol(S) \geq \phi \lambda |S|$

Derandomization: Unbalanced Cuts

First goal: ensure that sample $(1\pm\varepsilon)p$ for all unbal. cuts $\partial S : |S| \le \frac{\alpha}{\phi}$

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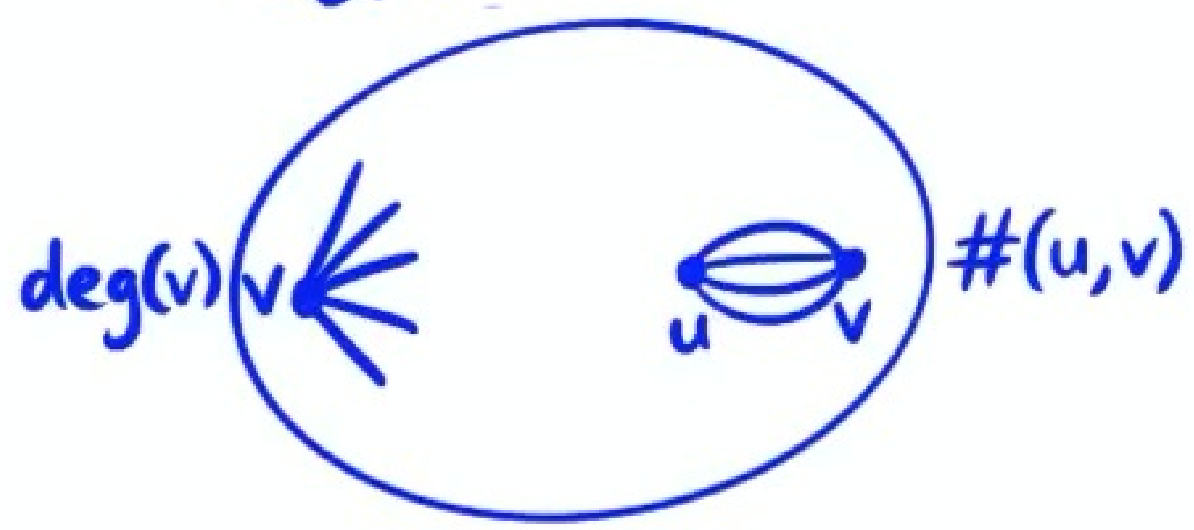

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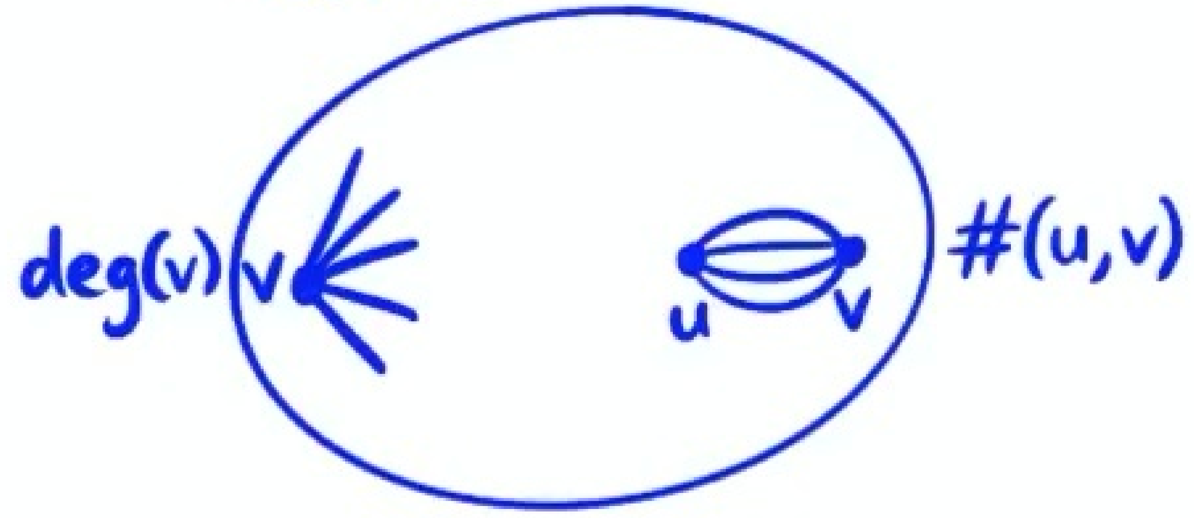

