# Detecting Feedback Vertex Sets of Size k in O*(2.7^k) Time

Jason Li

**With Jesper Nederlof (Utrecht Univ., Netherlands)**

**May 7, 2020**

# Introduction

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Want time FPT in k: f(k)*poly(n)

Goal in FPT setting: minimize function f(k).

poly(n) factor does not matter

# Prior Work

Downey and Fellows '92: $f(k) = k^{O(k)}$

Becker et al. [BBG'00]: f(k)=4^k, randomized

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This talk: (3-ε)^k, or how to break 3^k.

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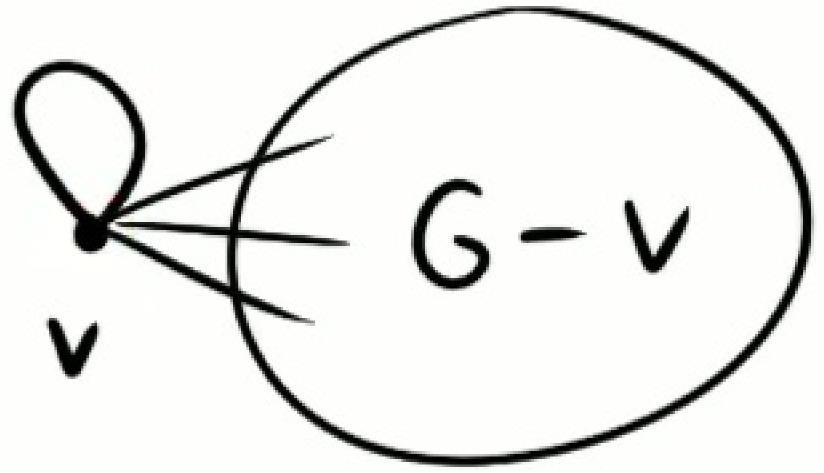

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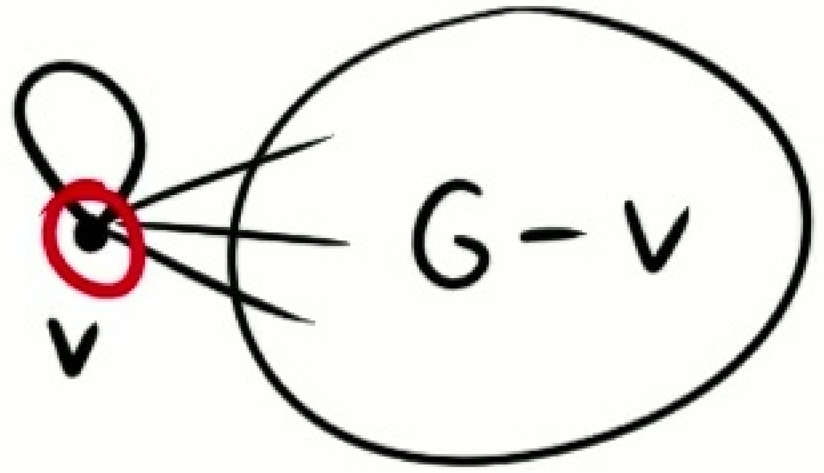


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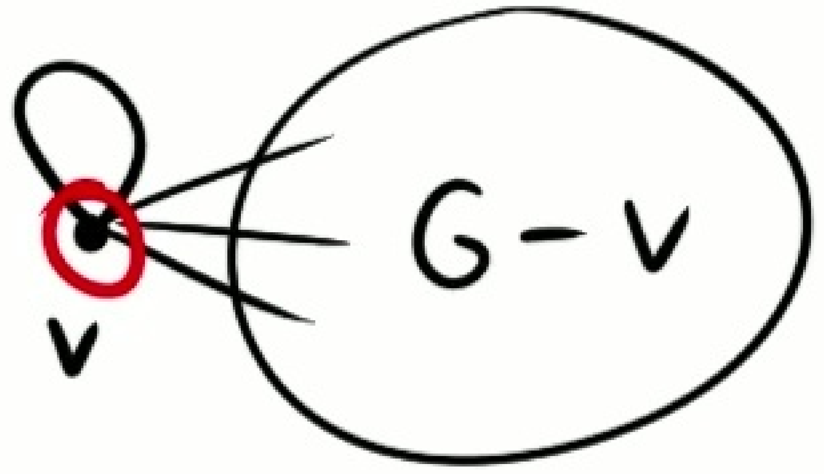


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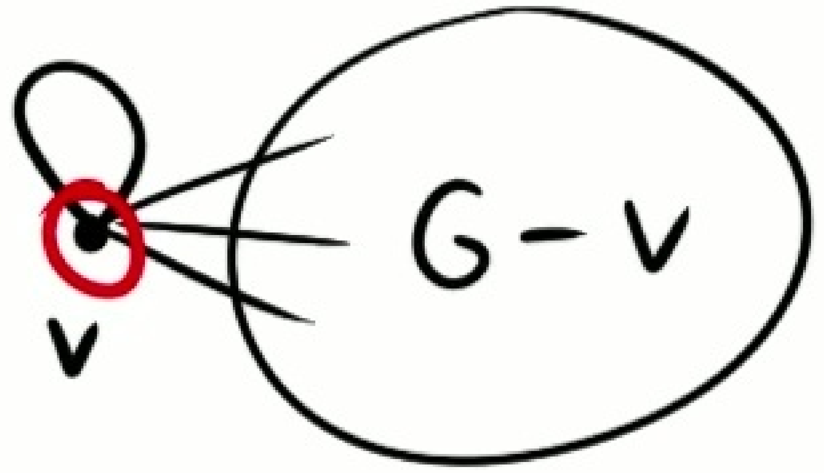


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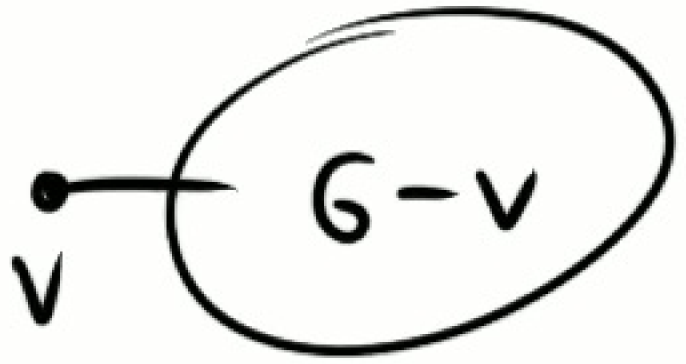

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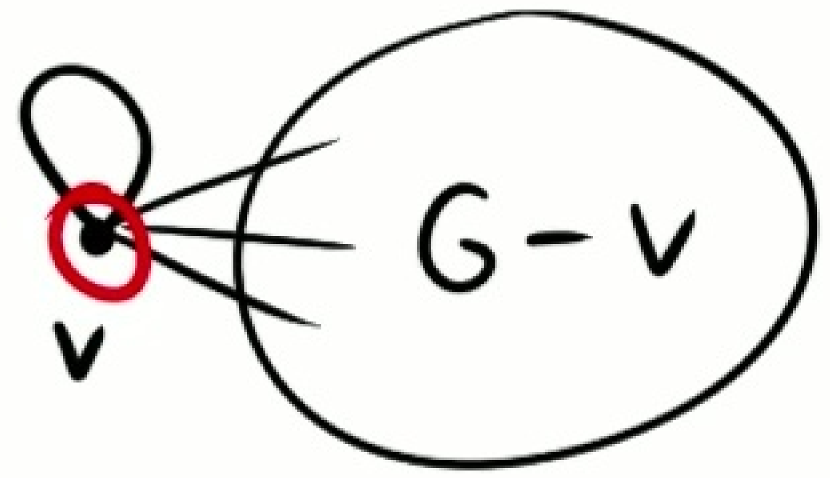


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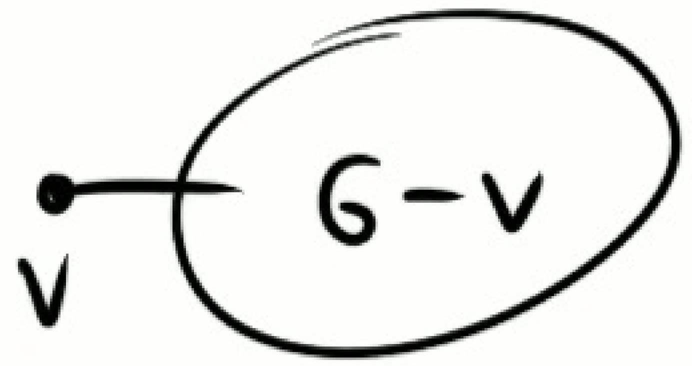


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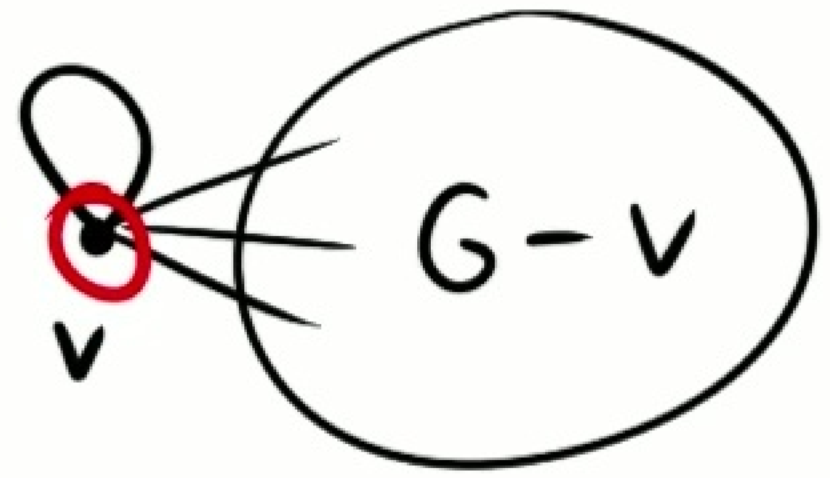


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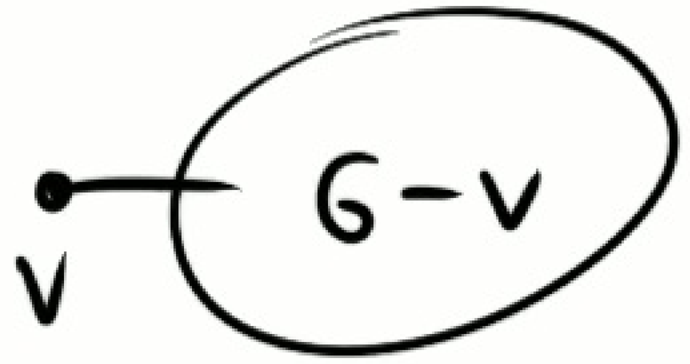


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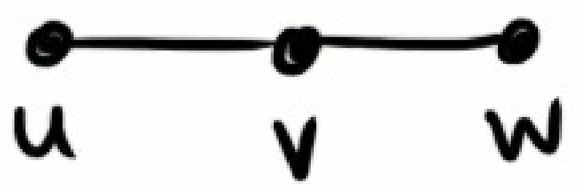

