Congestion-Approximators from the Bottom Up

Jason Li (CMU)
Joint with Satish Rao (Berkeley),
Di Wang (Google)

Max-Flow Problem

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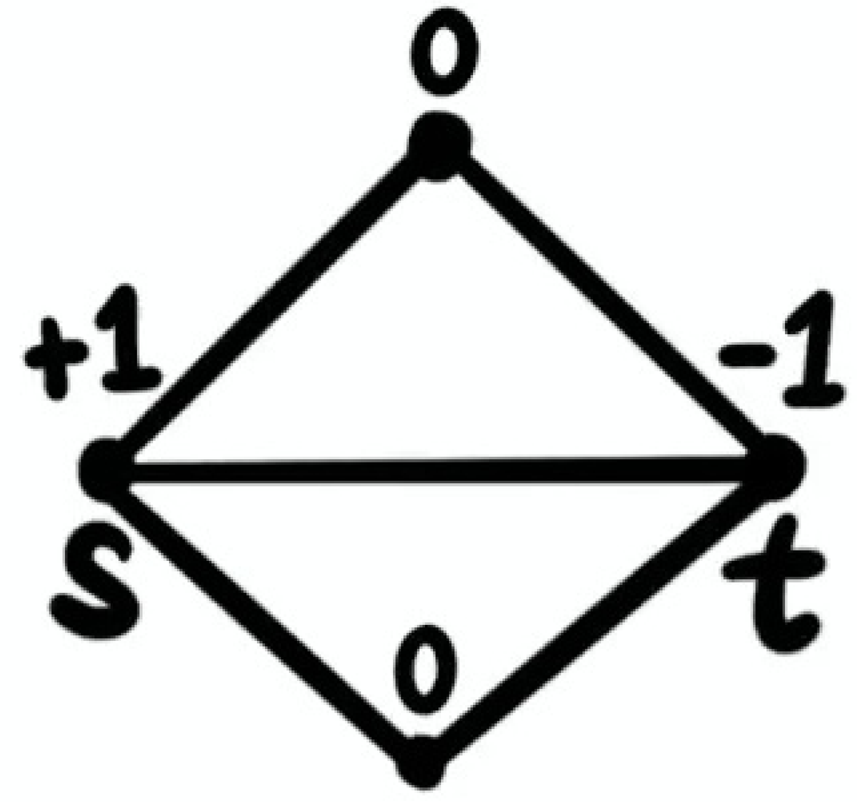

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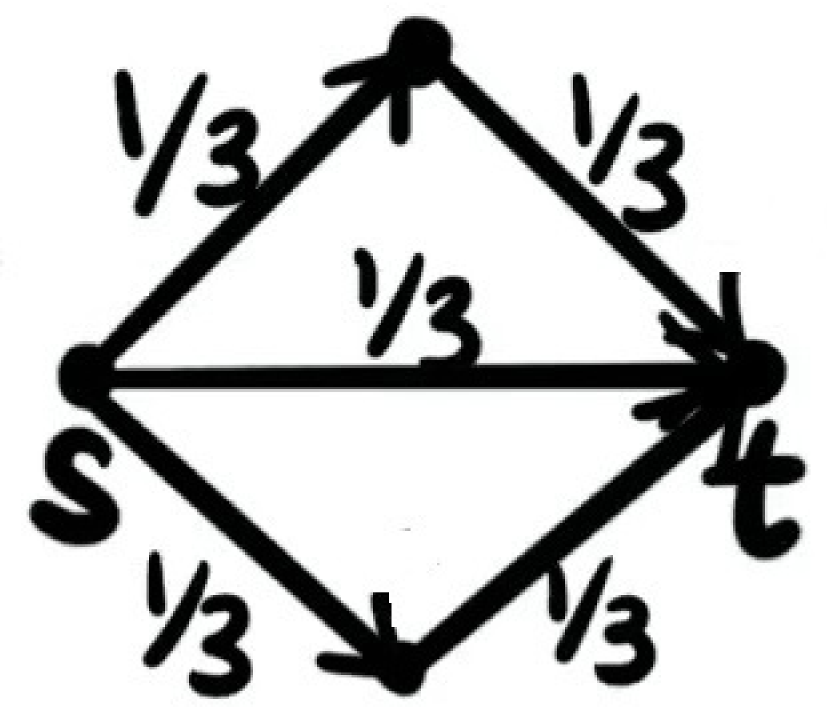

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congestion = 1/3

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Flow/Cut Duality

Given a subset of vertices S, consider

$$\frac{|b(S)|}{\partial S} \quad \begin{array}{l} \leftarrow \left|\sum_{v \in S} b_v\right| \\ \leftarrow \text{size of boundary} \end{array}$$

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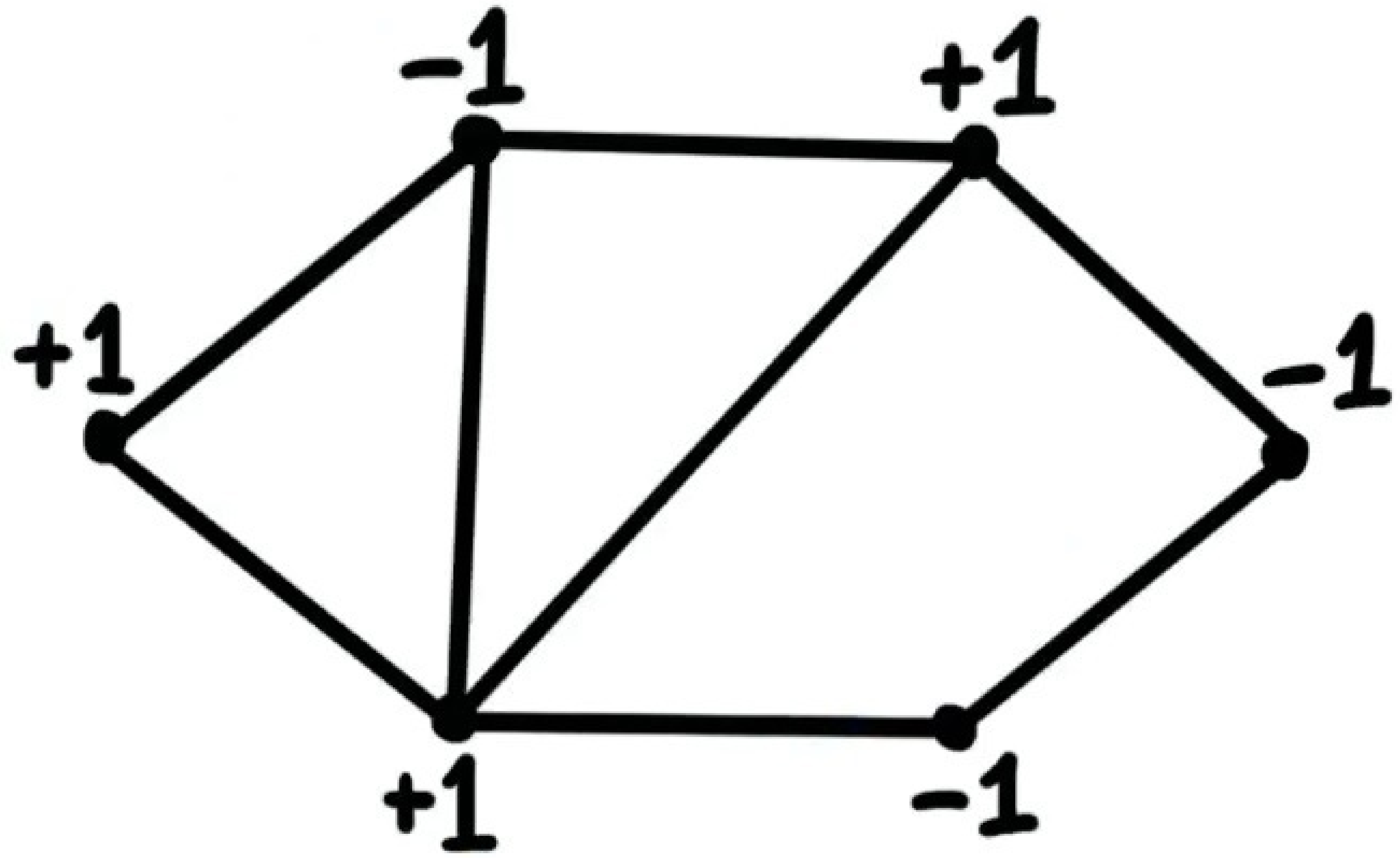