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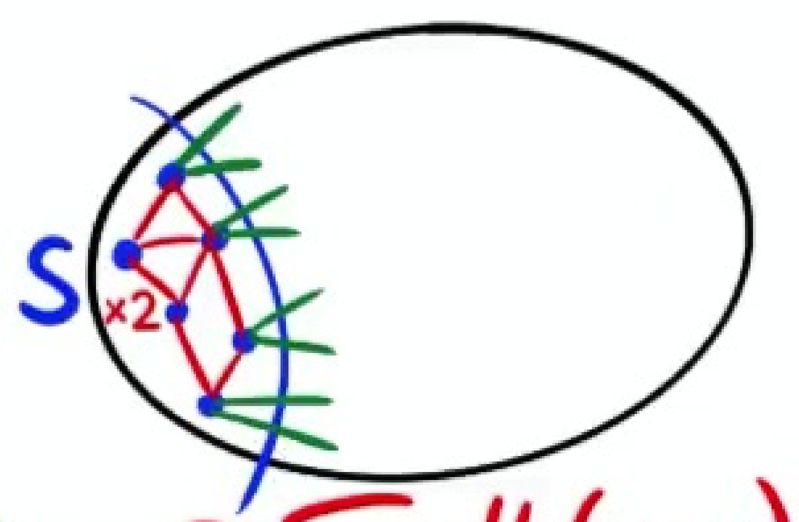
Proof: Let $|S| \le \frac{\alpha}{\phi}$



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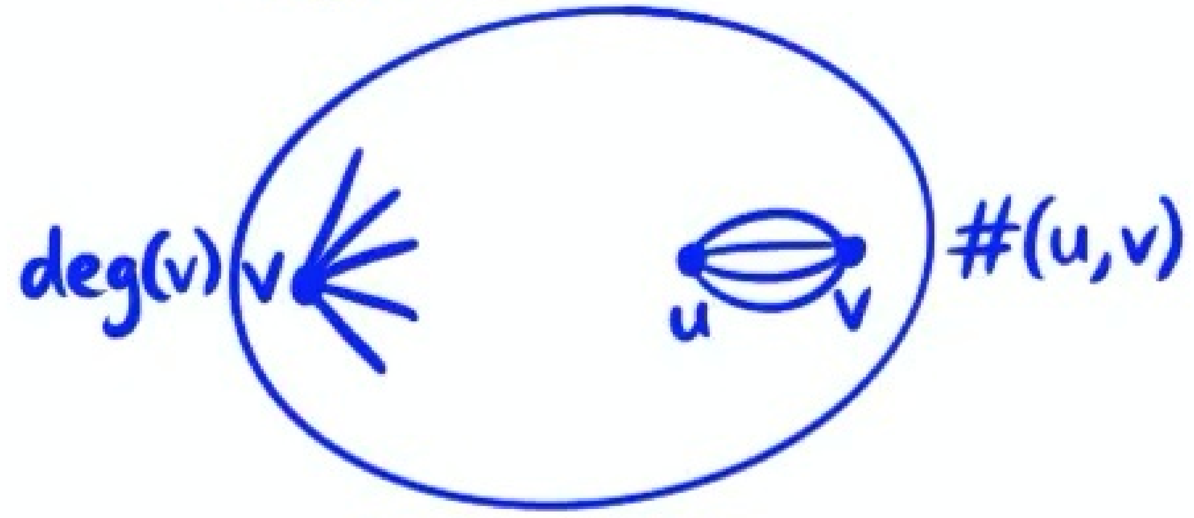

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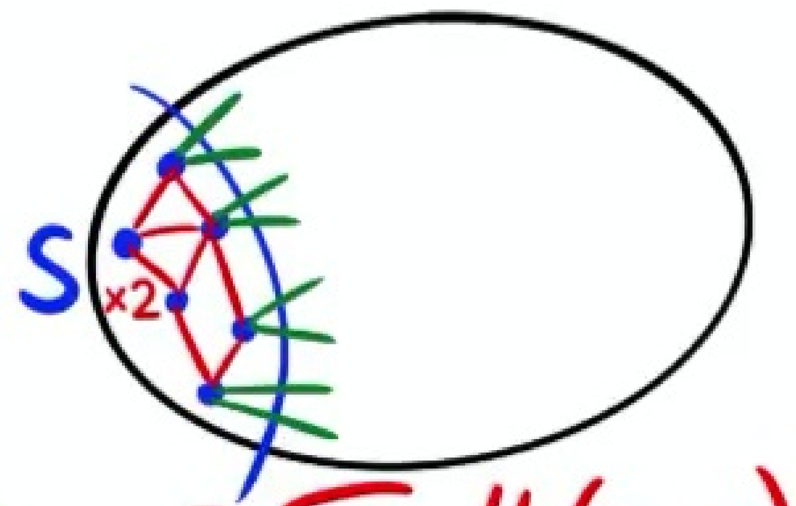