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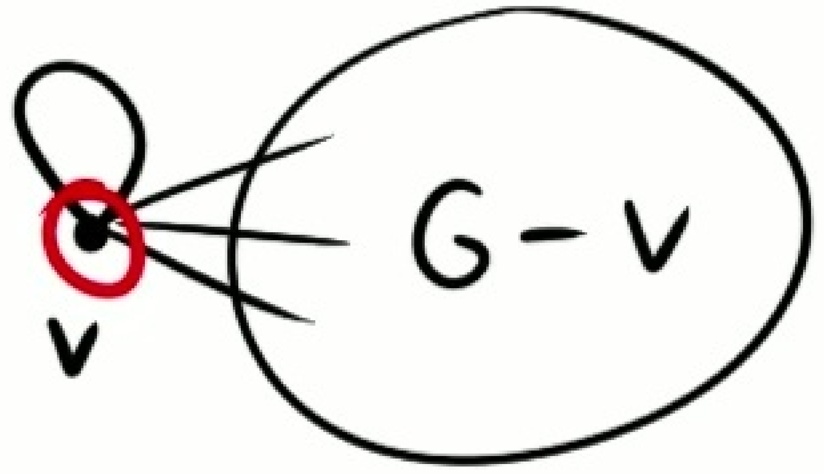


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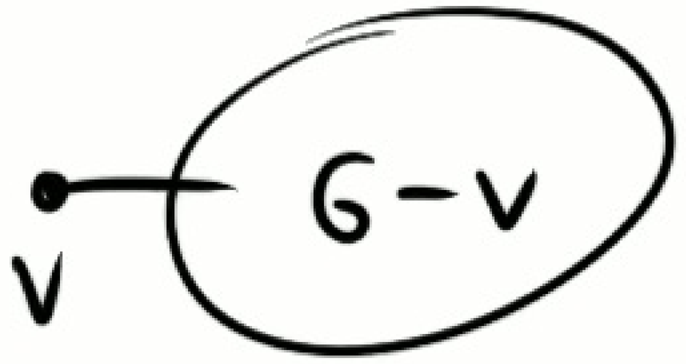


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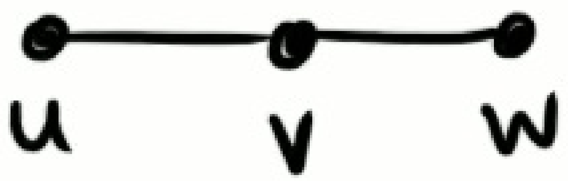


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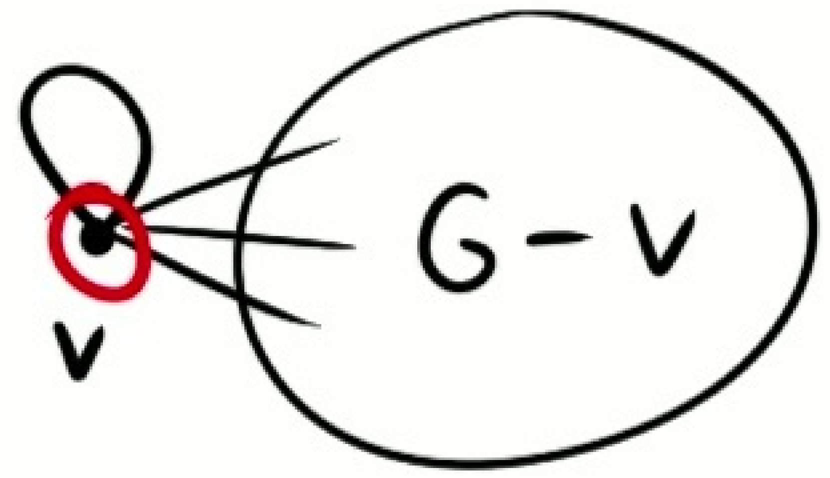


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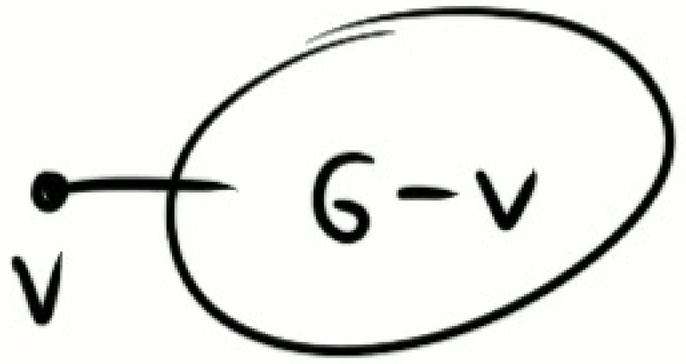


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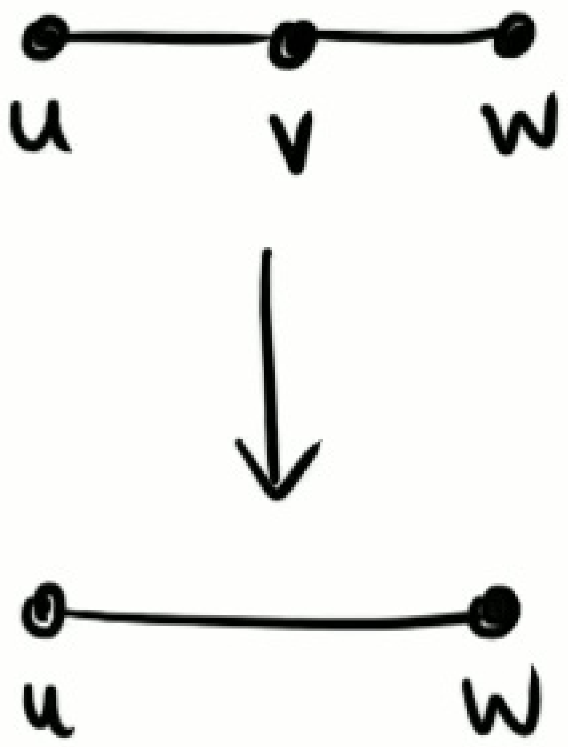


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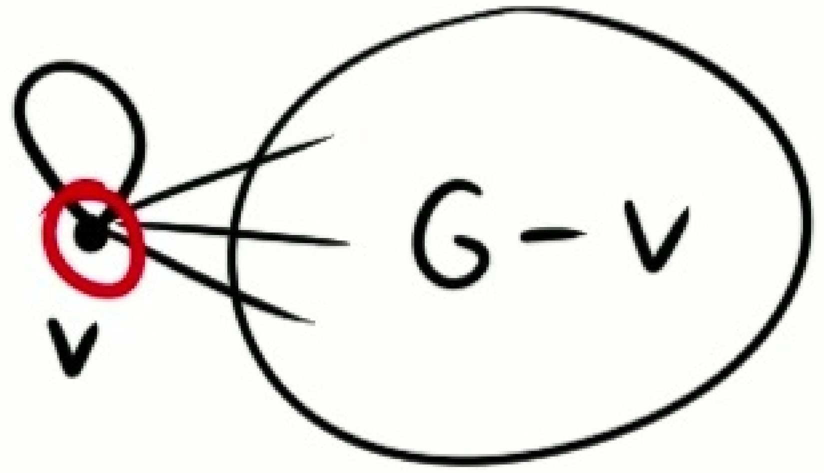


**When (1),(2),(3) no longer apply:**
- no self-loops
- minimum degree 3

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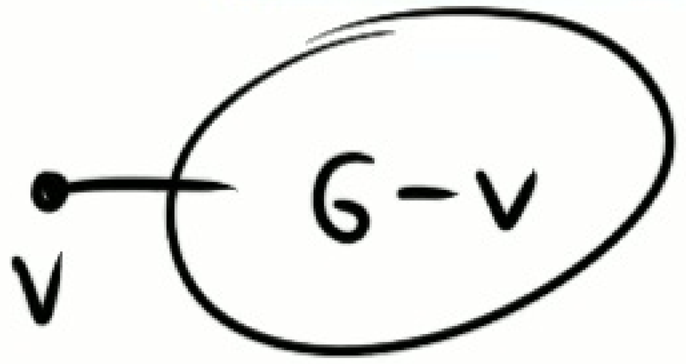


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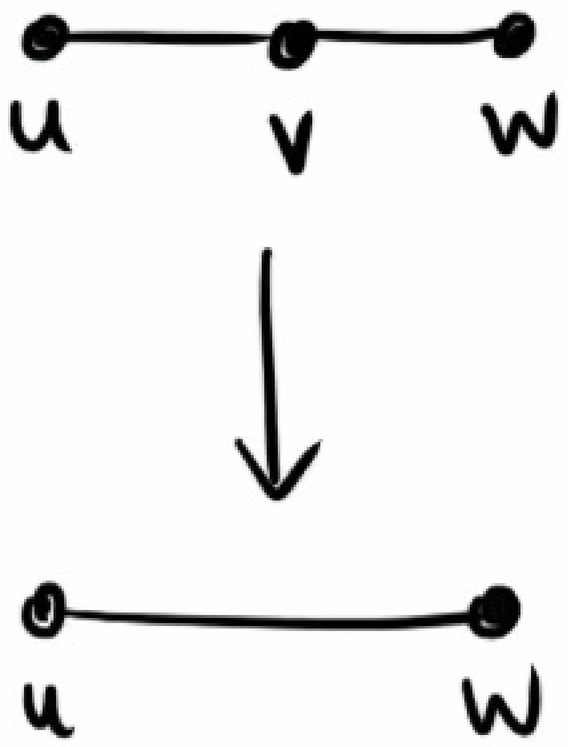