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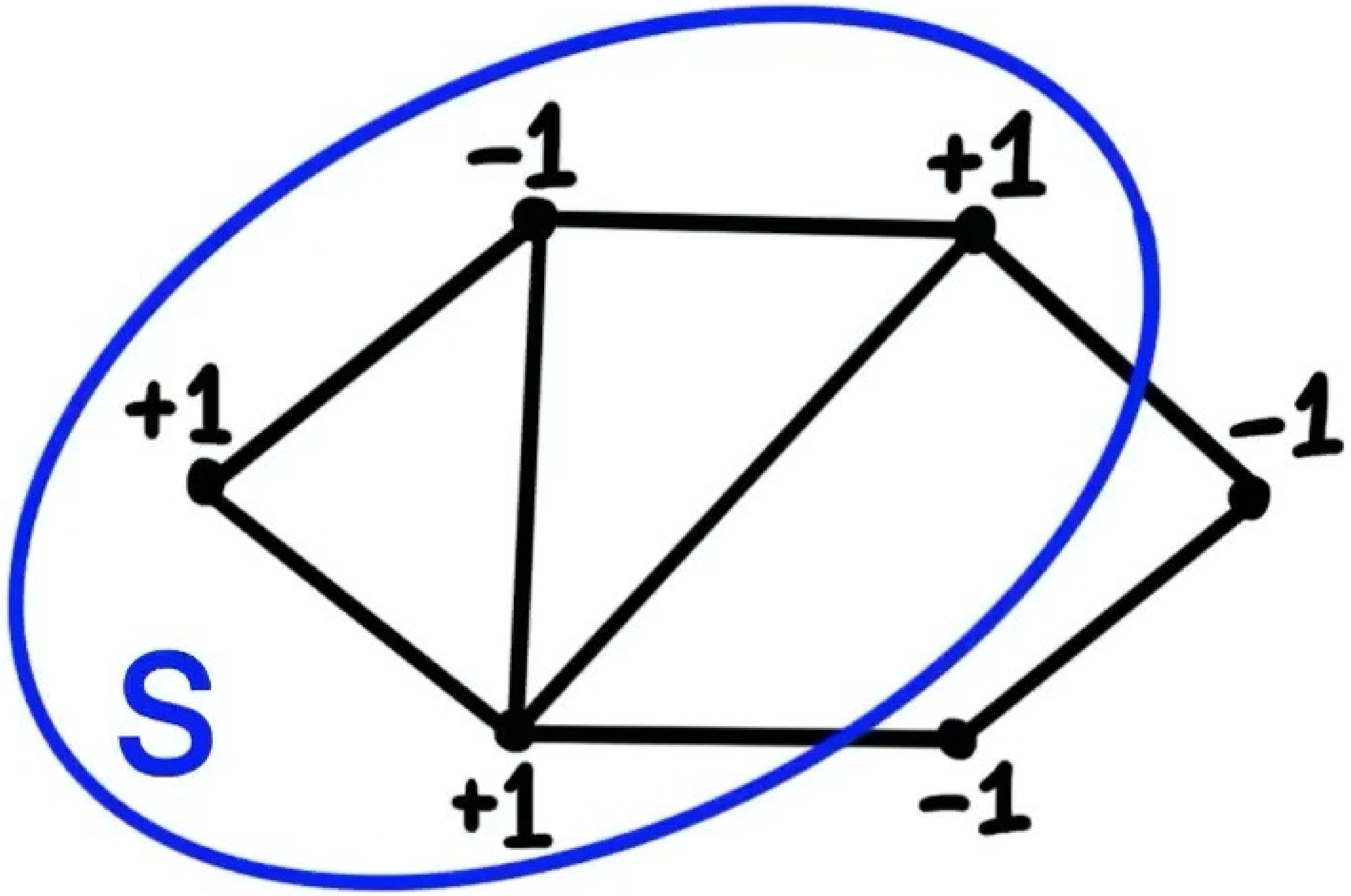

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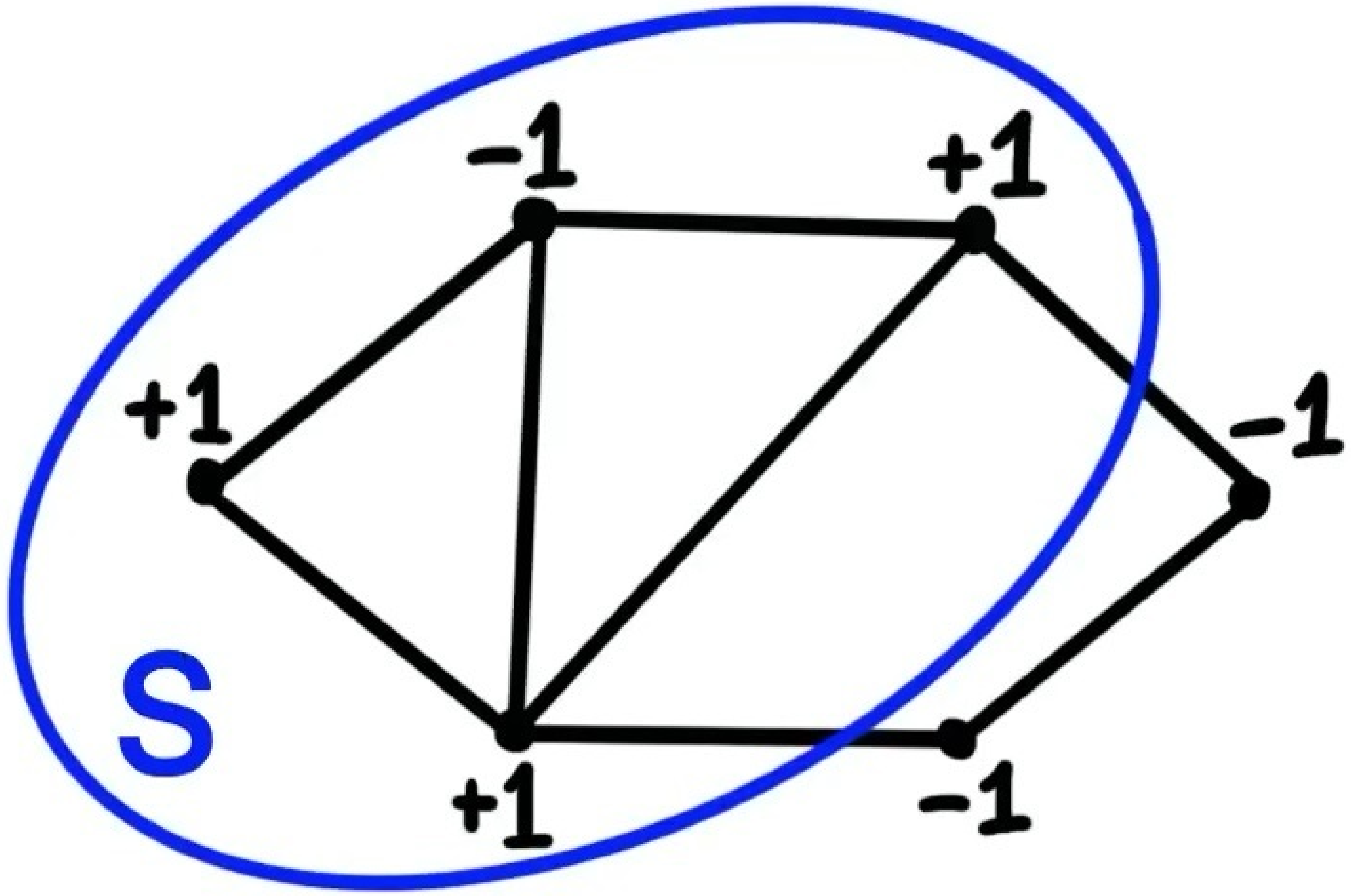

$\partial S = 2$

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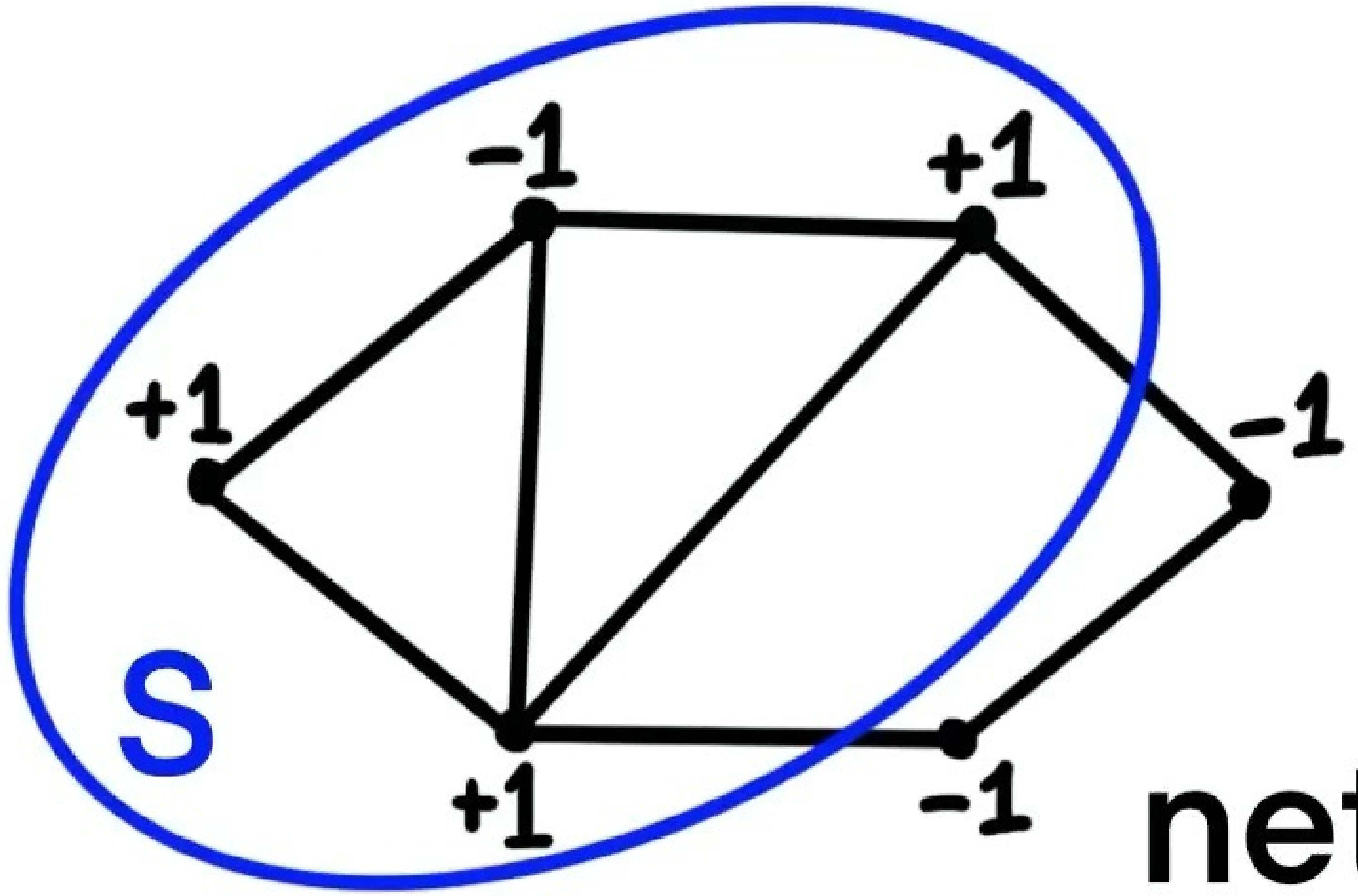

net |b(S)|=2 flow must
go across ∂S=2 edges

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→ any flow has congestion >= 1