$$|\partial S| = \sum_{v\in S} \deg(v) - 2\sum_{u,v\in S} \#(u,v)$$

$|S|^2 \le (\alpha/\phi)^2$ terms, each with

$\le \varepsilon(\phi/\alpha)^2\lambda$ additive error

$\Rightarrow \varepsilon\lambda$ total additive error

$(1+\varepsilon)$ multiplicative error

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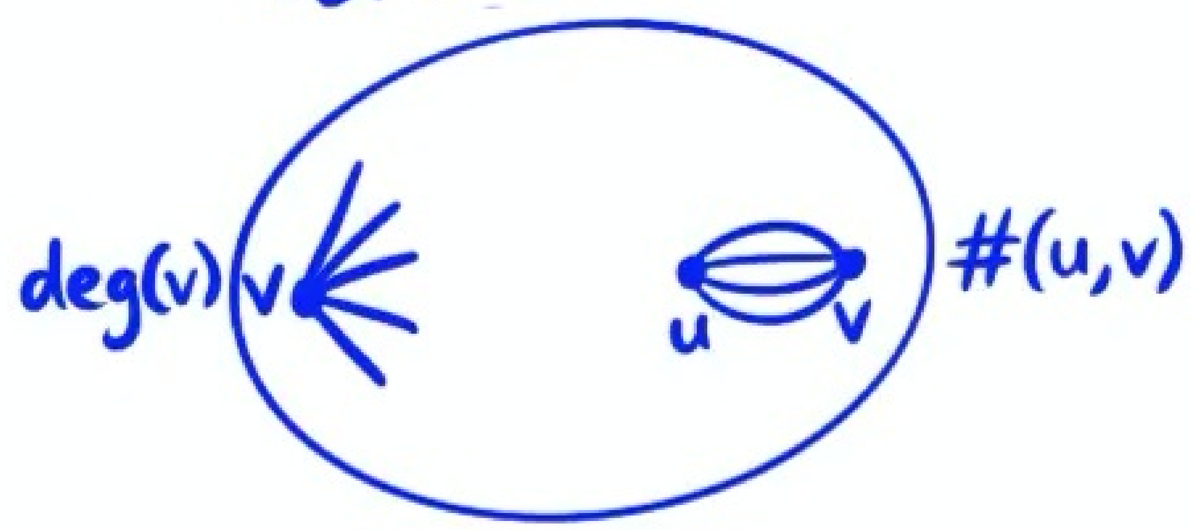
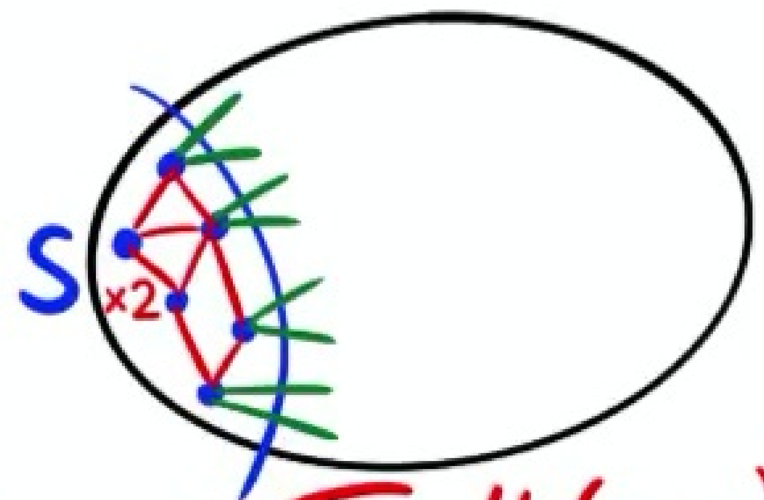


only n+m constraints!