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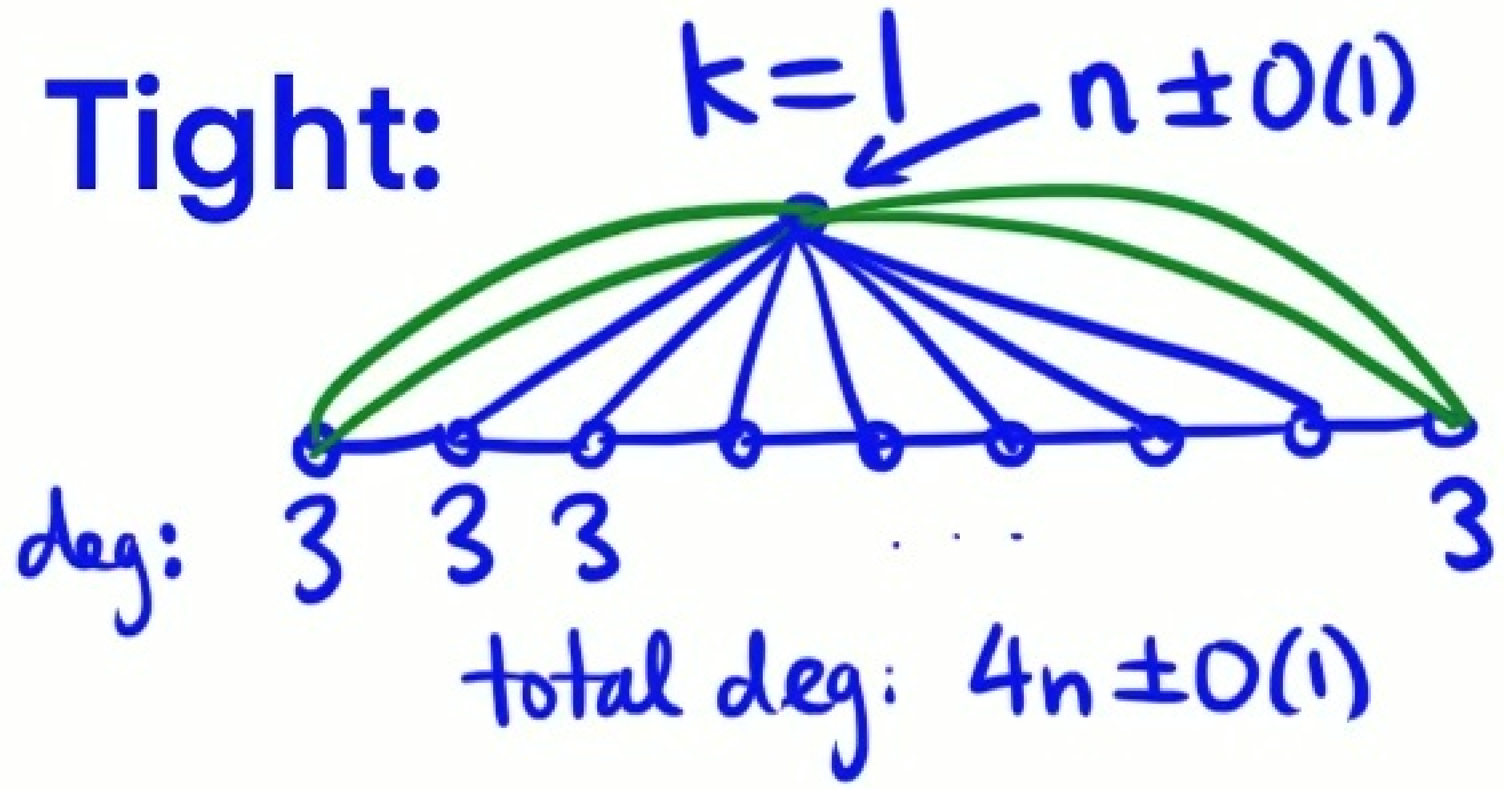

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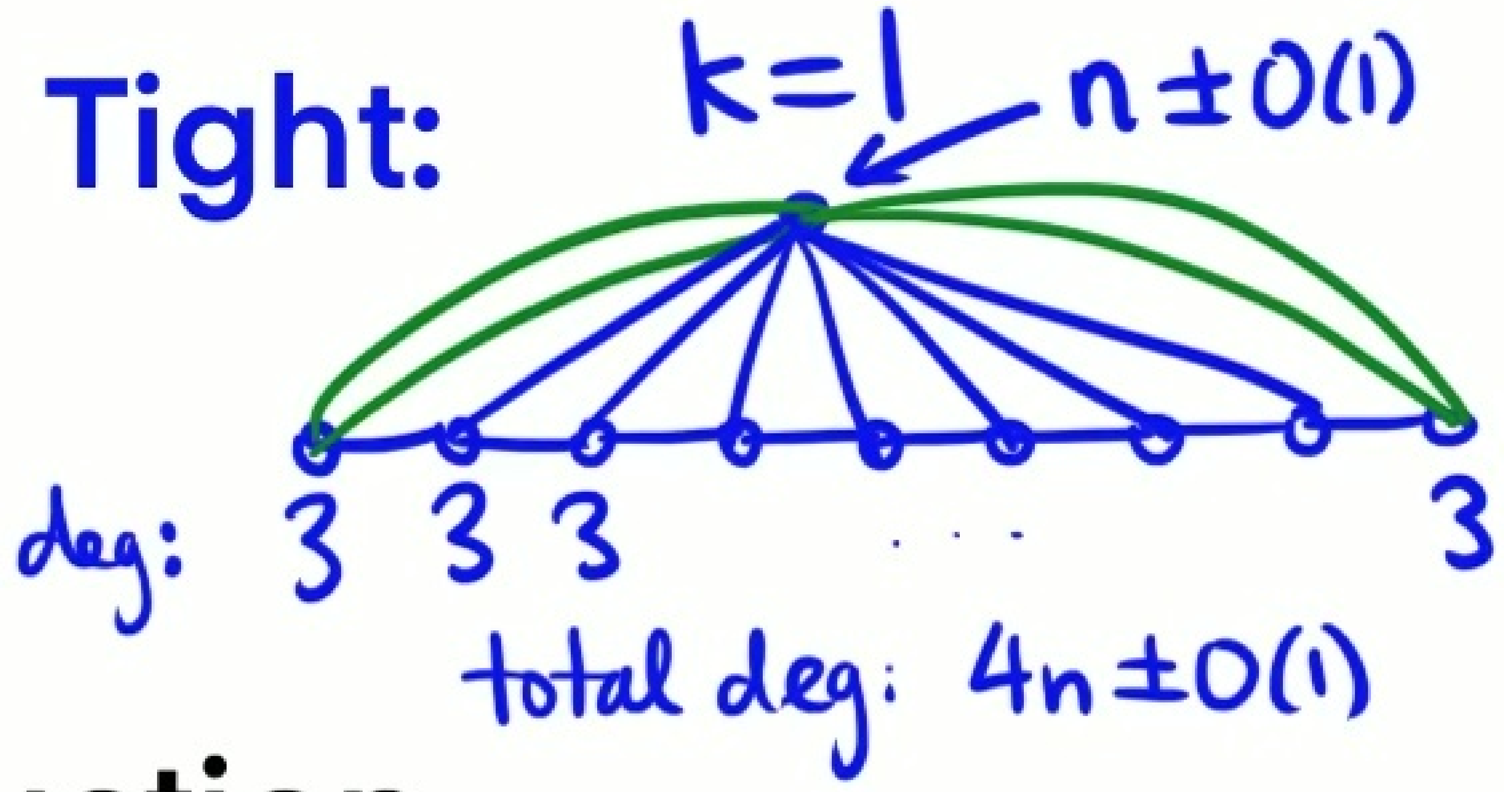


Prob. 1/4 to decrease k by 1 and preserve reduction

=> prob. 1/4^k to go all the way. Repeat 4^k times: O*(4^k) algo.

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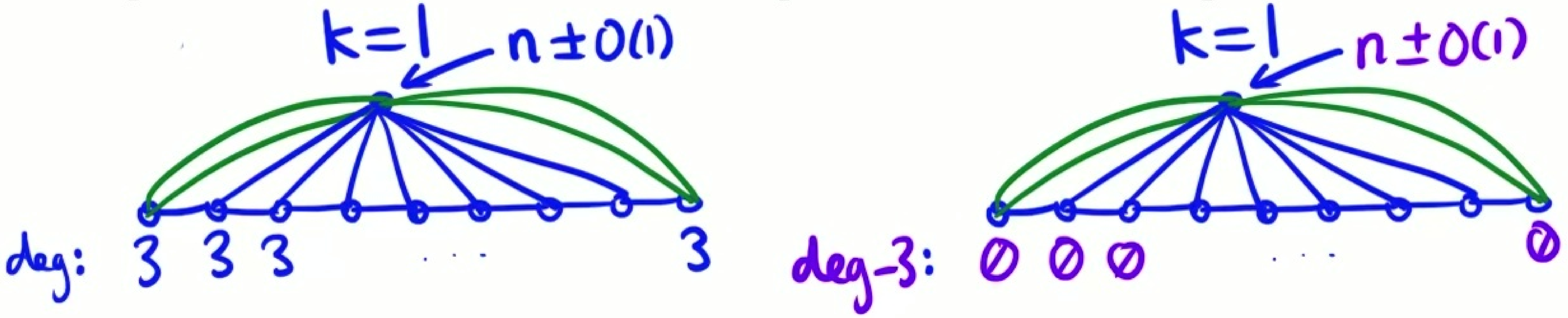


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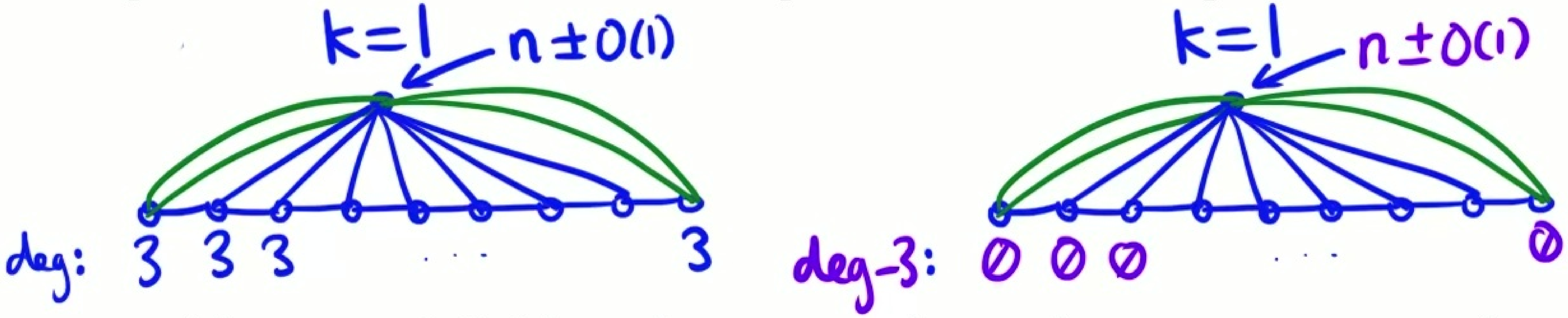


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Lemma: if $m \geq 100k$, then reduction succeeds w.p. 1/2.99

Repeat k times => 2.99^k algo

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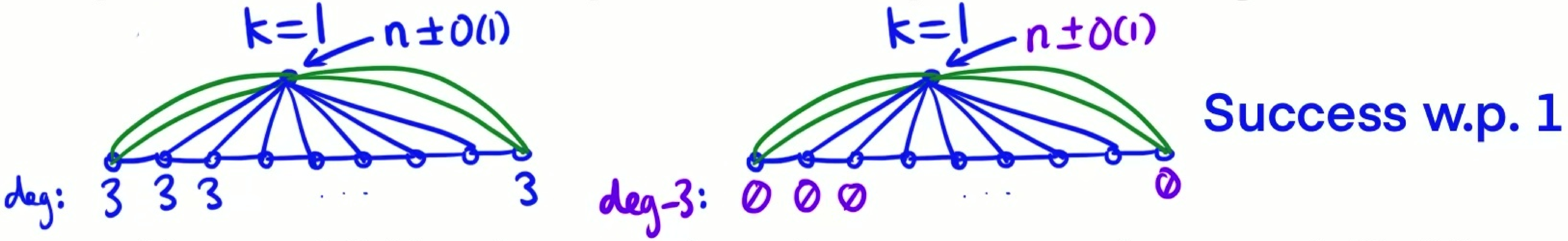


