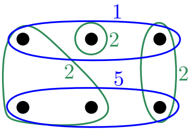
A Local Search-Based Approach for Set Covering

Jason Li (Simons Institute)
Joint with Anupam Gupta (CMU), Euiwoong Lee (UMich)

SODA 2023
January 23, 2023

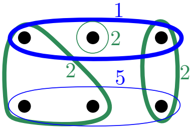
Set Cover

- Set Cover: given a family of sets $S \subseteq U$ with weights $w(S)$, find a minimum weight collection of sets that cover U



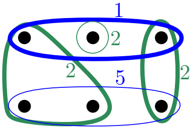
Set Cover

- Set Cover: given a family of sets $S \subseteq U$ with weights $w(S)$, find a minimum weight collection of sets that cover U



Set Cover

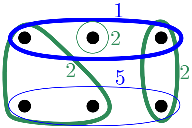
- Set Cover: given a family of sets $S \subseteq U$ with weights $w(S)$, find a minimum weight collection of sets that cover U



- Parameter k: all sets S have size $\leq k$ (here, $k = 3$)

Set Cover

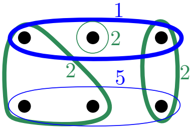
- Set Cover: given a family of sets $S \subseteq U$ with weights $w(S)$, find a minimum weight collection of sets that cover U



- Parameter k: all sets S have size $\leq k$ (here, $k = 3$)
- Folklore greedy algorithm: H_k-approximation where $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k} \approx \ln k + O(1)$

Set Cover

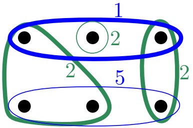
- Set Cover: given a family of sets $S \subseteq U$ with weights $w(S)$, find a minimum weight collection of sets that cover U



- Parameter k: all sets S have size $\leq k$ (here, $k = 3$)
- Folklore greedy algorithm: H_k-approximation where $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k} \approx \ln k + O(1)$
- Hard to approximate within $\ln k - O(\ln \ln k)$ [Feige'98,Trevisan'01]

Set Cover

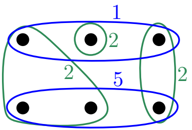
- Set Cover: given a family of sets $S \subseteq U$ with weights $w(S)$, find a minimum weight collection of sets that cover U



- Parameter k: all sets S have size $\leq k$ (here, $k = 3$)
- Folklore greedy algorithm: H_k-approximation where $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k} \approx \ln k + O(1)$
- Hard to approximate within $\ln k - O(\ln \ln k)$ [Feige'98,Trevisan'01]
- This talk: local search algorithm for set cover

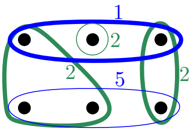
Exact Set Cover

- Exact Set Cover: given a family of sets $S \subseteq U$ with weights $w(S)$, find a minimum weight collection of sets that partition U

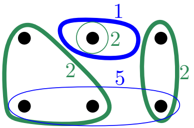


Exact Set Cover

- Exact Set Cover: given a family of sets $S \subseteq U$ with weights $w(S)$, find a minimum weight collection of sets that partition U



- Transformation: for each original set S, add all subsets of S, each of weight $w(S)$



Local Search

- Local search for set cover:

Local Search

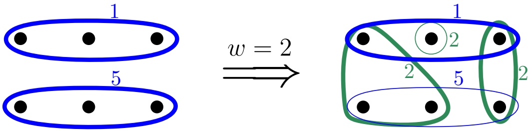
- Local search for set cover:
    - Maintain a feasible solution at all times

Local Search

- Local search for set cover (of width w):
  - Maintain a feasible solution at all times
  - Try all ways to add $\leq w$ disjoint sets $S_1, \ldots, S_w$ and prune existing sets to ensure exact cover.

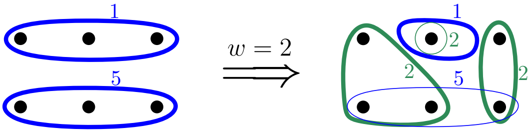
Local Search

- Local search for set cover (of width w):
  - Maintain a feasible solution at all times
  - Try all ways to add $\leq w$ disjoint sets $S_1, \ldots, S_w$ and prune existing sets to ensure exact cover.