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($b(V)=0$)

↑ quality

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This work: polylog-quality congestion approximator without recursive max-flow

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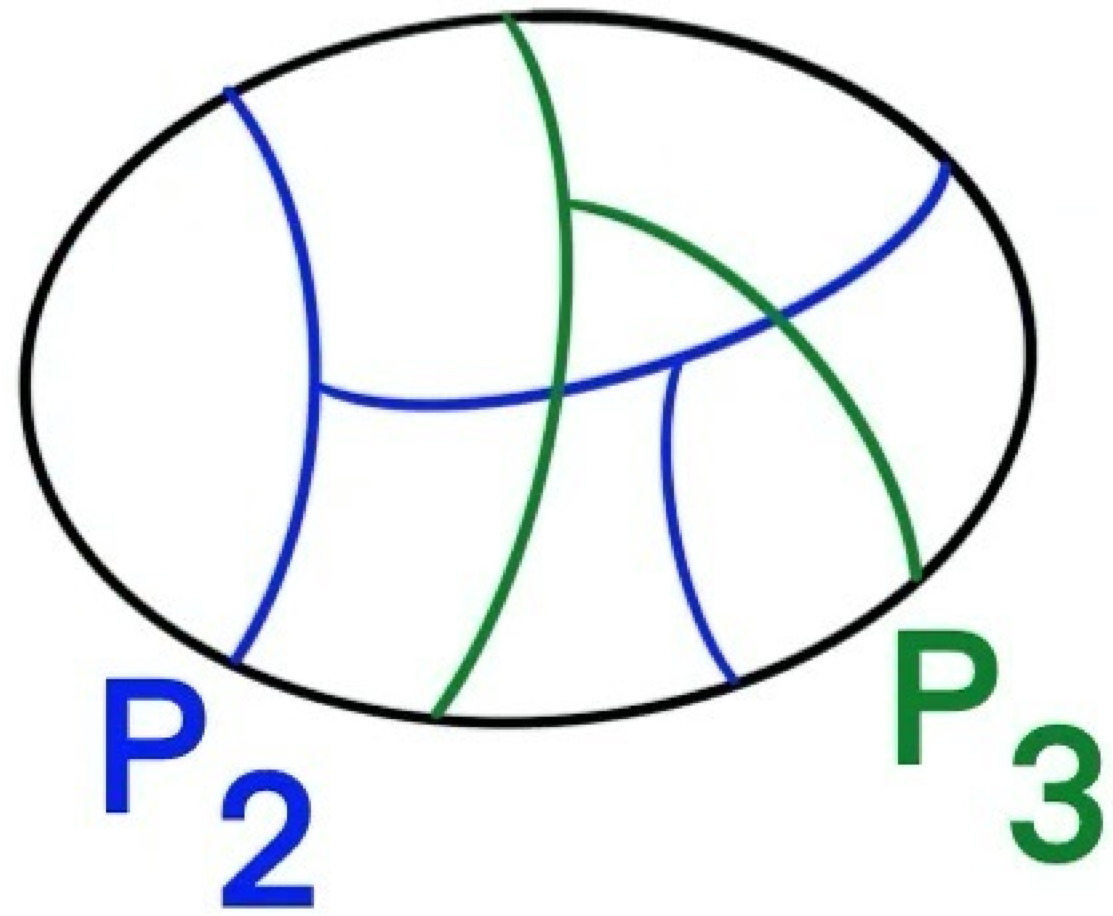
Theorem: Consider partitions $P_1, P_2, \ldots P_L$ of V s.t.



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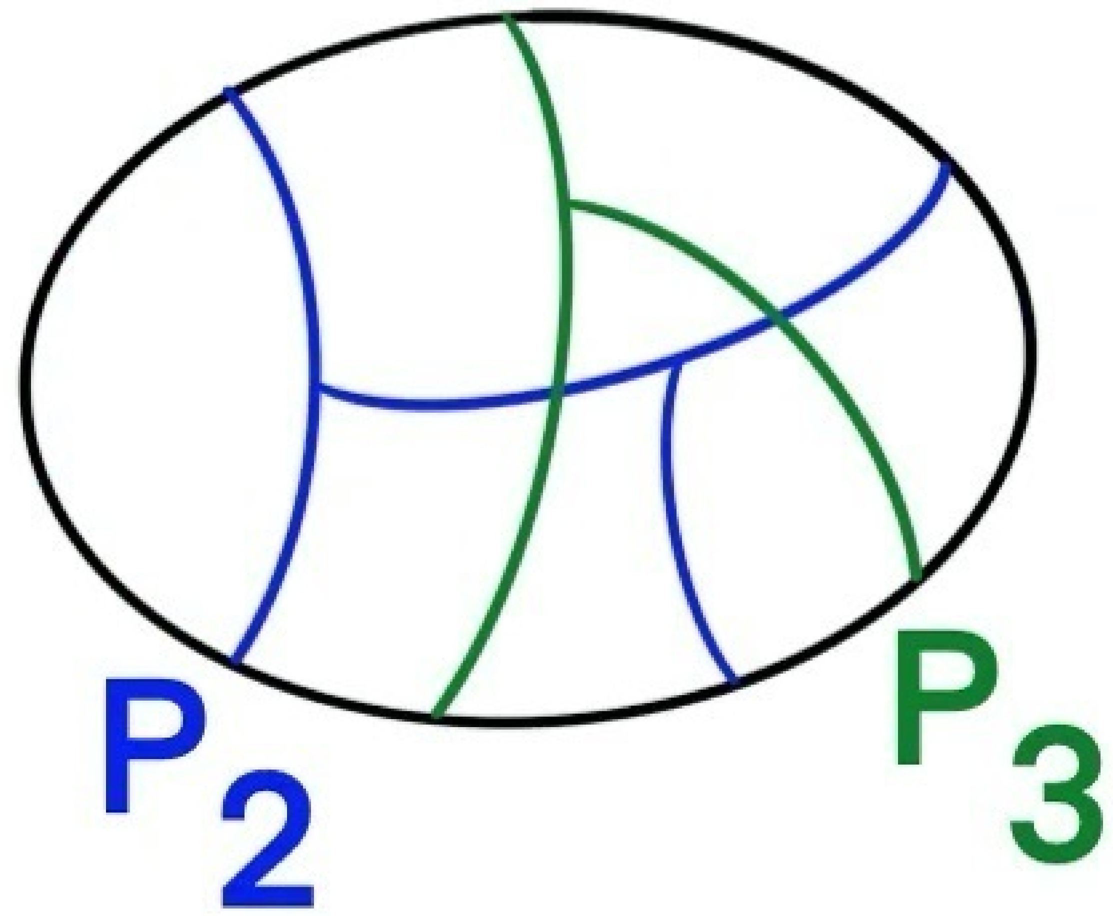

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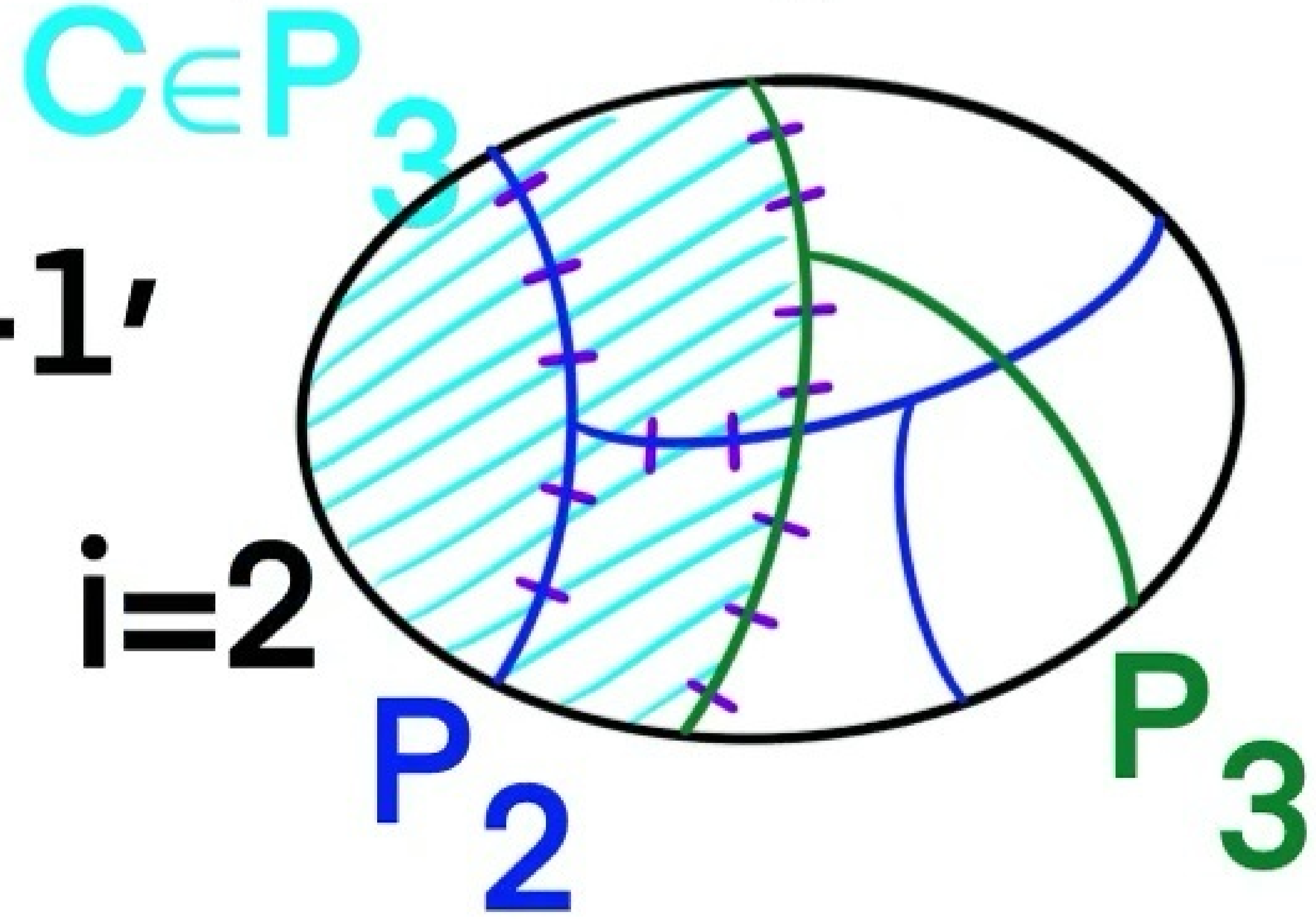