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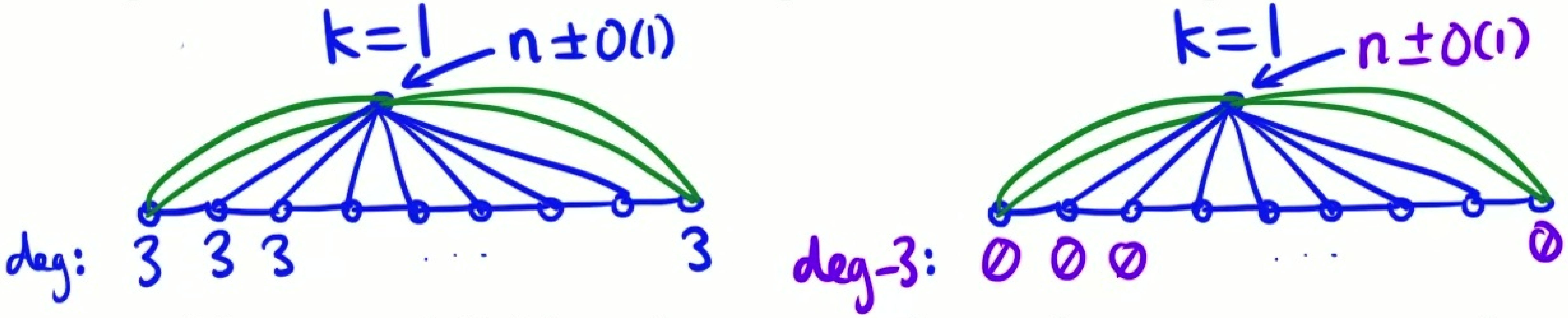


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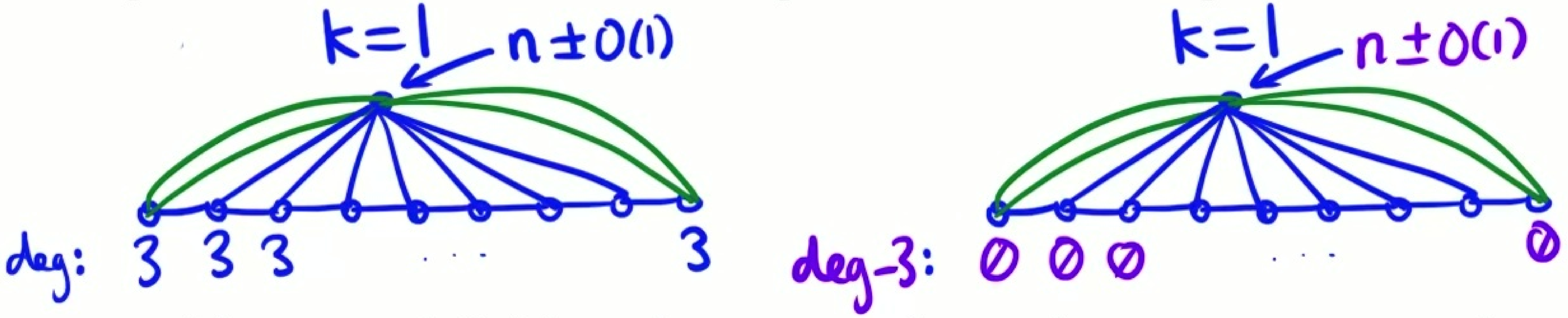


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G has treewidth (1-Ω(1))k ⇒ time $3^{(1-\Omega(1))k} = (3-\varepsilon)^k$.

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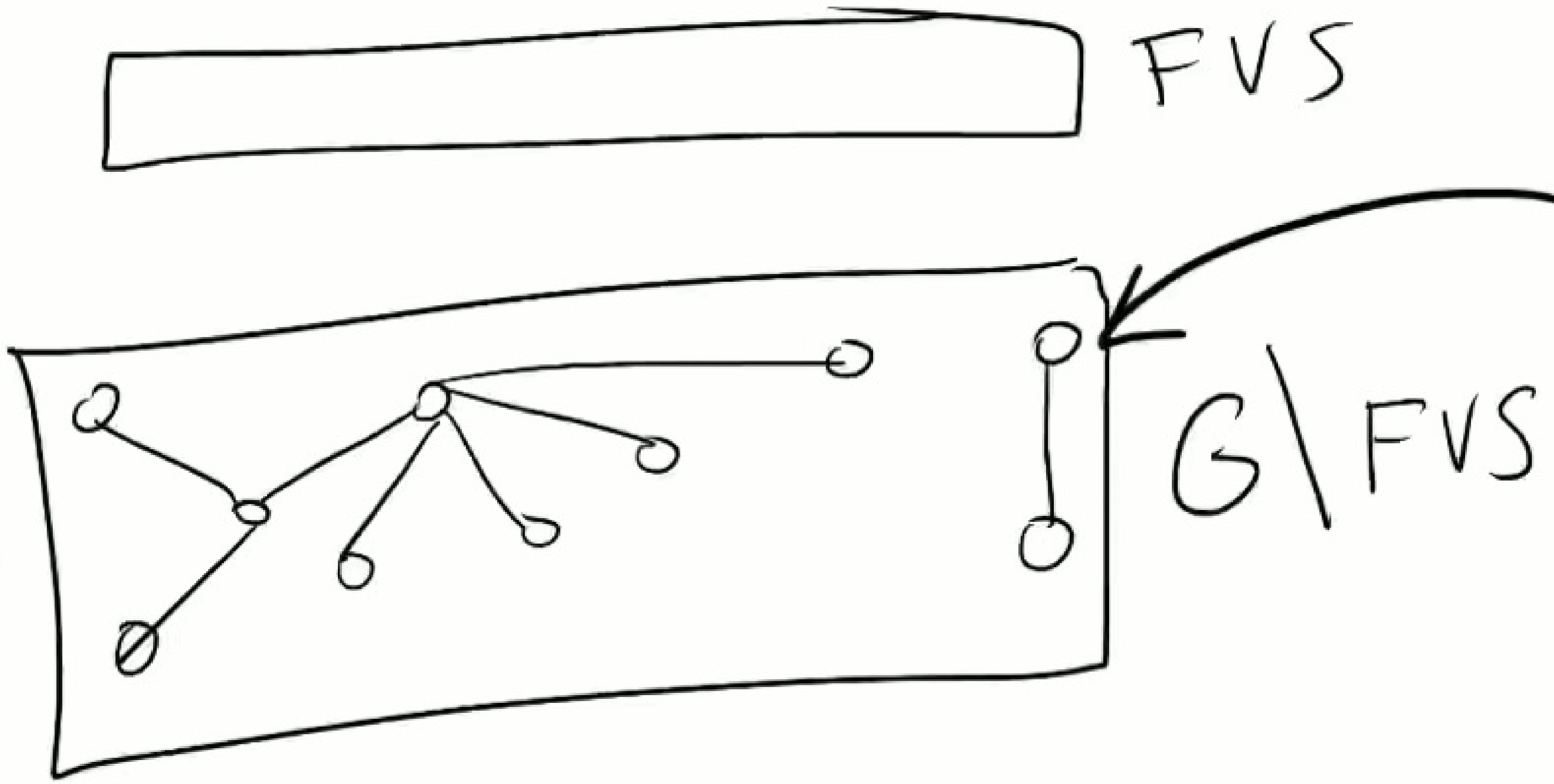


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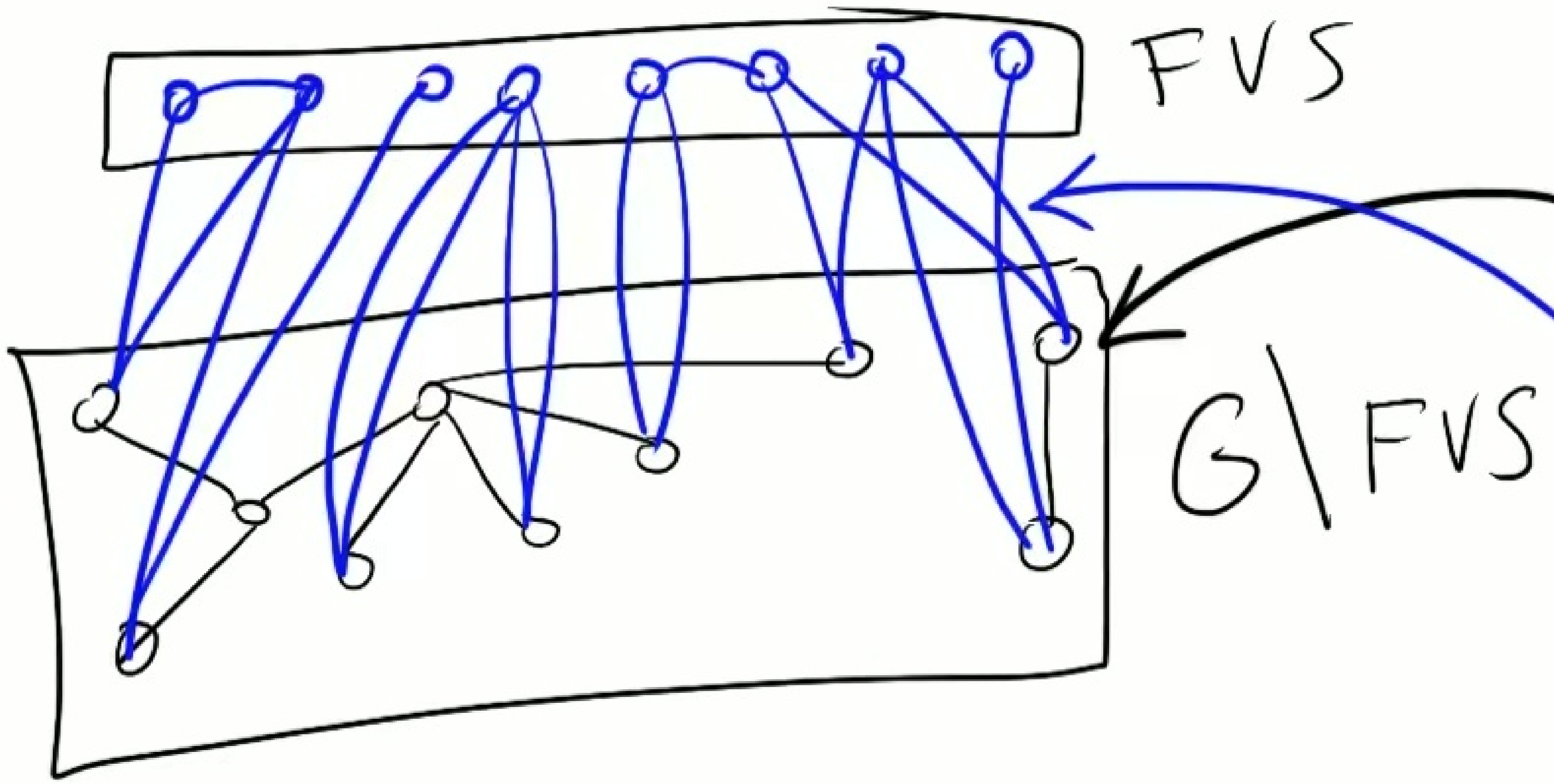