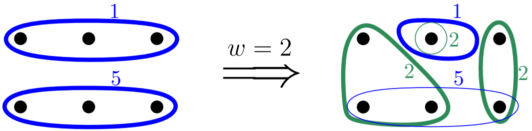
Local Search

- Local search for set cover (of width w):
  - Maintain a feasible solution at all times
  - Try all ways to add $\leq w$ disjoint sets $S_1, \ldots, S_w$ and prune existing sets to ensure exact cover.



Local Search

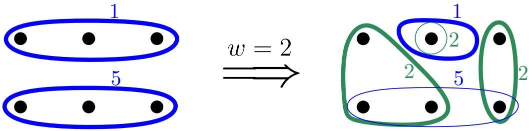
- Local search for set cover (of width w):
  - Maintain a feasible solution at all times
  - Try all ways to add $\leq w$ disjoint sets $S_1, \ldots, S_w$ and prune existing sets to ensure exact cover.



If feasible solution of lower cost, move to new solution.

Local Search

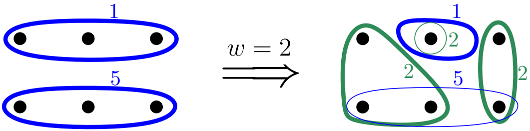
- Local search for set cover (of width w):
  - Maintain a feasible solution at all times
  - Try all ways to add $\leq w$ disjoint sets $S_1, \ldots, S_w$ and prune existing sets to ensure exact cover.



  If feasible solution of lower cost, move to new solution.
  - Repeat until no further improvement is possible.

Local Search

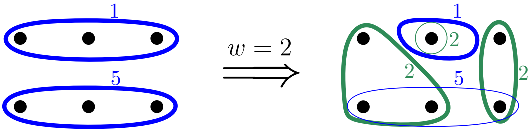
- Local search for set cover (of width w):
  - Maintain a feasible solution at all times
  - Try all ways to add $\leq w$ disjoint sets $S_1, \ldots, S_w$ and prune existing sets to ensure exact cover.



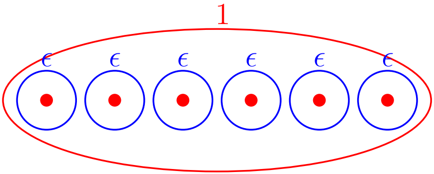
    If feasible solution of lower cost, move to new solution.
  - Repeat until no further improvement is possible.
- Does not achieve any constant approximation!

Local Search

- Local search for set cover (of width w):
  - Maintain a feasible solution at all times
  - Try all ways to add $\leq w$ disjoint sets $S_1, \ldots, S_w$ and prune existing sets to ensure exact cover.



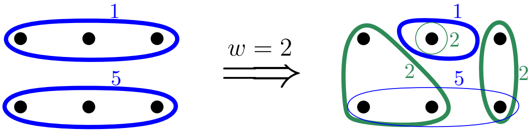
    If feasible solution of lower cost, move to new solution.
  - Repeat until no further improvement is possible.
- Does not achieve any constant approximation!



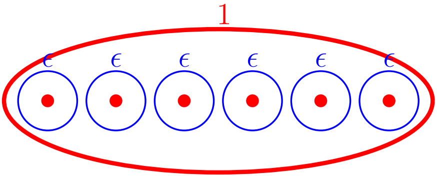
$\epsilon \ll 1/n, \; w \ll n$

Local Search

- Local search for set cover (of width w):
  - Maintain a feasible solution at all times
  - Try all ways to add $\leq w$ disjoint sets $S_1, \ldots, S_w$ and prune existing sets to ensure exact cover.



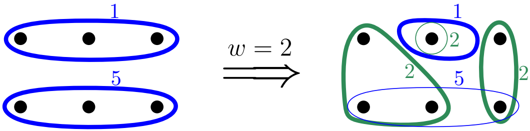
If feasible solution of lower cost, move to new solution.
  - Repeat until no further improvement is possible.
- Does not achieve any constant approximation!



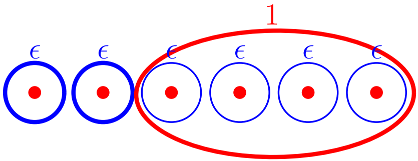
$\epsilon \ll 1/n, \; w \ll n$

Local Search

- Local search for set cover (of width w):
  - Maintain a feasible solution at all times
  - Try all ways to add $\leq w$ disjoint sets $S_1, \ldots, S_w$ and prune existing sets to ensure exact cover.