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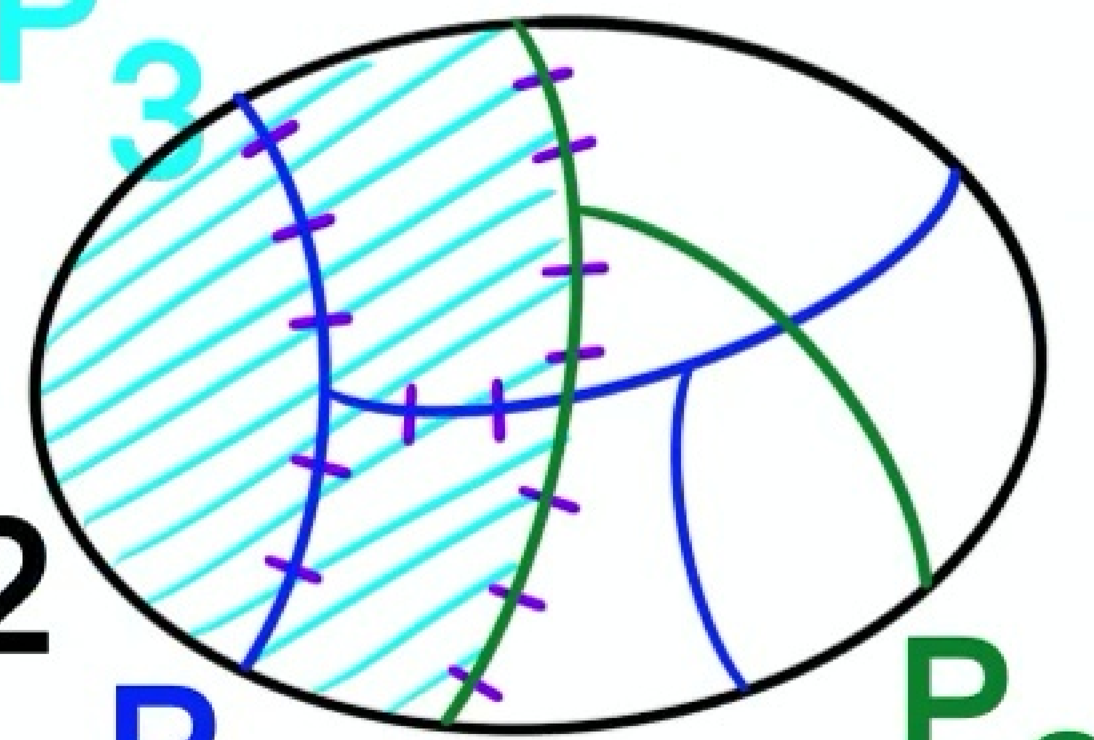

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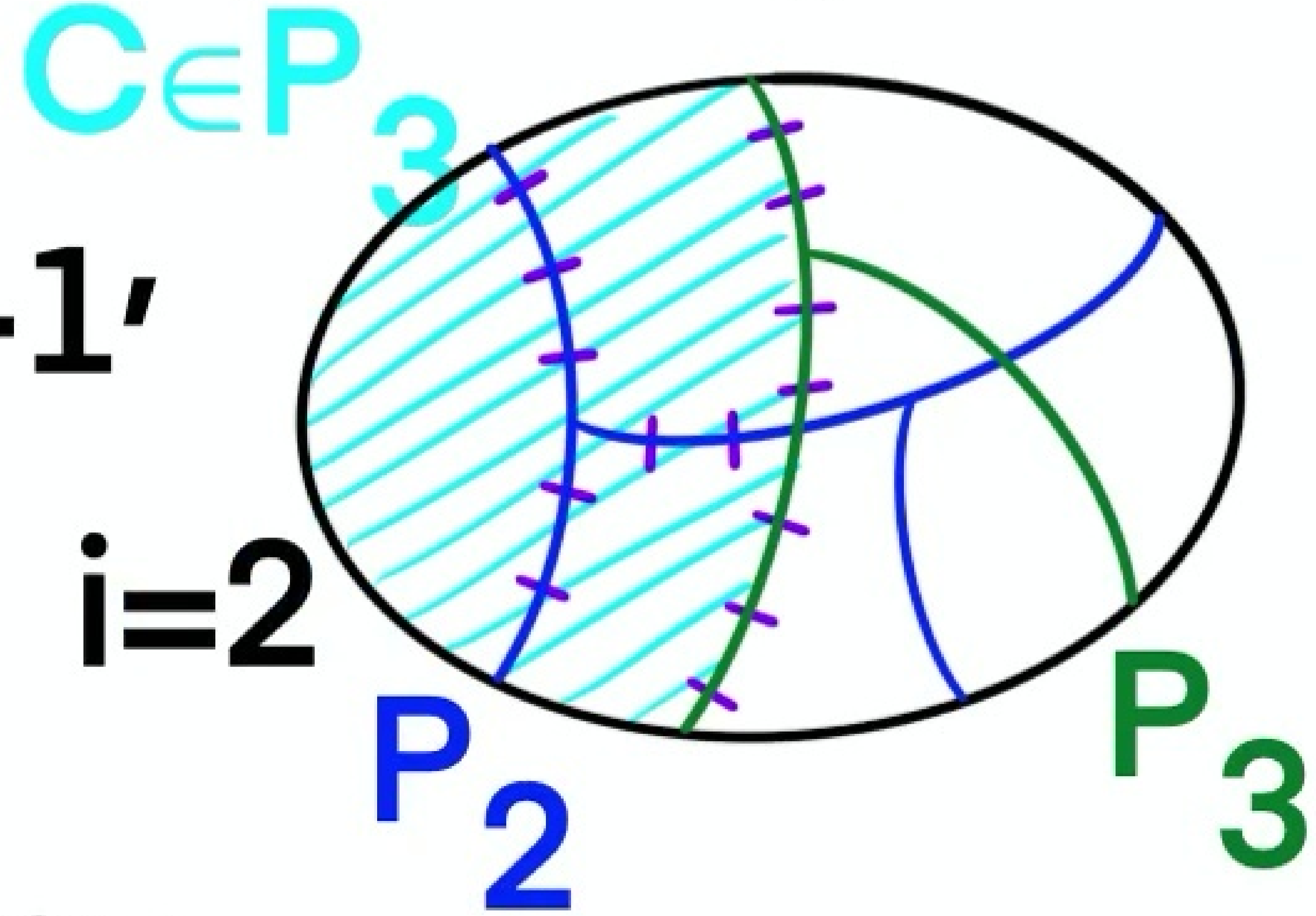

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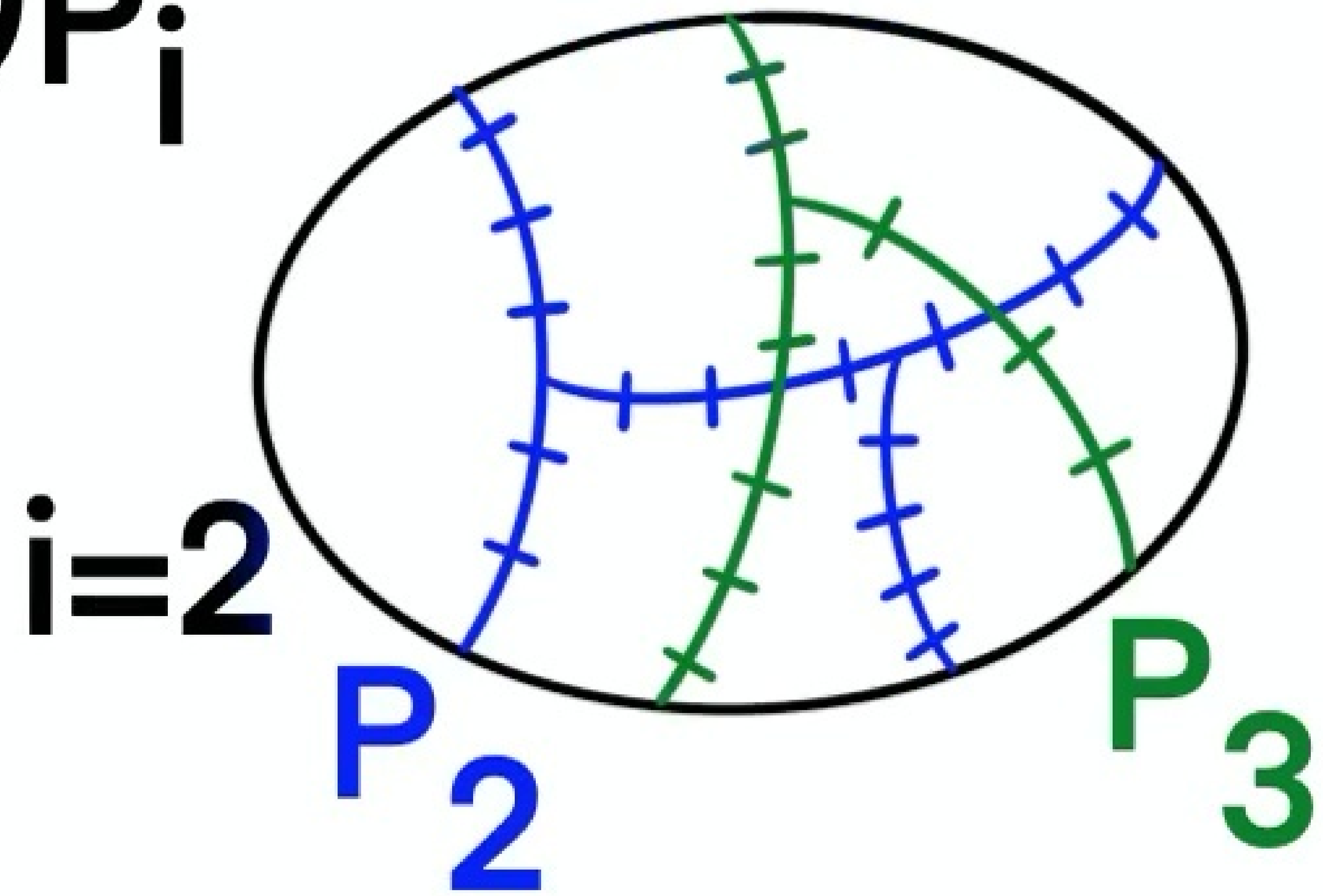