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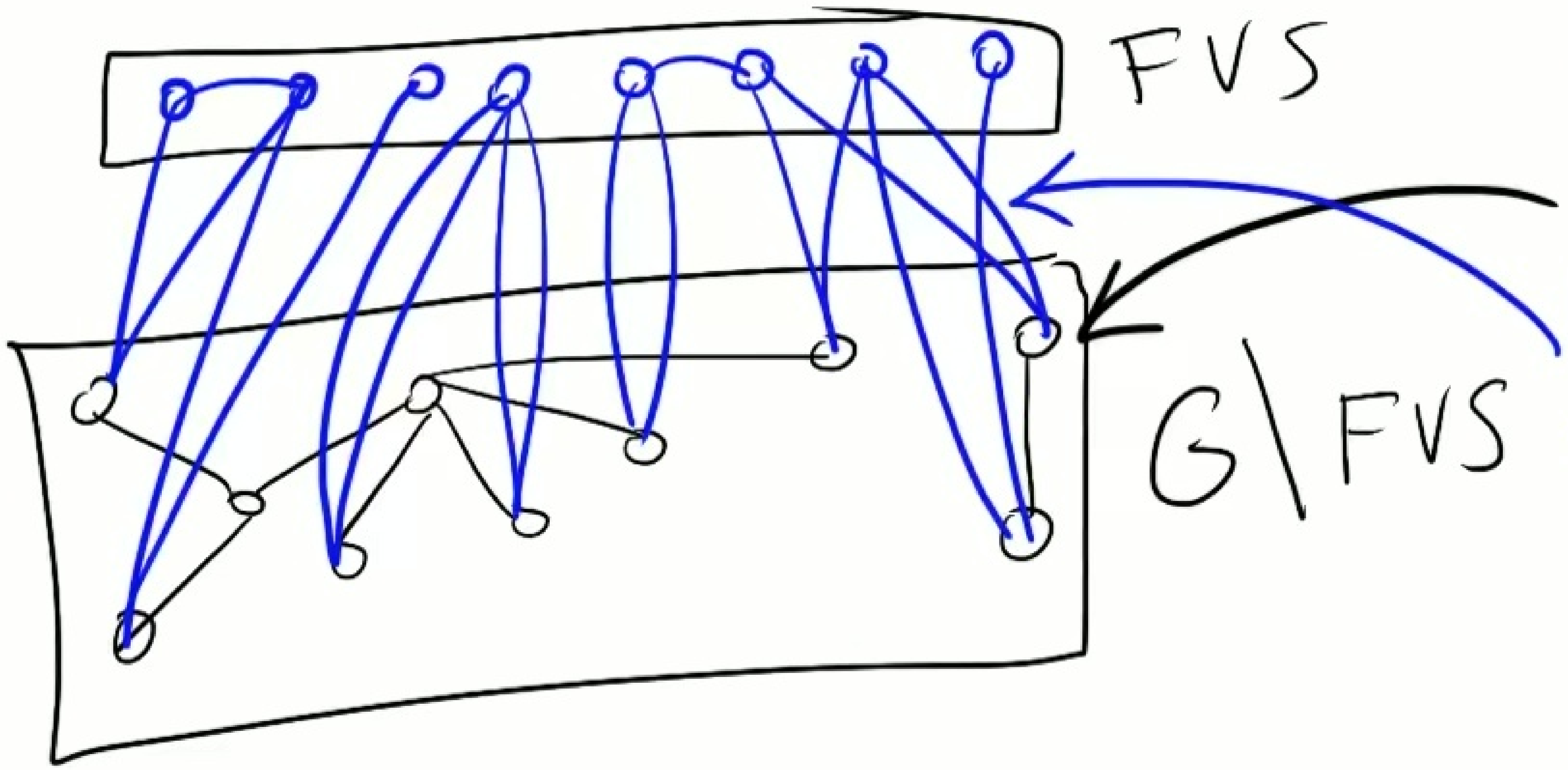


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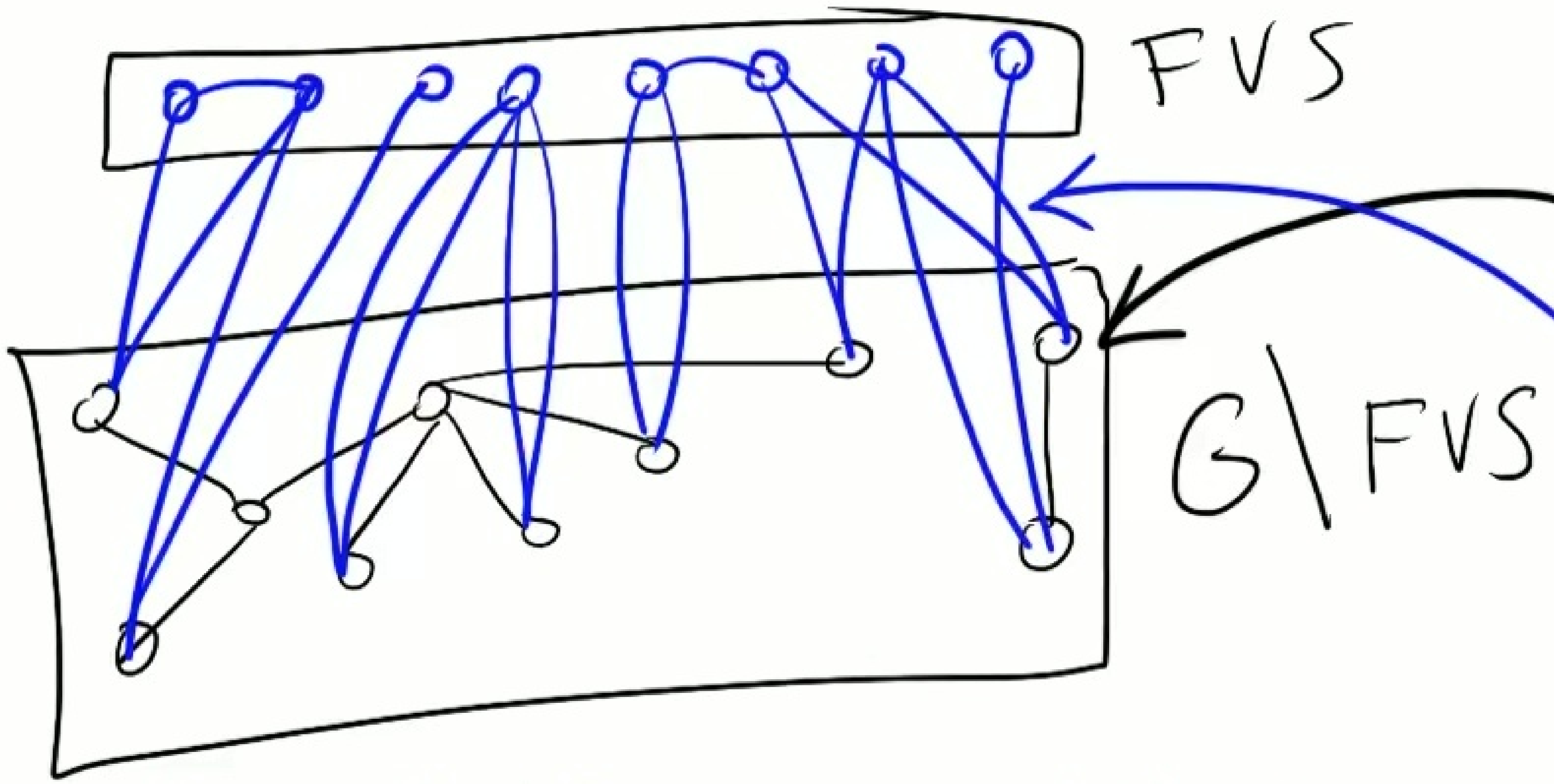


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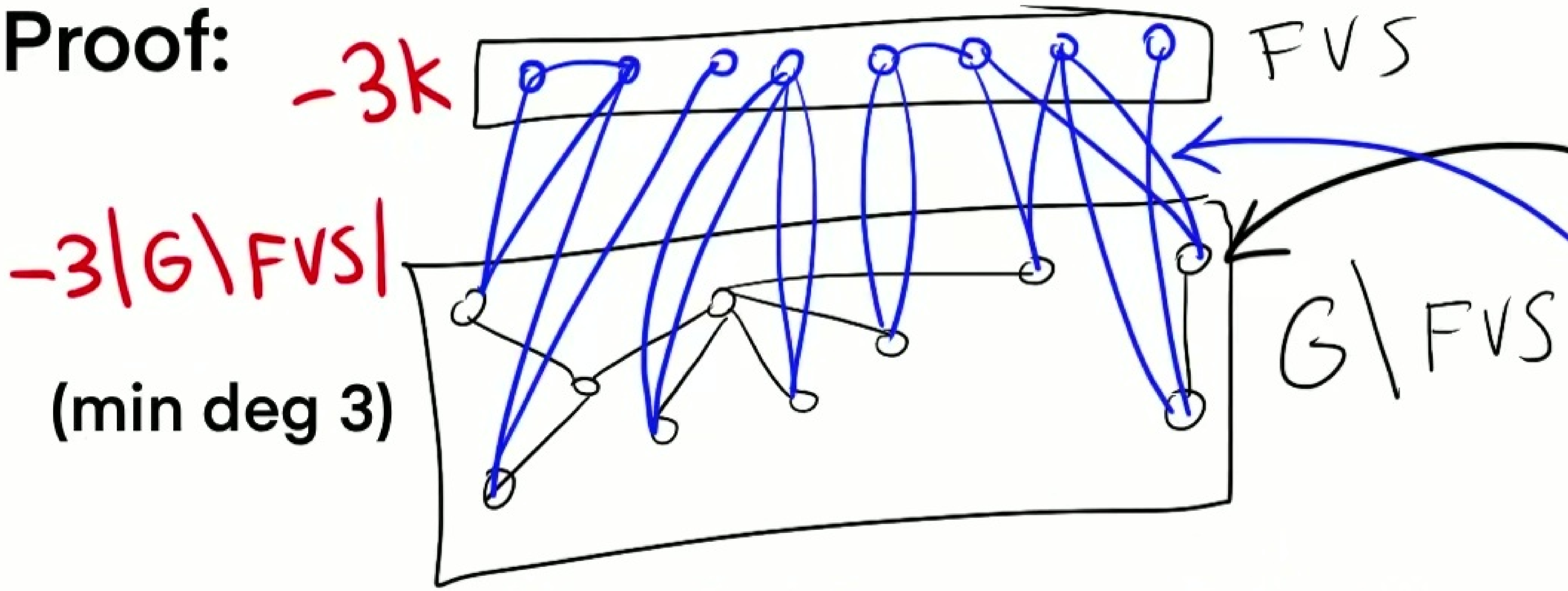