Proof: Let $|S| \le \frac{\alpha}{\phi}$



$$|\partial S| = \sum_{v\in S} \deg(v) - 2\sum_{u,v\in S} \#(u,v)$$

$|S|^2 \le (\alpha/\phi)^2$ terms, each with

$\le \varepsilon(\phi/\alpha)^2\lambda$ additive error

$\Rightarrow \varepsilon\lambda$ total additive error

$(1+\varepsilon)$ multiplicative error

Pessimistic estimators method: $\widetilde{O}(m)$ time

Recap: Deterministic Mincut

Thm: deterministic mincut in $m^{1+o(1)}$ time

Karger: reduces to computing mincut sparsifier

Deterministic sparsifier is hard: 2^n many cuts to preserve

Preconditioning assumption: input is expander

Locality assumption: mincut is unbalanced

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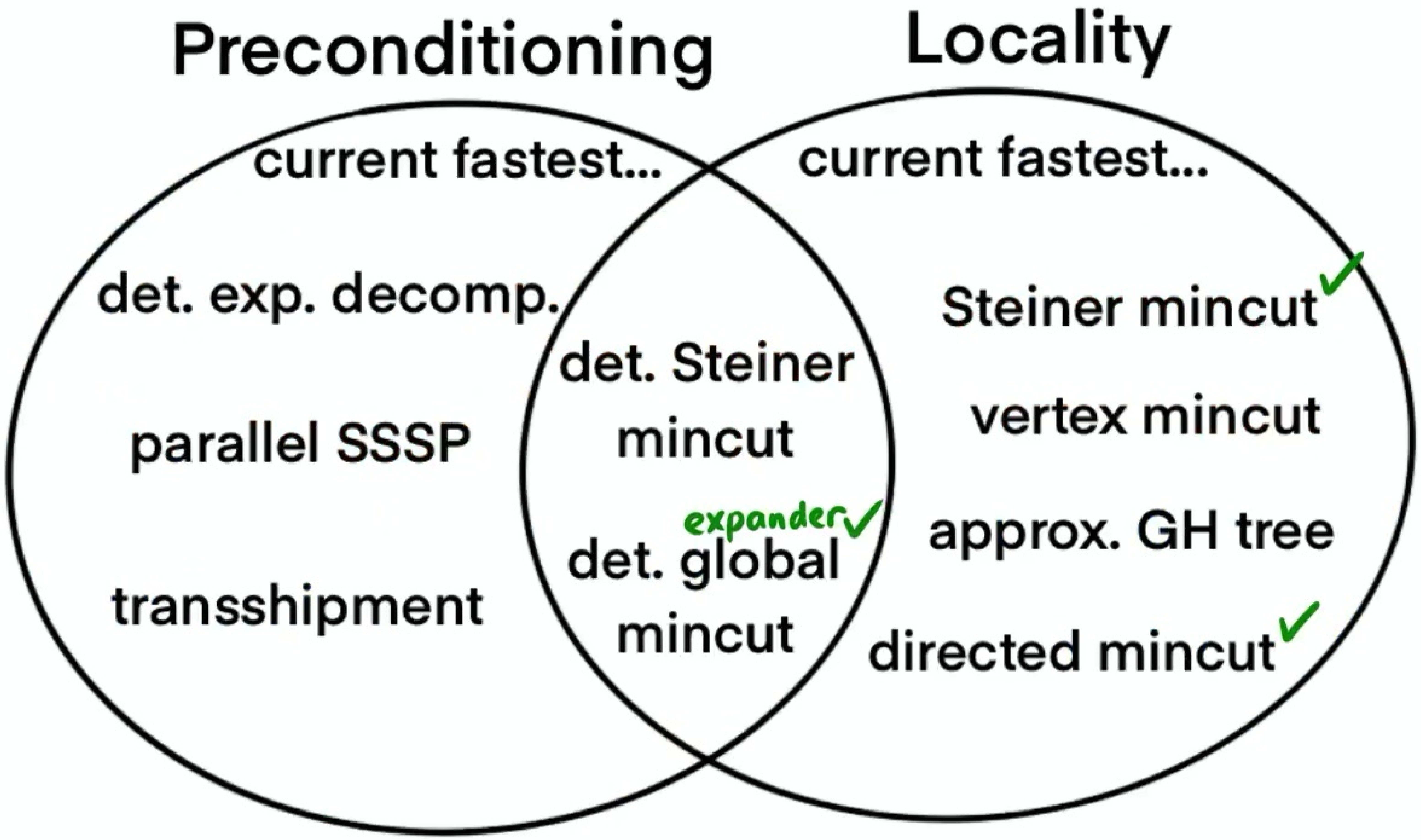
- Unbalanced cuts: only need to preserve deg(v) and #(u,v)
- Balanced cuts: overlay expander