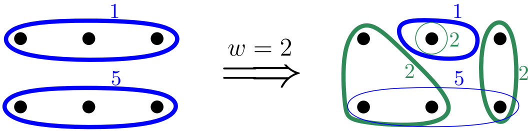
If feasible solution of lower cost, move to new solution.
  - Repeat until no further improvement is possible.
- Does not achieve any constant approximation!



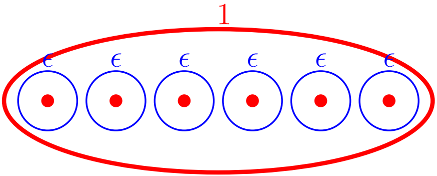
$$\epsilon \ll 1/n, \; w \ll n$$

Local Search

- Local search for set cover (of width w):
  - Maintain a feasible solution at all times
  - Try all ways to add $\leq w$ disjoint sets $S_1, \ldots, S_w$ and prune existing sets to ensure exact cover.



    If feasible solution of lower cost, move to new solution.
  - Repeat until no further improvement is possible.
- Does not achieve any constant approximation!



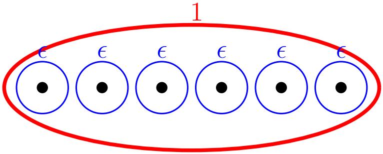
$$\epsilon \ll 1/n, \; w \ll n$$

Non-Oblivious Local Search

- What *should* local search do?

Non-Oblivious Local Search

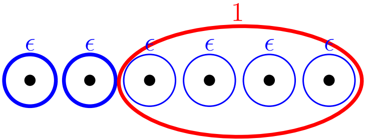
- What *should* local search do?



$\epsilon \ll 1/n$, $w \ll n$

Non-Oblivious Local Search

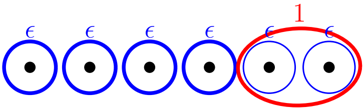
- What *should* local search do?



$\epsilon \ll 1/n,\ w \ll n$

Non-Oblivious Local Search

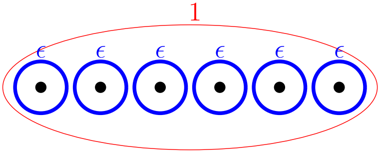
- What *should* local search do?



$$\epsilon \ll 1/n,\ w \ll n$$

Non-Oblivious Local Search

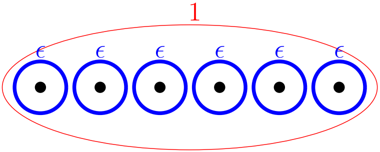
- What *should* local search do?



$\epsilon \ll 1/n$, $w \ll n$

Non-Oblivious Local Search

- What *should* local search do?

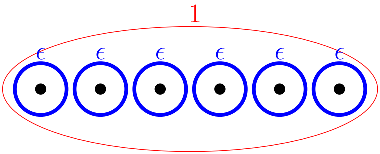


$\epsilon \ll 1/n$, $w \ll n$

- Define a custom potential function $\Phi(\mathcal{C})$ to measure progress: $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot f(|S|)$

Non-Oblivious Local Search

- What *should* local search do?

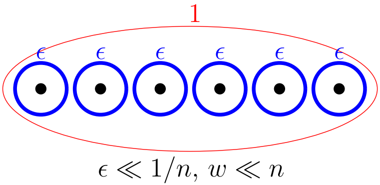


$\epsilon \ll 1/n$, $w \ll n$

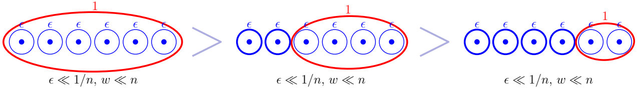
- Define a custom potential function $\Phi(\mathcal{C})$ to measure progress: $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot f(|S|)$
  - Approximates true objective: $w(\mathcal{C}) \leq \Phi(\mathcal{C}) \leq \alpha\, w(\mathcal{C})$ for approximation α, so enforce $1 \leq f(s) \leq \alpha$

Non-Oblivious Local Search

- What *should* local search do?