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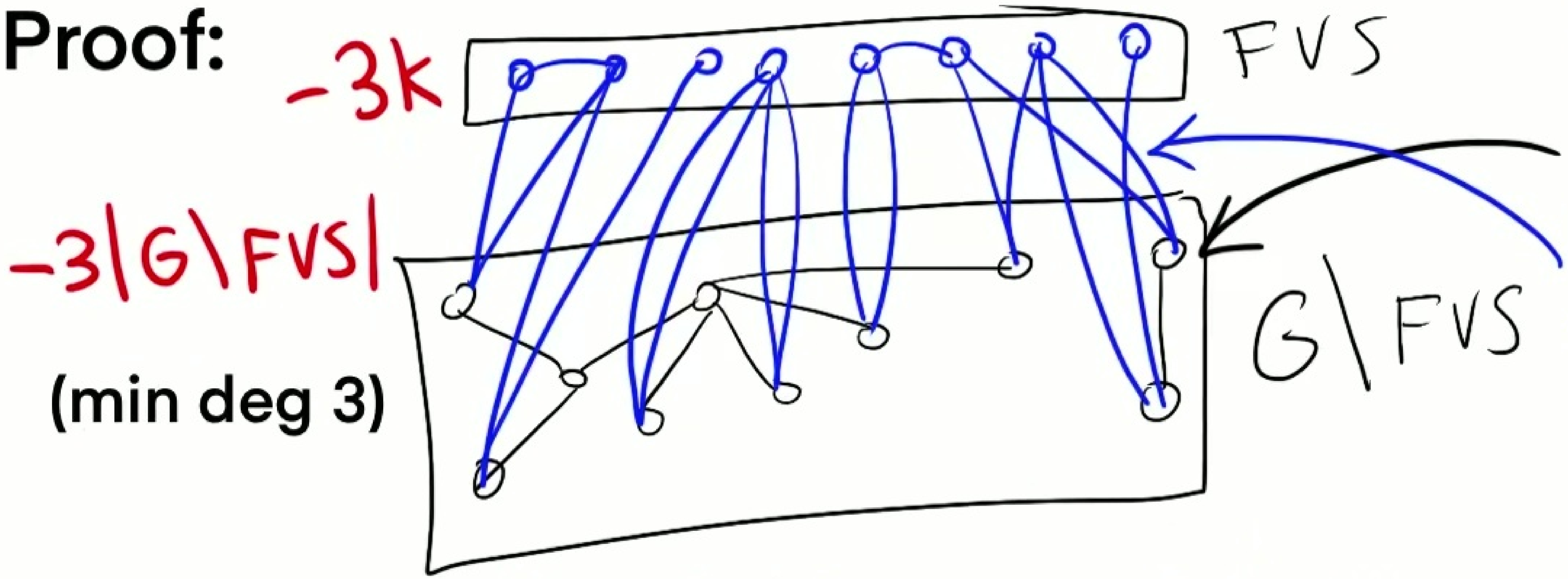


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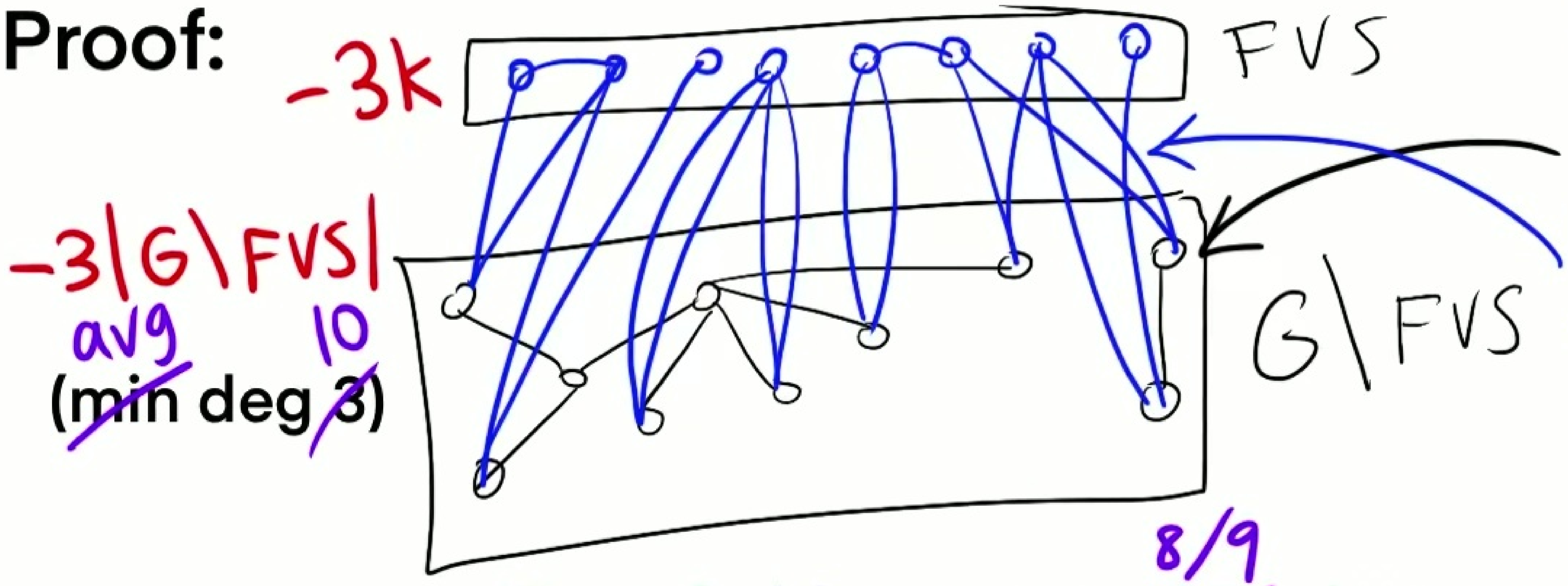


Acyclic : average degree <= 2

Must add >= ~~1~~ 8 edge per vertex in G\FVS on average

Acyclic: <= 1 edge in G\FVS per vertex in G\FVS on average

Sample blue w.p. >= ~~1/2~~ 8/9, then sample v in FVS w.p. 1/2 => 1/4 overall

Sample v with prob. ~ deg(v) ~~- 3~~ ?

- Suppose m >= 10n (high average degree)... Success prob >= 1/2.99

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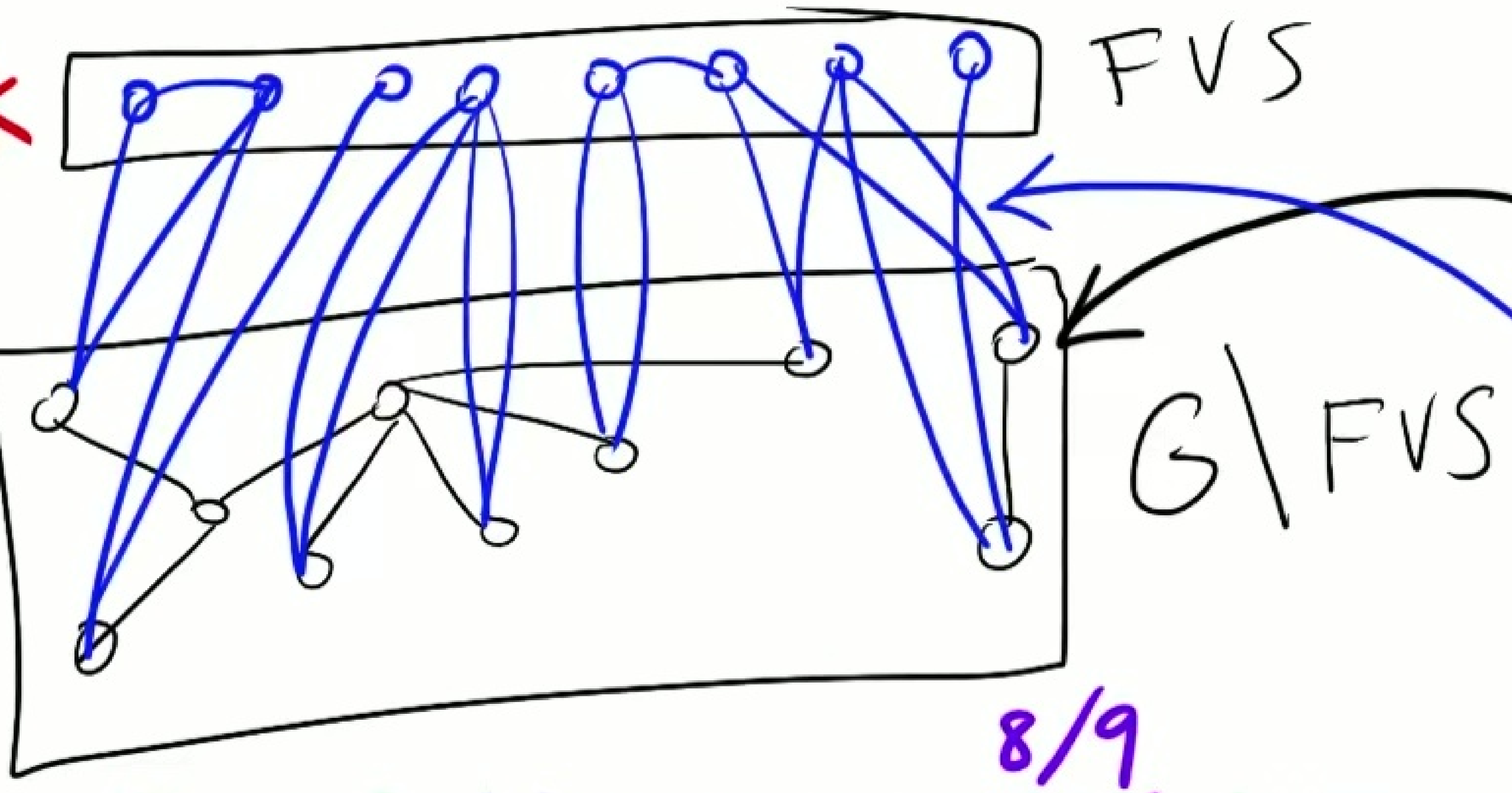
−3k

−3|G\FVS|

(~~min~~ avg deg ~~3~~ 10)

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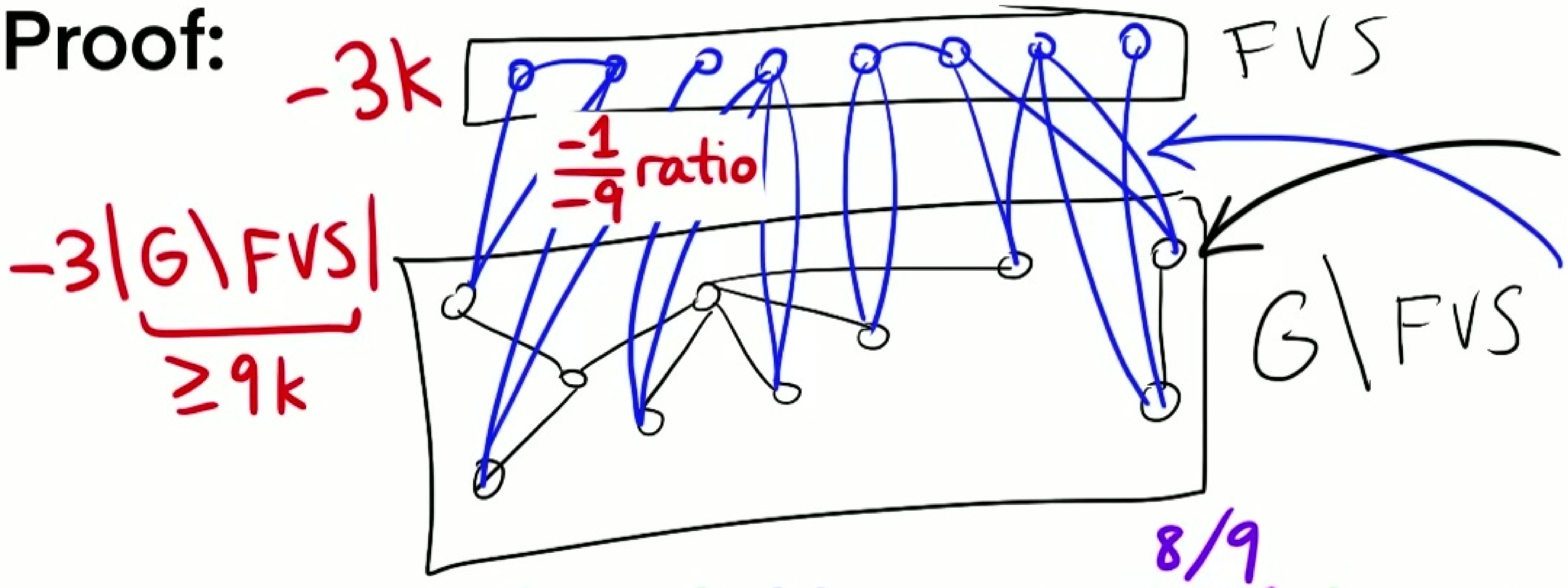