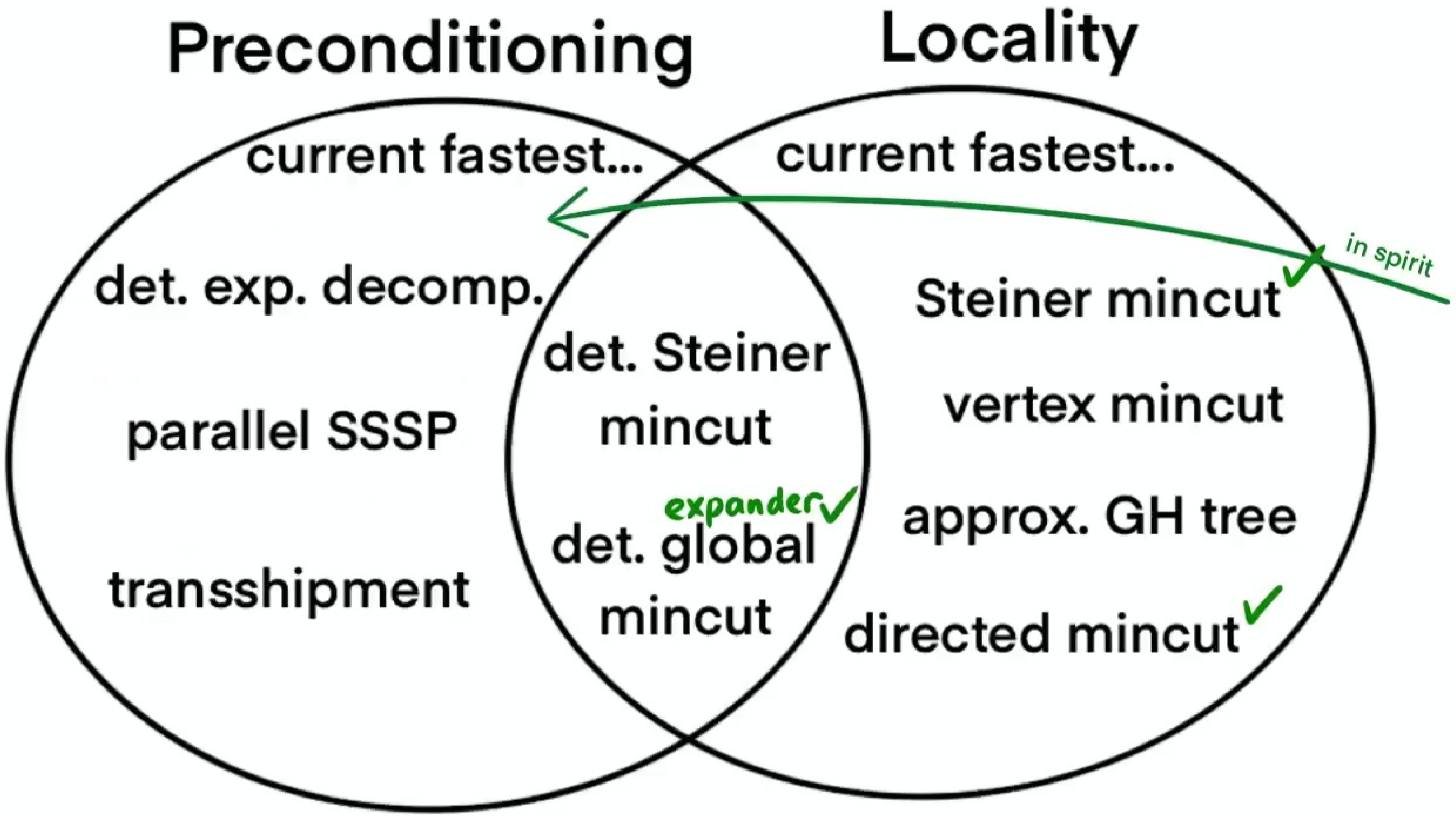
$\Rightarrow$ simple mincut sparsifier for expander

General graphs: expander decomposition

Summary


Preconditioning
Locality
current fastest...
current fastest...
det. exp. decomp.
parallel SSSP
transshipment
det. Steiner
mincut
expander✓
det. global
mincut
Steiner mincut✓
vertex mincut
approx. GH tree
directed mincut✓

Summary



Misc.