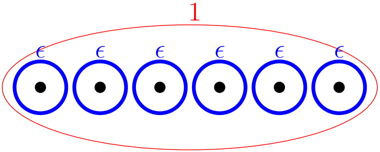


- Define a custom potential function $\Phi(\mathcal{C})$ to measure progress: $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot f(|S|)$
  - Approximates true objective: $w(\mathcal{C}) \leq \Phi(\mathcal{C}) \leq \alpha\, w(\mathcal{C})$ for approximation α, so enforce $1 \leq f(s) \leq \alpha$



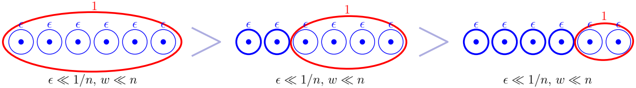
Non-Oblivious Local Search

- What *should* local search do?



$\epsilon \ll 1/n,\ w \ll n$

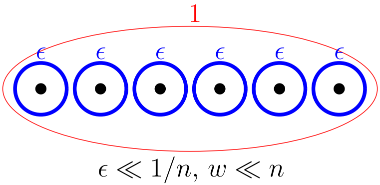
- Define a custom potential function $\Phi(\mathcal{C})$ to measure progress: $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot f(|S|)$
  - Approximates true objective: $w(\mathcal{C}) \leq \Phi(\mathcal{C}) \leq \alpha\, w(\mathcal{C})$ for approximation α, so enforce $1 \leq f(s) \leq \alpha$



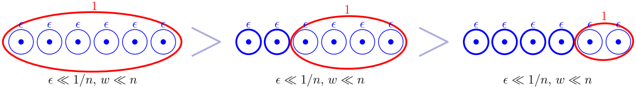
  - $f(s)$ should be increasing on s

Non-Oblivious Local Search

- What *should* local search do?



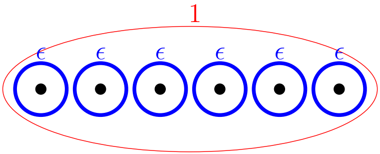
- Define a custom potential function $\Phi(\mathcal{C})$ to measure progress: $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot f(|S|)$
  - Approximates true objective: $w(\mathcal{C}) \leq \Phi(\mathcal{C}) \leq \alpha\, w(\mathcal{C})$ for approximation α, so enforce $1 \leq f(s) \leq \alpha$



  - $f(s)$ should be increasing on s
    - $\alpha = \ln k$: try $f(s) = H_s = 1 + \frac{1}{2} + \cdots + \frac{1}{s}$?

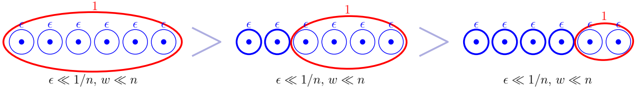
Non-Oblivious Local Search

- What *should* local search do?



$\epsilon \ll 1/n,\ w \ll n$

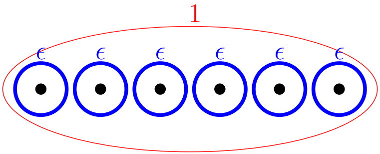
- Define a custom potential function $\Phi(\mathcal{C})$ to measure progress: $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot f(|S|)$
  - Approximates true objective: $w(\mathcal{C}) \leq \Phi(\mathcal{C}) \leq \alpha\, w(\mathcal{C})$ for approximation α, so enforce $1 \leq f(s) \leq \alpha$



  - $f(s)$ should be increasing on s
    - $\alpha = \ln k$: try $f(s) = H_s = 1 + \frac{1}{2} + \cdots + \frac{1}{s}$?
    - Inspired by [Traub-Zenklusen'21] on Steiner tree by non-oblivious local search

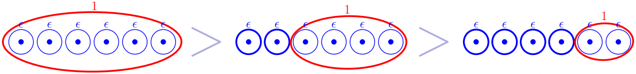
Non-Oblivious Local Search

- What *should* local search do?



$\epsilon \ll 1/n,\, w \ll n$

- Define a custom potential function $\Phi(\mathcal{C})$ to measure progress: $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot f(|S|)$
  - Approximates true objective: $w(\mathcal{C}) \leq \Phi(\mathcal{C}) \leq \alpha\, w(\mathcal{C})$ for approximation α, so enforce $1 \leq f(s) \leq \alpha$



  - $f(s)$ should be increasing on s
    - $\alpha = \ln k$: try $f(s) = H_s = 1 + \frac{1}{2} + \cdots + \frac{1}{s}$?
    - Inspired by [Traub-Zenklusen'21] on Steiner tree by non-oblivious local search
    - Φ is called the Rosenthal potential

Width-1 Local Search

- Theorem: width-1 local search with the Rosenthal potential is an H_k-approximation

Width-1 Local Search

- Theorem: width-1 local search with the Rosenthal potential is an H_k-approximation
- Proof: Let $\mathcal{C}^*$ be the optimal solution.