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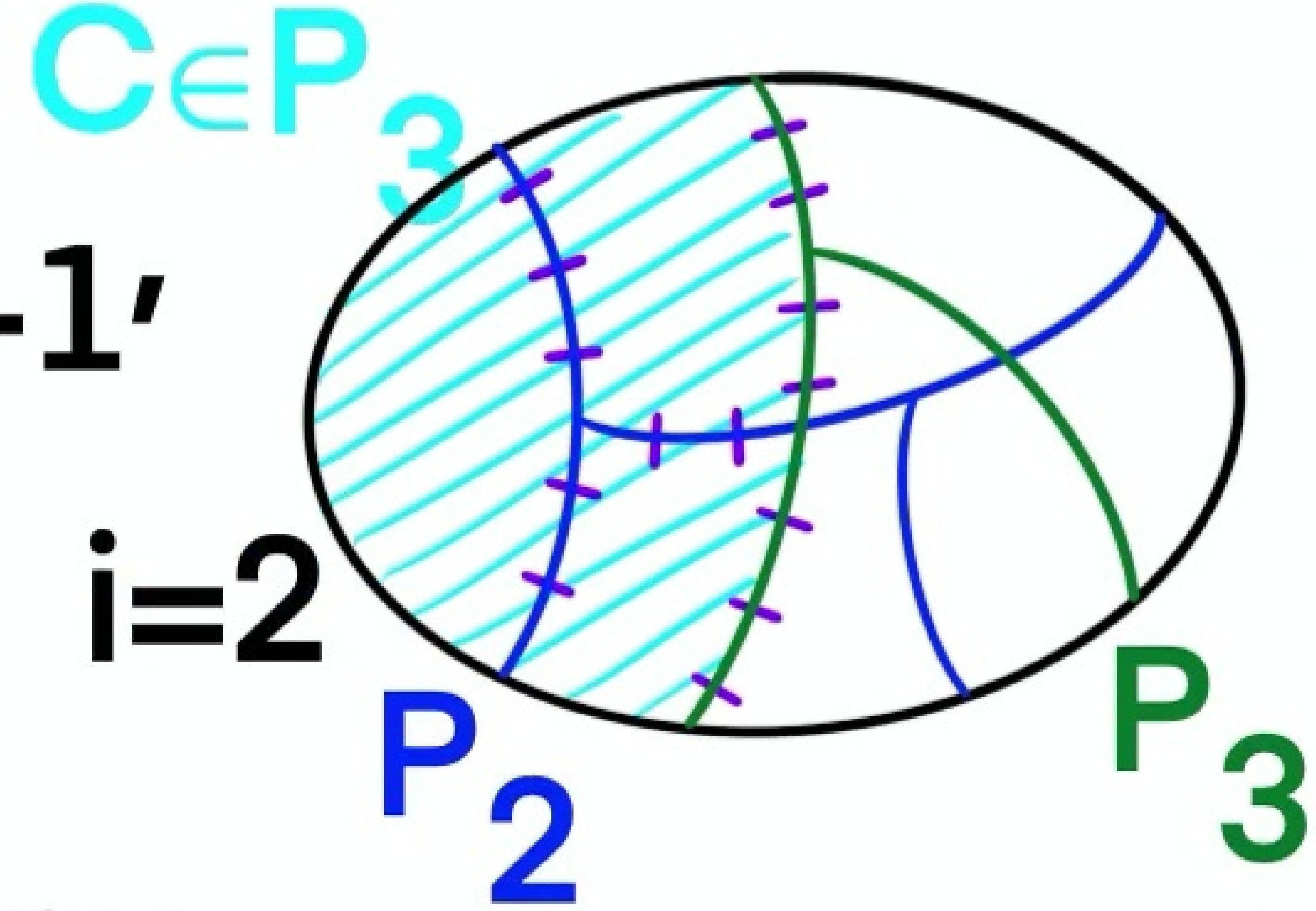

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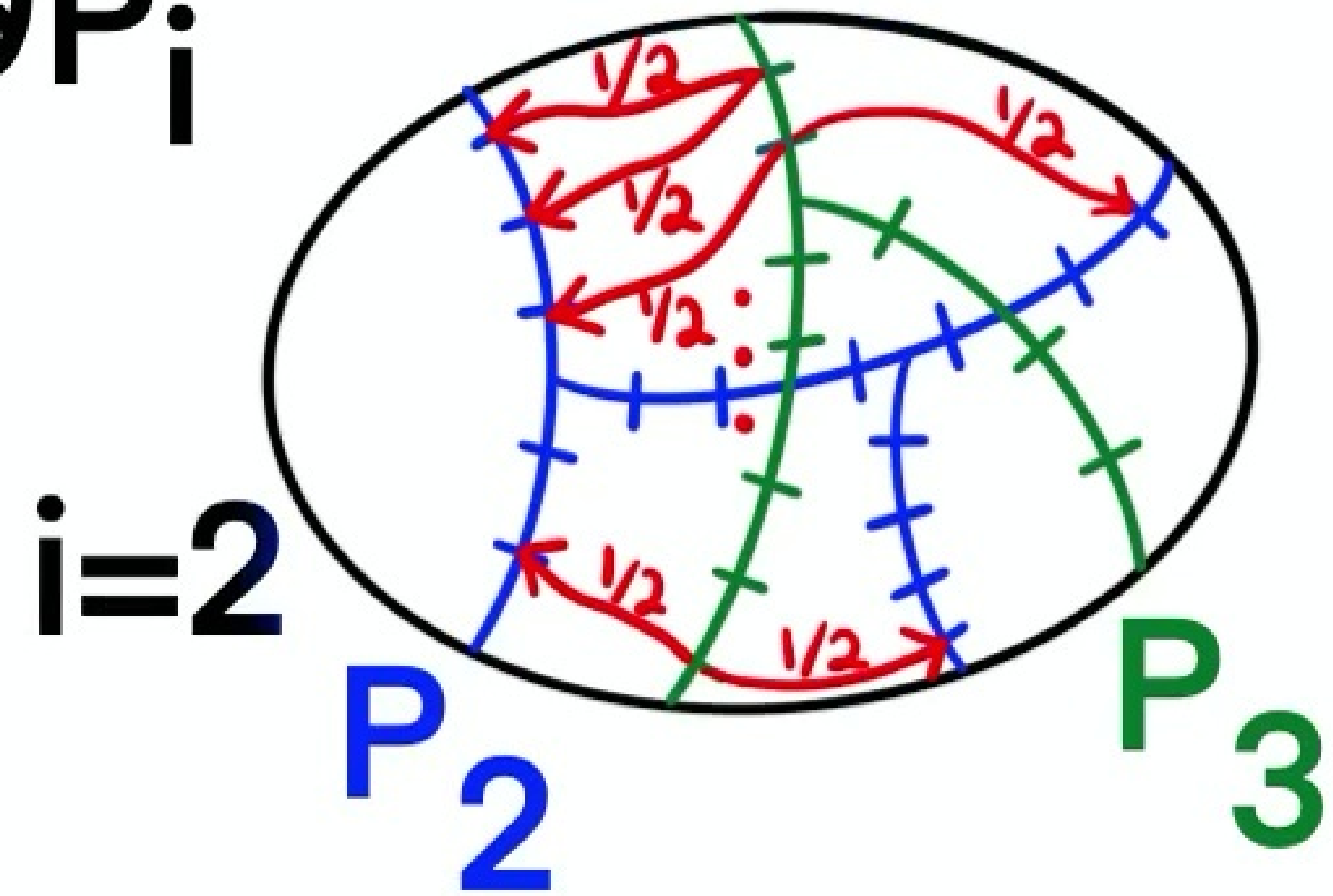