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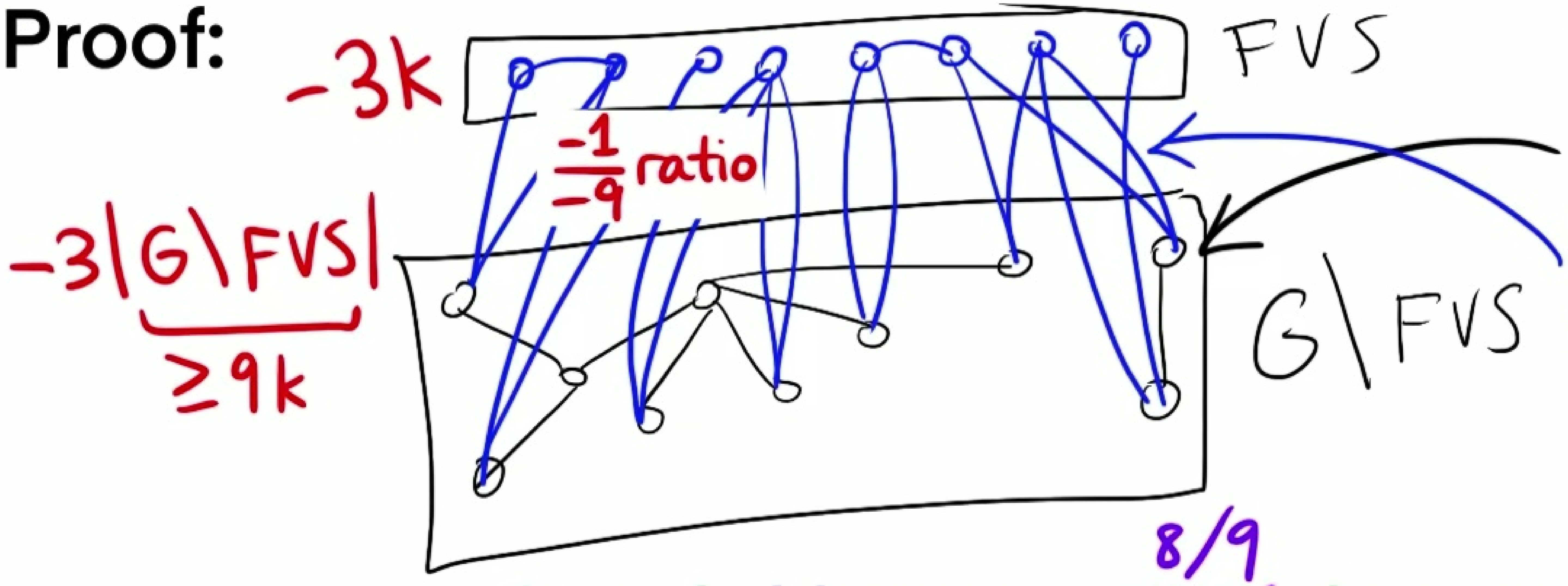


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If m >= 100k, then either m >= 10n or n >= 10k, so success prob >= 1/2.99

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Original problem: given graph G, find FVS size k, or determine none exist.

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Idea: if $S_i$ is FVS of size k to $G[\{v_1,\ldots,v_i\}]$, then

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Solve on $(G[\{v_1,\ldots,v_{i+1}\}], S_i \cup \{v_{i+1}\})$ to get

FVS $S_{i+1}$ of size k on $G[\{v_1,\ldots,v_{i+1}\}]$. Repeat

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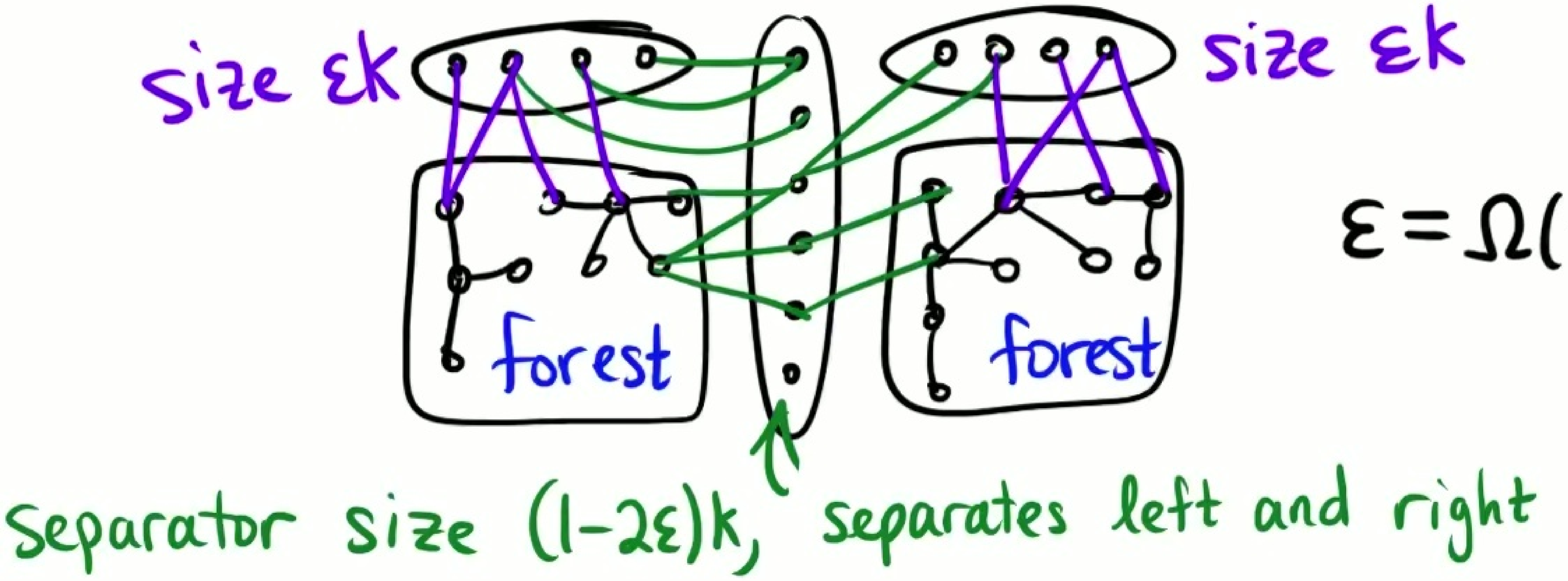


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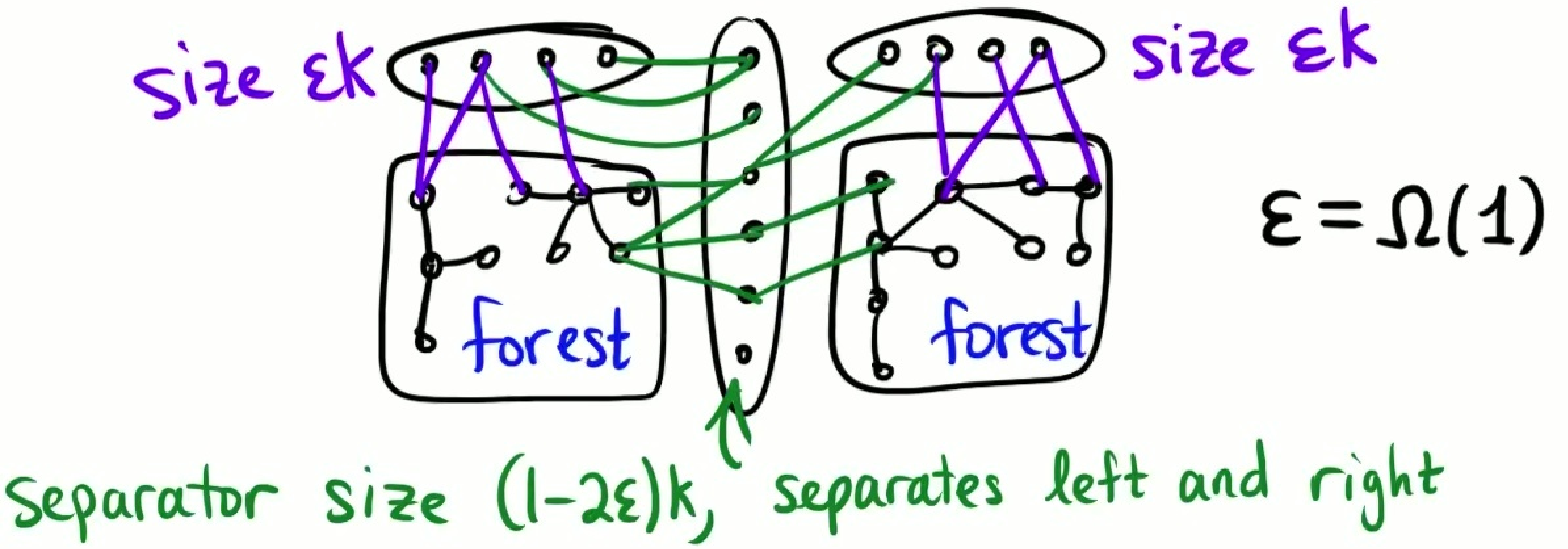