Width-1 Local Search

- Theorem: width-1 local search with the Rosenthal potential is an H_k-approximation
- Proof: Let $\mathcal{C}^*$ be the optimal solution.
  - Recall that $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot H_{|S|}$.

Width-1 Local Search

- Theorem: width-1 local search with the Rosenthal potential is an H_k-approximation
- Proof: Let $\mathcal{C}^*$ be the optimal solution.
  - Recall that $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot H_{|S|}$.
  - Adding in any $S^* \in \mathcal{C}^*$ and pruning does not improve local search:

$$\underbrace{w(S^*) H_{|S^*|}}_{\text{increase in } \Phi \text{ from } S^*} \geq \sum_{S \in \mathcal{C}} \underbrace{w(S) \left(H_{|S|} - H_{|S \setminus S^*|} \right)}_{\text{savings from pruning } S \to S \setminus S^*} .$$

Width-1 Local Search

- Theorem: width-1 local search with the Rosenthal potential is an H_k-approximation
- Proof: Let $\mathcal{C}^*$ be the optimal solution.
  - Recall that $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot H_{|S|}$.
  - Adding in any $S^* \in \mathcal{C}^*$ and pruning does not improve local search:

$$\underbrace{w(S^*) H_{|S^*|}}_{\text{increase in } \Phi \text{ from } S^*} \geq \sum_{S \in \mathcal{C}} \underbrace{w(S)\left(H_{|S|} - H_{|S \setminus S^*|}\right)}_{\text{savings from pruning } S \to S \setminus S^*} .$$

- $H_{|S|} - H_{|S \setminus S^*|} = \frac{1}{|S \setminus S^*|+1} + \frac{1}{|S \setminus S^*|+2} + \cdots + \frac{1}{|S|} \geq |S \cap S^*| \cdot \frac{1}{|S|}$

Width-1 Local Search

- Theorem: width-1 local search with the Rosenthal potential is an H_k-approximation
- Proof: Let $\mathcal{C}^*$ be the optimal solution.
  - Recall that $\Phi(\mathcal{C}) = \sum_{S\in\mathcal{C}} w(S)\cdot H_{|S|}$.
  - Adding in any $S^* \in \mathcal{C}^*$ and pruning does not improve local search:

$$\underbrace{w(S^*)H_{|S^*|}}_{\text{increase in } \Phi \text{ from } S^*} \geq \sum_{S\in\mathcal{C}} \underbrace{w(S)\left(H_{|S|} - H_{|S\setminus S^*|}\right)}_{\text{savings from pruning } S \to S\setminus S^*} .$$

- $H_{|S|} - H_{|S\setminus S^*|} = \frac{1}{|S\setminus S^*|+1} + \frac{1}{|S\setminus S^*|+2} + \cdots + \frac{1}{|S|} \geq |S\cap S^*|\cdot\frac{1}{|S|}$
  $\implies w(S^*)H_{|S^*|} \geq \sum_{S\in\mathcal{C}} w(S)|S\cap S^*|\cdot\frac{1}{|S|}$

Width-1 Local Search

- Theorem: width-1 local search with the Rosenthal potential is an H_k-approximation
- Proof: Let $\mathcal{C}^*$ be the optimal solution.
  - Recall that $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot H_{|S|}$.
  - Adding in any $S^* \in \mathcal{C}^*$ and pruning does not improve local search:

$$\underbrace{w(S^*)H_{|S^*|}}_{\text{increase in } \Phi \text{ from } S^*} \geq \sum_{S \in \mathcal{C}} \underbrace{w(S)\left(H_{|S|} - H_{|S \setminus S^*|}\right)}_{\text{savings from pruning } S \to S \setminus S^*} .$$