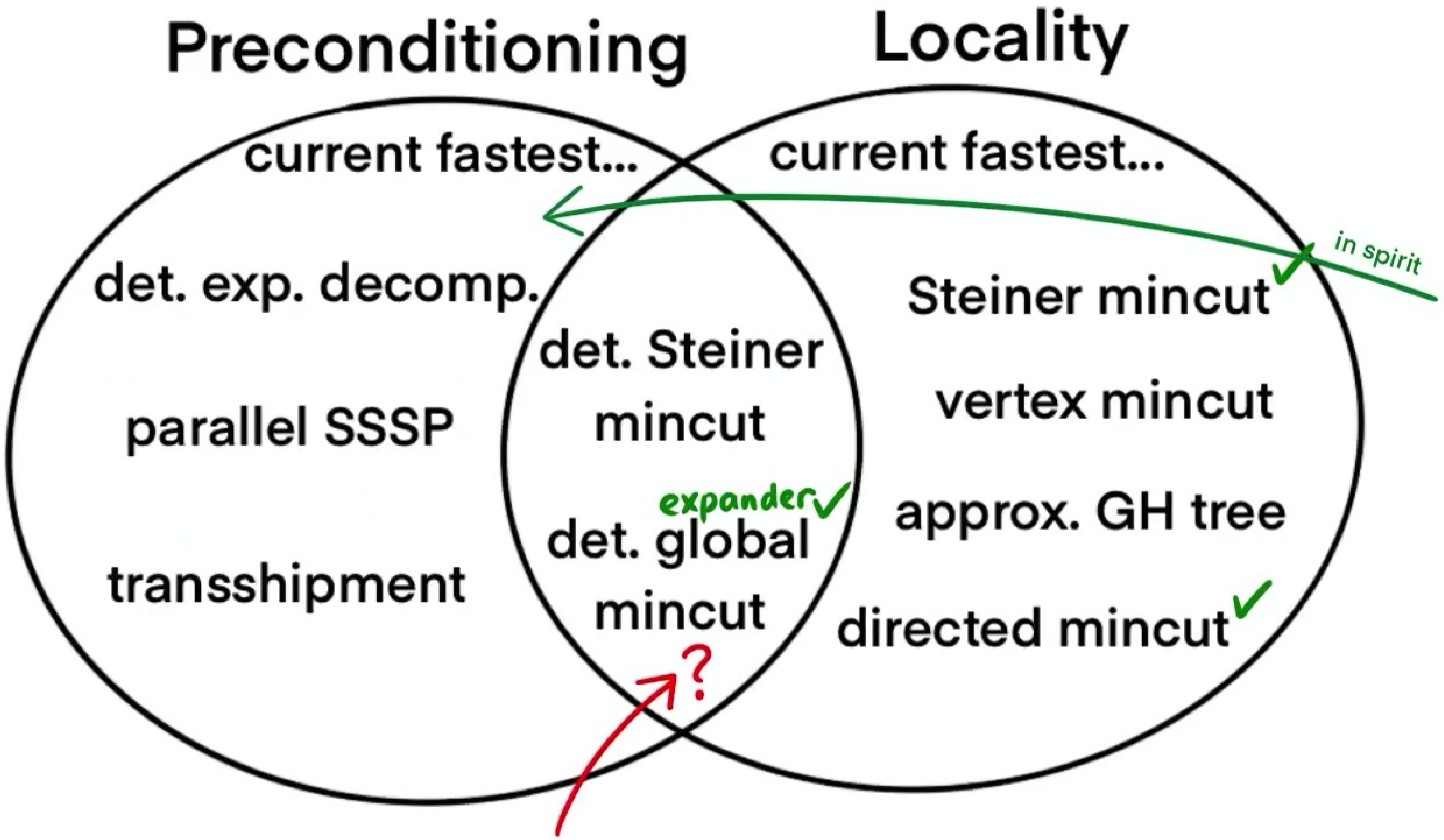
current fastest...

minimum k-cut

constant-approx. planar sparsest cut

Feedback Vertex Set (FPT)

Summary



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Feedback Vertex Set (FPT)

Future work: Gomory-Hu tree in polylog(n) max-flows?

Know: GH tree for expanders in polylog(n) max-flows (Min. Iso. Cuts)

Don’t know general case ⇒ expander case reduction!