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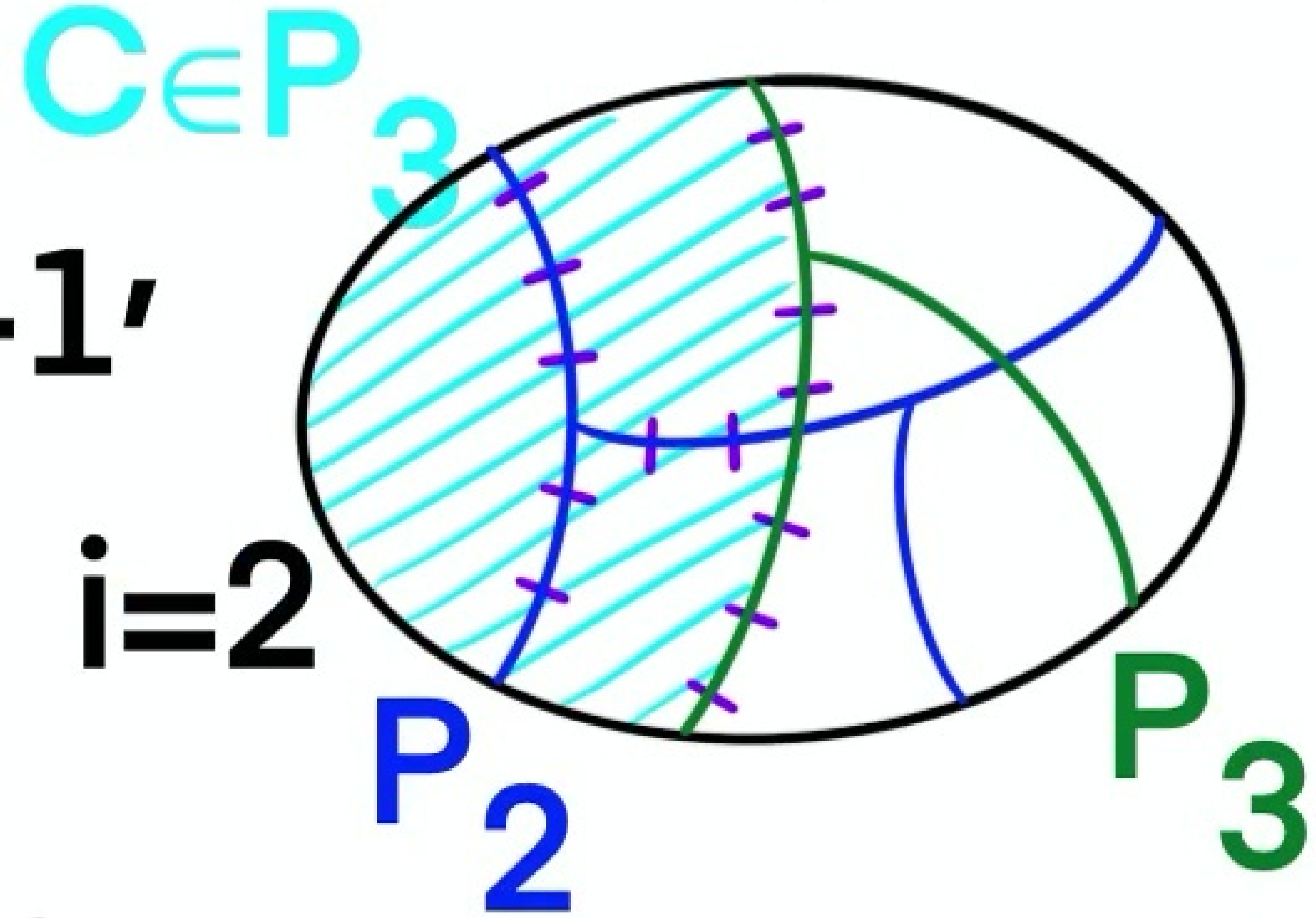

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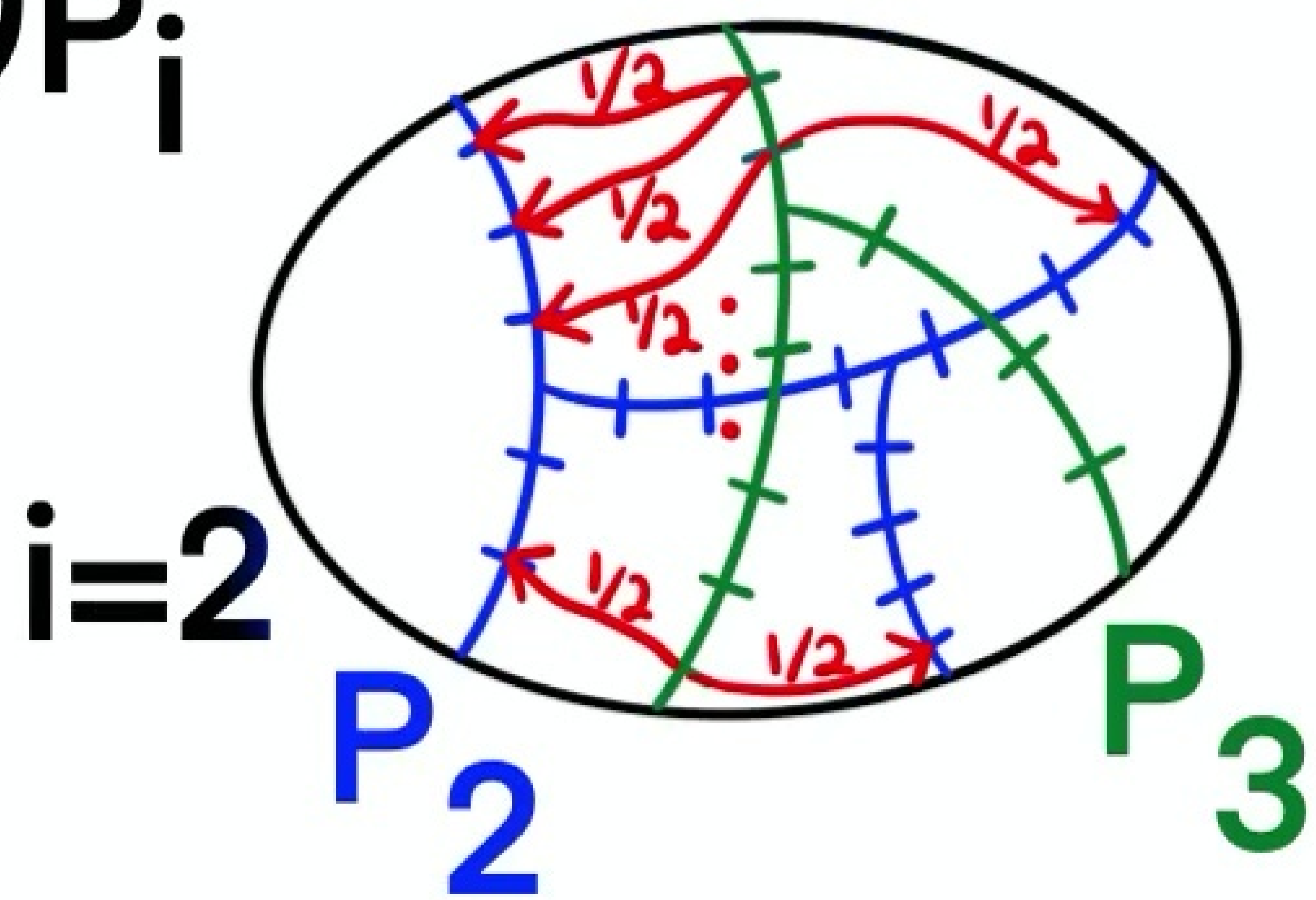

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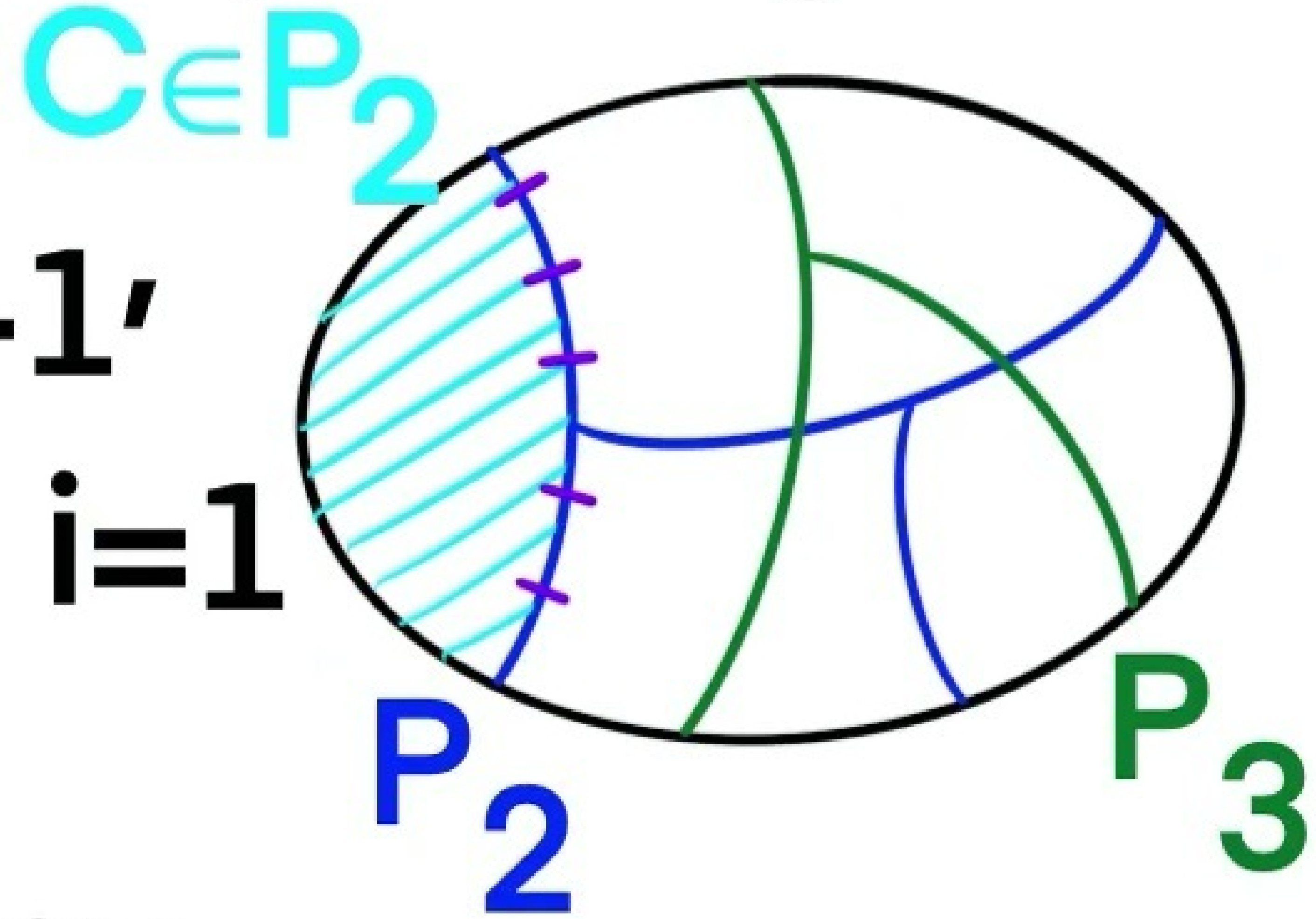