- $H_{|S|} - H_{|S \setminus S^*|} = \frac{1}{|S \setminus S^*|+1} + \frac{1}{|S \setminus S^*|+2} + \cdots + \frac{1}{|S|} \geq |S \cap S^*| \cdot \frac{1}{|S|}$
  $\implies w(S^*)H_{|S^*|} \geq \sum_{S \in \mathcal{C}} w(S)|S \cap S^*| \cdot \frac{1}{|S|}$

$$\sum_{S^* \in \mathcal{C}^*} w(S^*)H_{|S^*|} \geq \sum_{S^* \in \mathcal{C}^*} \sum_{S \in \mathcal{C}} w(S)|S \cap S^*| \cdot \frac{1}{|S|}$$

$$= \sum_{S \in \mathcal{C}} w(S) \underbrace{\sum_{S^* \in \mathcal{C}^*} |S \cap S^*| \cdot \frac{1}{|S|}}_{=1}$$

Width-1 Local Search

- Theorem: width-1 local search with the Rosenthal potential is an H_k-approximation
- Proof: Let $\mathcal{C}^*$ be the optimal solution.
  - Recall that $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot H_{|S|}$.
  - Adding in any $S^* \in \mathcal{C}^*$ and pruning does not improve local search:

$$\underbrace{w(S^*)H_{|S^*|}}_{\text{increase in } \Phi \text{ from } S^*} \geq \sum_{S \in \mathcal{C}} \underbrace{w(S)\left(H_{|S|} - H_{|S \setminus S^*|}\right)}_{\text{savings from pruning } S \to S \setminus S^*} .$$

- $H_{|S|} - H_{|S \setminus S^*|} = \frac{1}{|S \setminus S^*|+1} + \frac{1}{|S \setminus S^*|+2} + \cdots + \frac{1}{|S|} \geq |S \cap S^*| \cdot \frac{1}{|S|}$

$$\implies w(S^*)H_{|S^*|} \geq \sum_{S \in \mathcal{C}} w(S)|S \cap S^*| \cdot \frac{1}{|S|}$$

$$\sum_{S^* \in \mathcal{C}^*} w(S^*)H_{|S^*|} \geq \sum_{S^* \in \mathcal{C}^*} \sum_{S \in \mathcal{C}} w(S)|S \cap S^*| \cdot \frac{1}{|S|}$$

$$= \sum_{S \in \mathcal{C}} w(S) \underbrace{\sum_{S^* \in \mathcal{C}^*} |S \cap S^*| \cdot \frac{1}{|S|}}_{=1} = ALG$$

Width-1 Local Search

- Theorem: width-1 local search with the Rosenthal potential is an H_k-approximation
- Proof: Let $\mathcal{C}^*$ be the optimal solution.
  - Recall that $\Phi(\mathcal{C}) = \sum_{S \in \mathcal{C}} w(S) \cdot H_{|S|}$.
  - Adding in any $S^* \in \mathcal{C}^*$ and pruning does not improve local search:

$$\underbrace{w(S^*)H_{|S^*|}}_{\text{increase in } \Phi \text{ from } S^*} \geq \sum_{S \in \mathcal{C}} \underbrace{w(S)\left(H_{|S|} - H_{|S \setminus S^*|}\right)}_{\text{savings from pruning } S \to S \setminus S^*} .$$

- $H_{|S|} - H_{|S \setminus S^*|} = \frac{1}{|S \setminus S^*|+1} + \frac{1}{|S \setminus S^*|+2} + \cdots + \frac{1}{|S|} \geq |S \cap S^*| \cdot \frac{1}{|S|}$
  $\implies w(S^*)H_{|S^*|} \geq \sum_{S \in \mathcal{C}} w(S) |S \cap S^*| \cdot \frac{1}{|S|}$

$$H_k \cdot OPT \geq \sum_{S^* \in \mathcal{C}^*} w(S^*)H_{|S^*|} \geq \sum_{S^* \in \mathcal{C}^*} \sum_{S \in \mathcal{C}} w(S)|S \cap S^*| \cdot \frac{1}{|S|}$$

$$= \sum_{S \in \mathcal{C}} w(S) \underbrace{\sum_{S^* \in \mathcal{C}^*} |S \cap S^*| \cdot \frac{1}{|S|}}_{=1} = ALG \quad \square$$

Width-2 Local Search

- Theorem: width-2 local search with the Rosenthal potential is an $H_k(1 - \Theta(1/k^2))$-approximation

Width-2 Local Search

- Theorem: width-2 local search with the Rosenthal potential is an $H_k(1 - \Theta(1/k^2))$-approximation
- Where was the width-1 proof tight?

Width-2 Local Search

- Theorem: width-2 local search with the Rosenthal potential is an $H_k(1 - \Theta(1/k^2))$-approximation
- Where was the width-1 proof tight?
- $H_{|S|} - H_{|S \setminus S^*|} = \frac{1}{|S \setminus S^*|+1} + \frac{1}{|S \setminus S^*|+2} + \cdots + \frac{1}{|S|} \geq |S \cap S^*| \cdot \frac{1}{|S|}$

Width-2 Local Search

- Theorem: width-2 local search with the Rosenthal potential is an $H_k(1 - \Theta(1/k^2))$-approximation
- Where was the width-1 proof tight?
- $H_{|S|} - H_{|S \setminus S^*|} = \frac{1}{|S \setminus S^*|+1} + \frac{1}{|S \setminus S^*|+2} + \cdots + \frac{1}{|S|} \geq |S \cap S^*| \cdot \frac{1}{|S|}$
- Tight if $|S \cap S^*| = 1$. What if $|S \cap S^*| = 2$?

Width-2 Local Search

- Theorem: width-2 local search with the Rosenthal potential is an $H_k(1 - \Theta(1/k^2))$-approximation
- Where was the width-1 proof tight?
- $H_{|S|} - H_{|S \setminus S^*|} = \frac{1}{|S \setminus S^*|+1} + \frac{1}{|S \setminus S^*|+2} + \cdots + \frac{1}{|S|} \geq |S \cap S^*| \cdot \frac{1}{|S|}$
- Tight if $|S \cap S^*| = 1$. What if $|S \cap S^*| = 2$?
  - $\frac{1}{|S|-1} + \frac{1}{|S|} = \frac{2}{|S|} + \Theta(\frac{1}{|S|^2}) \geq (1 + \Theta(\frac{1}{|S|})) \cdot \frac{1}{|S|}$

Width-2 Local Search

- Theorem: width-2 local search with the Rosenthal potential is an $H_k(1 - \Theta(1/k^2))$-approximation
- Where was the width-1 proof tight?
- $H_{|S|} - H_{|S \setminus S^*|} = \frac{1}{|S \setminus S^*|+1} + \frac{1}{|S \setminus S^*|+2} + \cdots + \frac{1}{|S|} \geq |S \cap S^*| \cdot \frac{1}{|S|}$
- Tight if $|S \cap S^*| = 1$. What if $|S \cap S^*| = 2$?
  - $\frac{1}{|S|-1} + \frac{1}{|S|} = \frac{2}{|S|} + \Theta(\frac{1}{|S|^2}) \geq (1 + \Theta(\frac{1}{|S|})) \cdot \frac{1}{|S|}$
- More generally, $H_{|S|} - H_{|S \setminus S^*|} \geq (1 + \Theta(\frac{1}{|S|})) \cdot \frac{1}{|S|}$ as long as $|S \cap S^*| \geq 2$

Width-2 Local Search

- Theorem: width-2 local search with the Rosenthal potential is an $H_k(1-\Theta(1/k^2))$-approximation
- Where was the width-1 proof tight?
- $H_{|S|}-H_{|S\setminus S^*|}=\frac{1}{|S\setminus S^*|+1}+\frac{1}{|S\setminus S^*|+2}+\cdots+\frac{1}{|S|}\geq |S\cap S^*|\cdot\frac{1}{|S|}$
- Tight if $|S\cap S^*|=1$. What if $|S\cap S^*|=2$?
  - $\frac{1}{|S|-1}+\frac{1}{|S|}=\frac{2}{|S|}+\Theta(\frac{1}{|S|^2})\geq(1+\Theta(\frac{1}{|S|}))\cdot\frac{1}{|S|}$
- More generally, $H_{|S|}-H_{|S\setminus S^*|}\geq(1+\Theta(\frac{1}{|S|}))\cdot\frac{1}{|S|}$ as long as $|S\cap S^*|\geq 2$
- Adding in any $S_1^*,S_2^*\in\mathcal{C}^*$ and pruning does not improve local search:

$$\underbrace{w(S^*)H_{|S_1^*\cup S_2^*|}}_{\text{increase in }\Phi\text{ from }S^*}\geq\sum_{S\in\mathcal{C}}\underbrace{w(S)\left(H_{|S|}-H_{|S\setminus(S_1^*\cup S_2^*)|}\right)}_{\text{savings from pruning }S\to S\setminus(S_1^*\cup S_2^*)}.$$