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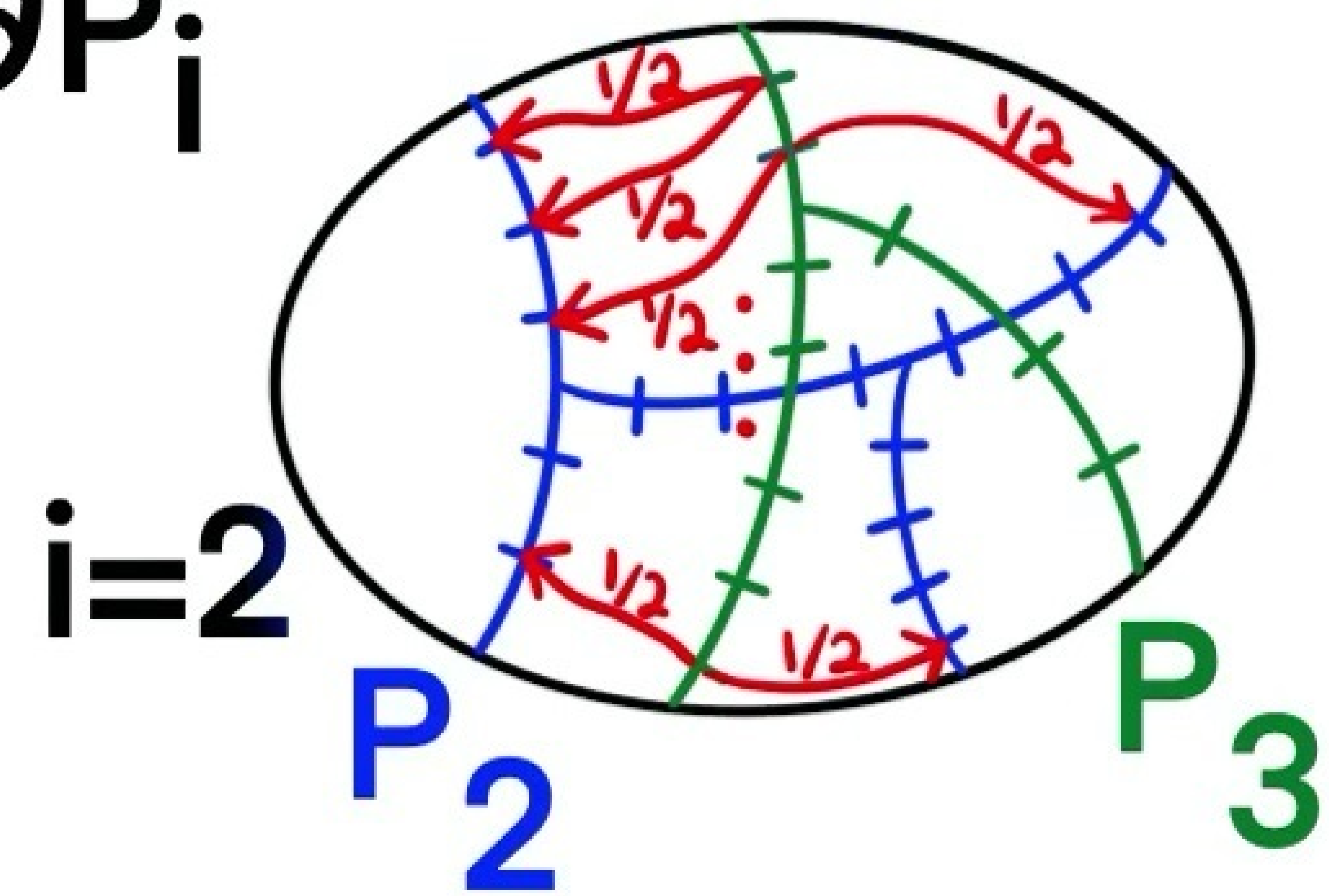

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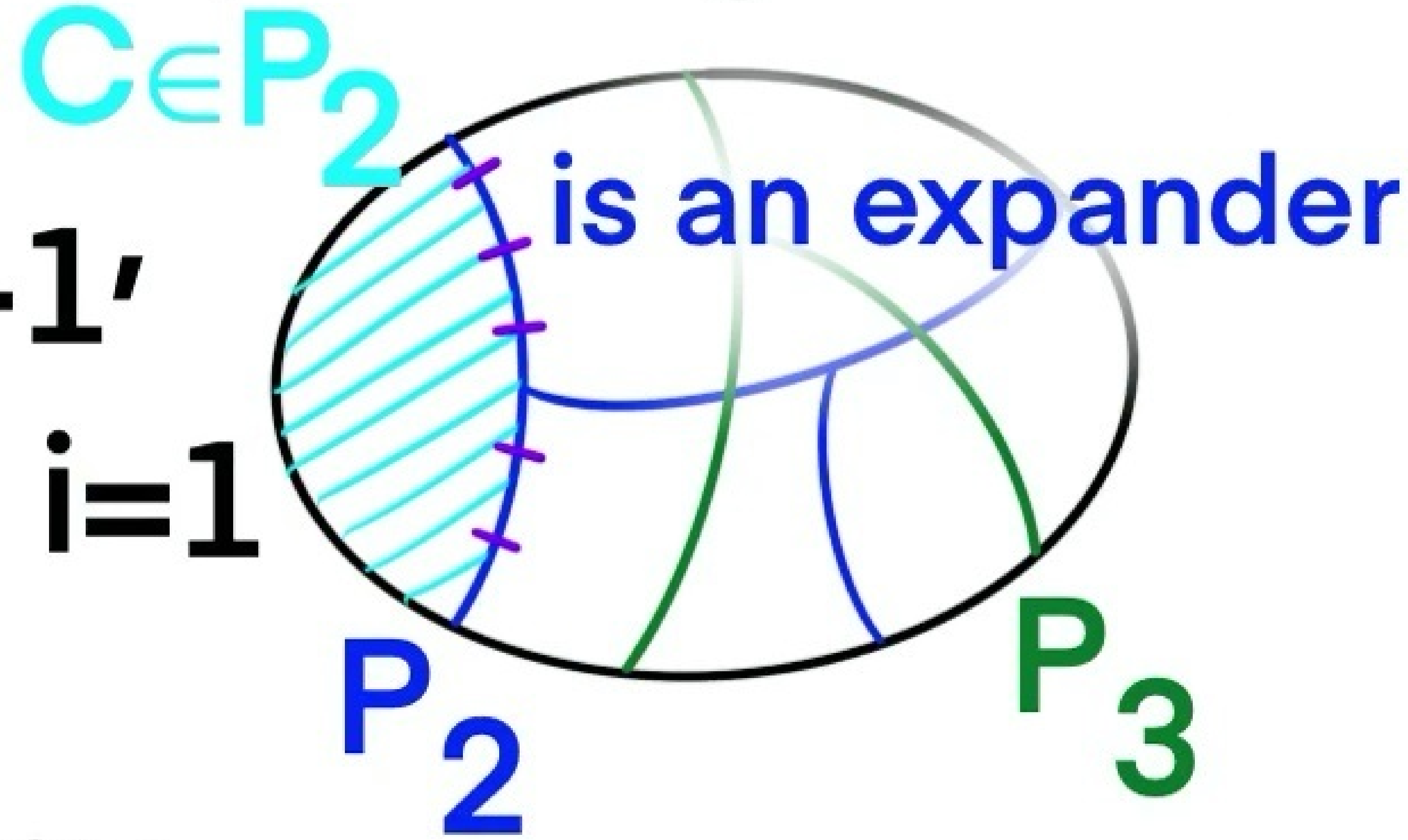

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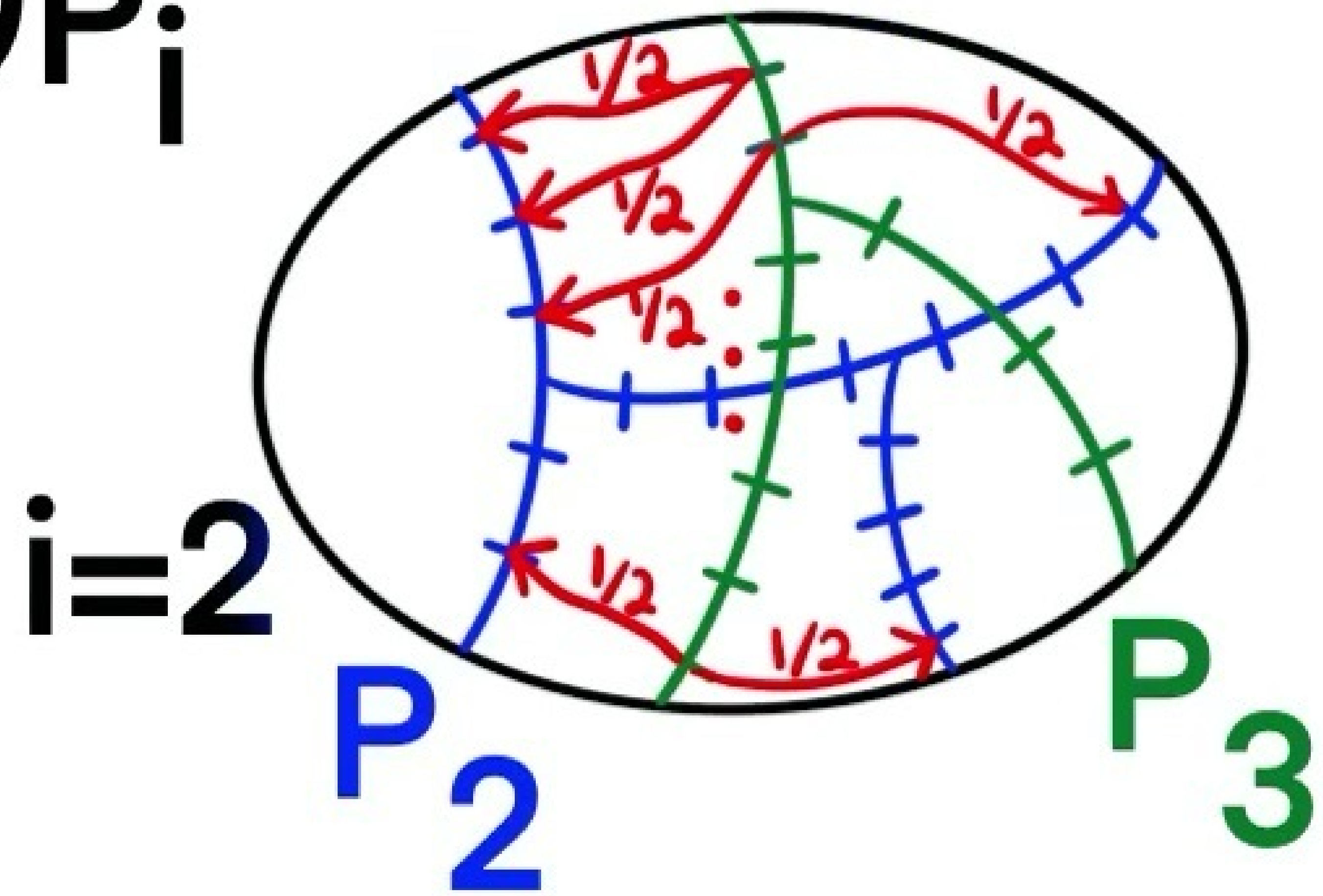