Width-2 Local Search

- Theorem: width-2 local search with the Rosenthal potential is an $H_k(1 - \Theta(1/k^2))$-approximation
- Where was the width-1 proof tight?
- $H_{|S|} - H_{|S \setminus S^*|} = \frac{1}{|S \setminus S^*|+1} + \frac{1}{|S \setminus S^*|+2} + \cdots + \frac{1}{|S|} \geq |S \cap S^*| \cdot \frac{1}{|S|}$
- Tight if $|S \cap S^*| = 1$. What if $|S \cap S^*| = 2$?
  - $\frac{1}{|S|-1} + \frac{1}{|S|} = \frac{2}{|S|} + \Theta(\frac{1}{|S|^2}) \geq (1 + \Theta(\frac{1}{|S|})) \cdot \frac{1}{|S|}$
- More generally, $H_{|S|} - H_{|S \setminus S^*|} \geq (1 + \Theta(\frac{1}{|S|})) \cdot \frac{1}{|S|}$ as long as $|S \cap S^*| \geq 2$
- Adding in any $S_1^*, S_2^* \in \mathcal{C}^*$ and pruning does not improve local search:

$$\underbrace{w(S^*)H_{|S_1^* \cup S_2^*|}}_{\text{increase in } \Phi \text{ from } S^*} \geq \sum_{S \in \mathcal{C}} \underbrace{w(S)\left(H_{|S|} - H_{|S \setminus (S_1^* \cup S_2^*)|}\right)}_{\text{has } (1+\Theta(\frac{1}{|S|})) \text{ ``often enough''}}.$$

Width-2 Local Search

- Theorem: width-2 local search with the Rosenthal potential is an $H_k(1-\Theta(1/k^2))$-approximation
- Where was the width-1 proof tight?
- $H_{|S|} - H_{|S\setminus S^*|} = \frac{1}{|S\setminus S^*|+1} + \frac{1}{|S\setminus S^*|+2} + \cdots + \frac{1}{|S|} \geq |S\cap S^*| \cdot \frac{1}{|S|}$
- Tight if $|S\cap S^*| = 1$. What if $|S\cap S^*| = 2$?
  - $\frac{1}{|S|-1} + \frac{1}{|S|} = \frac{2}{|S|} + \Theta(\frac{1}{|S|^2}) \geq (1+\Theta(\frac{1}{|S|}))\cdot\frac{1}{|S|}$
- More generally, $H_{|S|} - H_{|S\setminus S^*|} \geq (1+\Theta(\frac{1}{|S|}))\cdot\frac{1}{|S|}$ as long as $|S\cap S^*| \geq 2$
- Adding in any $S_1^*, S_2^* \in \mathcal{C}^*$ and pruning does not improve local search:

$$\underbrace{w(S^*)H_{|S_1^*\cup S_2^*|}}_{\text{increase in }\Phi\text{ from }S^*} \geq \sum_{S\in\mathcal{C}} \underbrace{w(S)\left(H_{|S|} - H_{|S\setminus(S_1^*\cup S_2^*)|}\right)}_{\text{has }(1+\Theta(\frac{1}{|S|}))\text{ “often enough”}}.$$

- Repeat analysis from before: $ALG \leq (1-\Theta(1/k^2))H_k OPT$

Further Improvements

- Can fine-tune function $f(s)$ to get approximation $H_k - \Theta(\frac{1}{k})$

Further Improvements

- Can fine-tune function $f(s)$ to get approximation $H_k - \Theta(\frac{1}{k})$
- Width-k local search gives approximation $H_k - \Theta(\frac{\log^2 k}{k})$

Further Improvements

- Can fine-tune function $f(s)$ to get approximation $H_k - \Theta(\frac{1}{k})$
- Width-k local search gives approximation $H_k - \Theta(\frac{\log^2 k}{k})$
- Best possible bounds for non-oblivious local search with potential function $\Phi = \sum_{S \in \mathcal{C}} w(S) \cdot f(|S|)$

Further Improvements

- Can fine-tune function $f(s)$ to get approximation $H_k - \Theta(\frac{1}{k})$
- Width-k local search gives approximation $H_k - \Theta(\frac{\log^2 k}{k})$
- Best possible bounds for non-oblivious local search with potential function $\Phi = \sum_{S \in \mathcal{C}} w(S) \cdot f(|S|)$
- Open: approximation $H_k - \Theta(1)$? (Known for unweighted set cover!)