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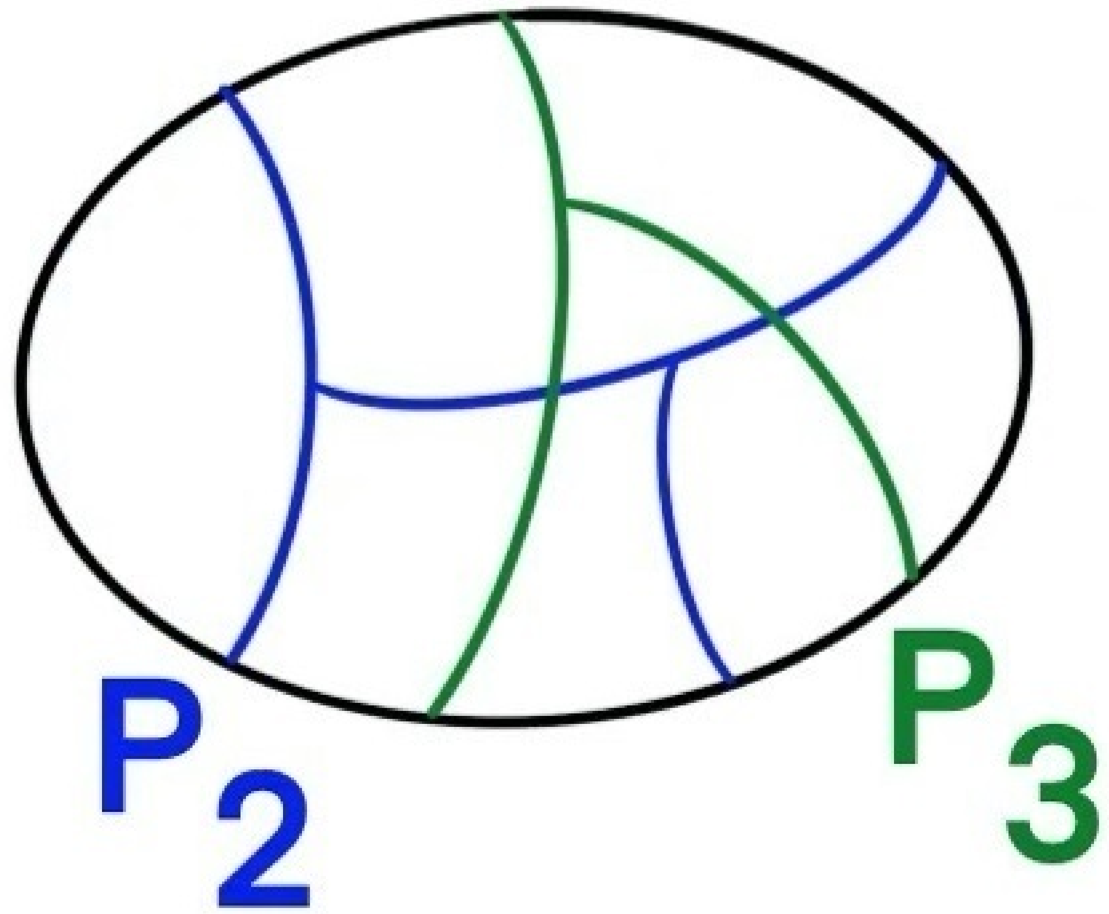

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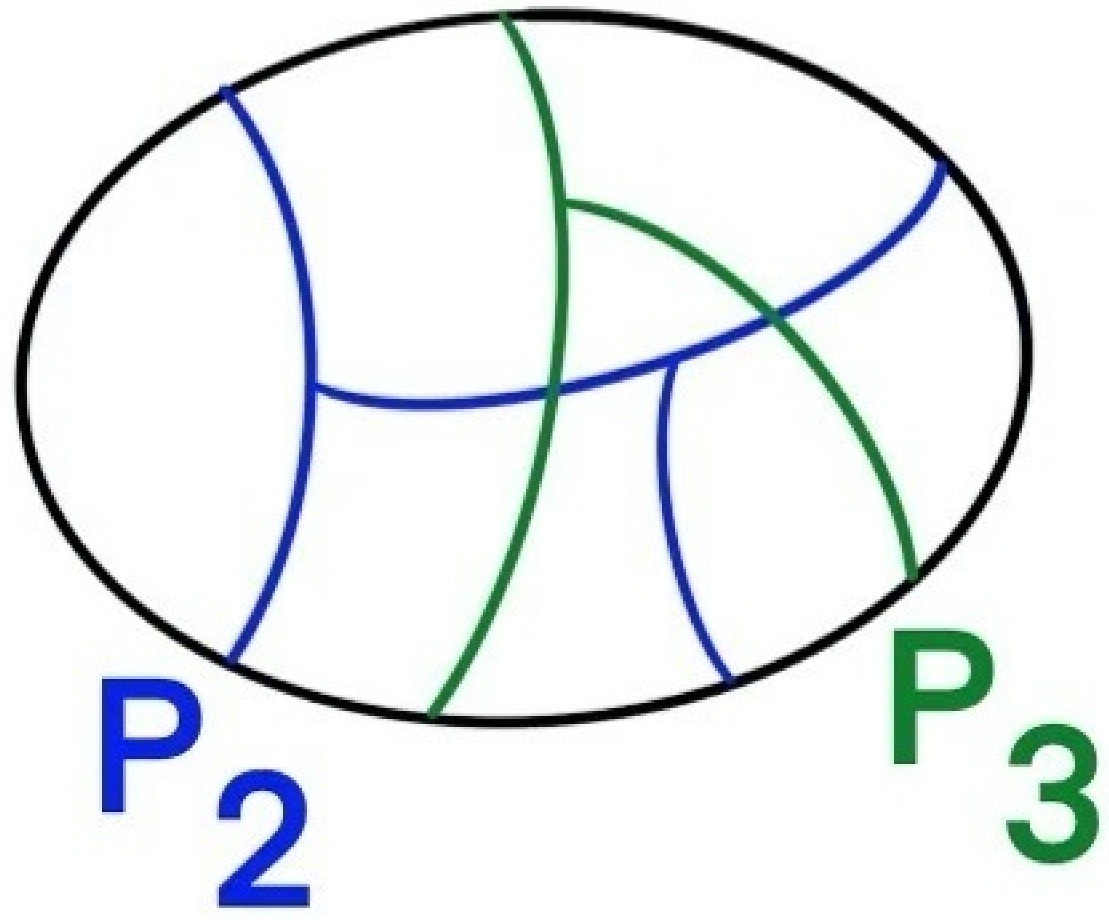

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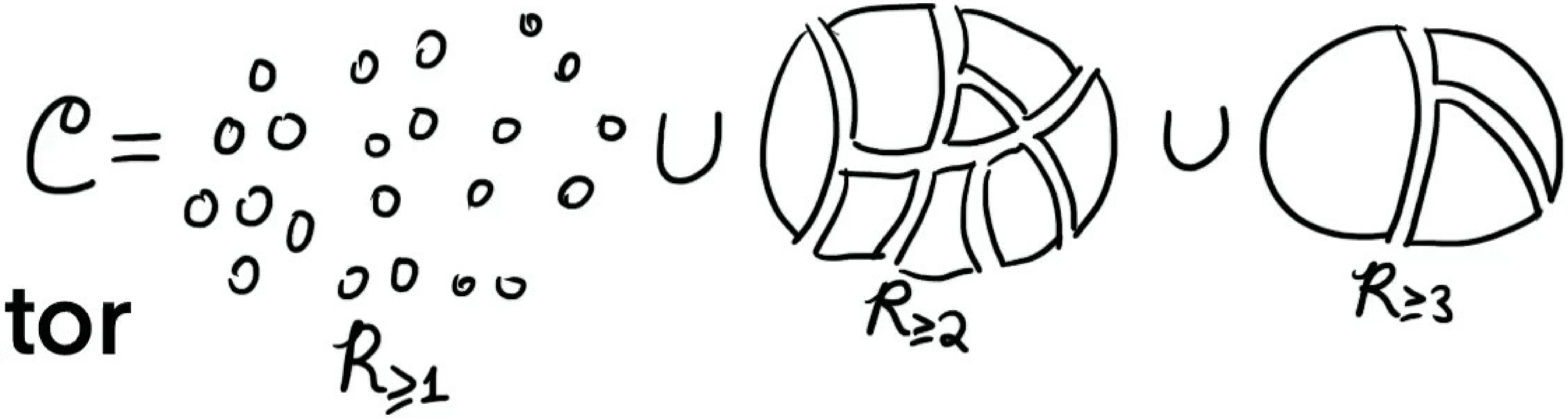

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Max-flow calls required, but use structure of $P_1, \ldots, P_{i-1}$ to build "pseudo"-congestion approximator sufficient for the specialized max-flow calls

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Number of polylog factors still high (unspecified).
Open question: reduce number of polylog factors?