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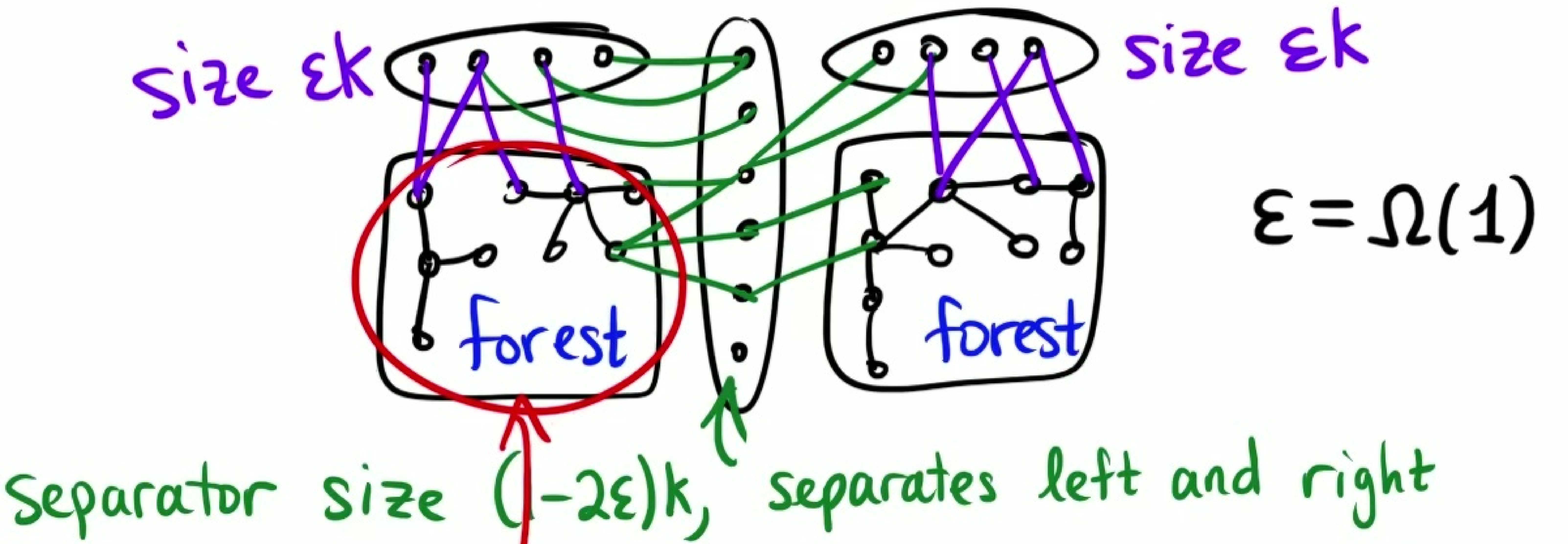


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tw( ) = 1 (forest)

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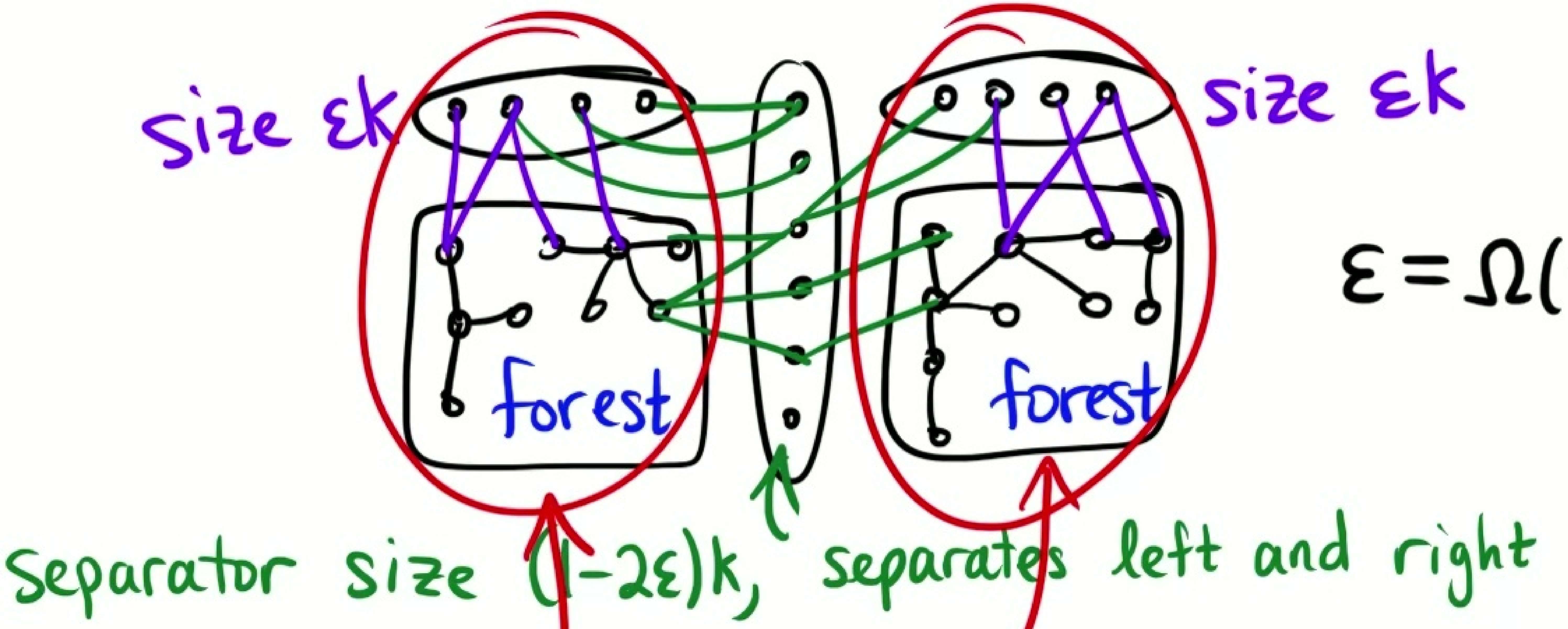


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$tw(\quad) \le 1 + \varepsilon k$ (forest + εk vertices)

$tw(G) \le 1 + \varepsilon k + (1-2\varepsilon)k$ (add last $(1-2\varepsilon)k$ vertices)

$= (1-\Omega(1))k$

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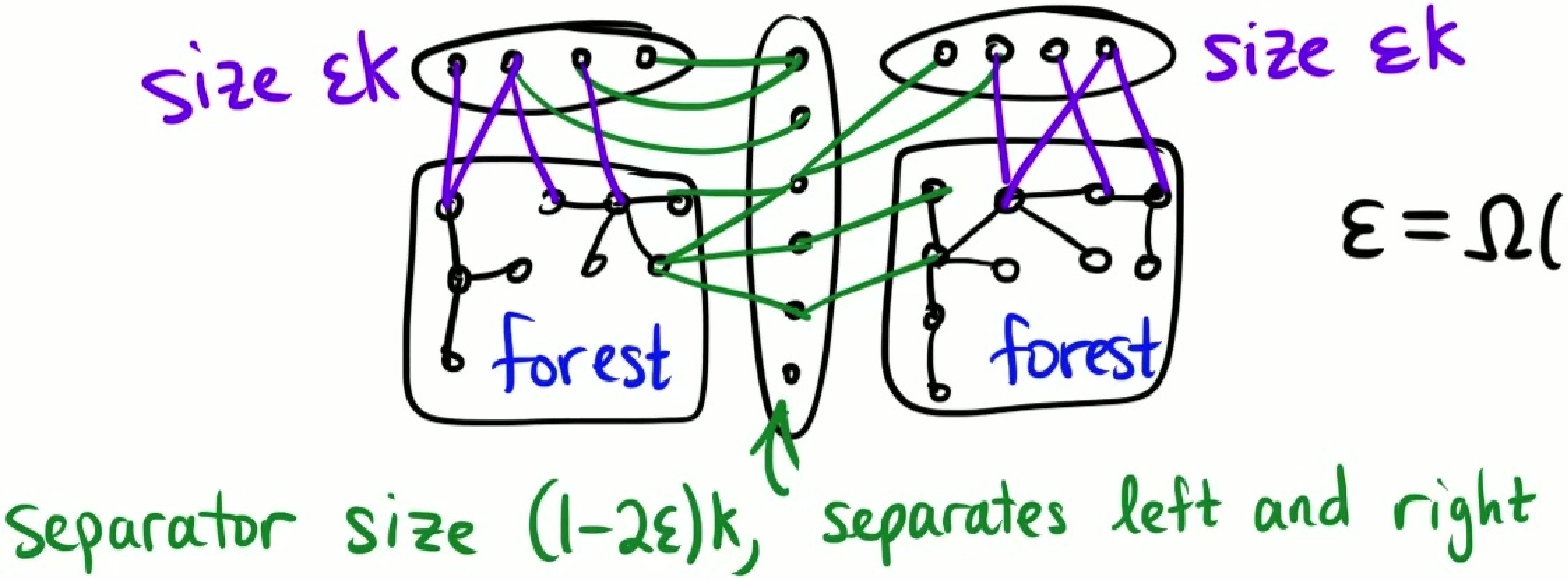


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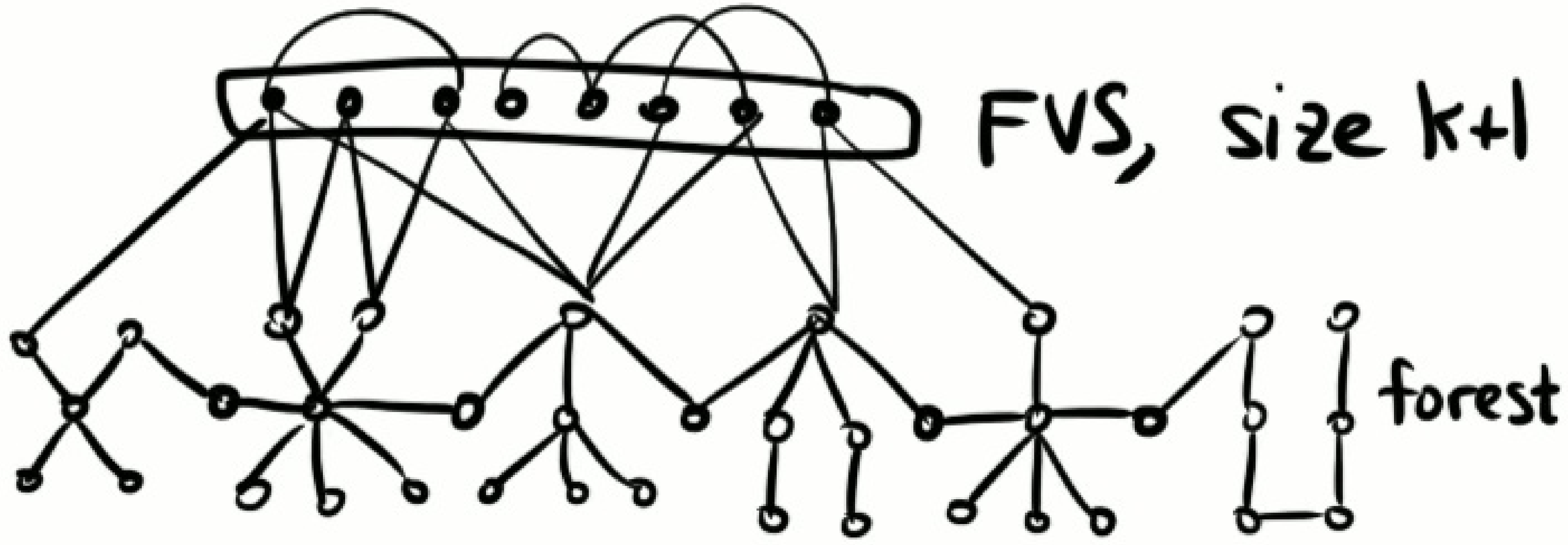

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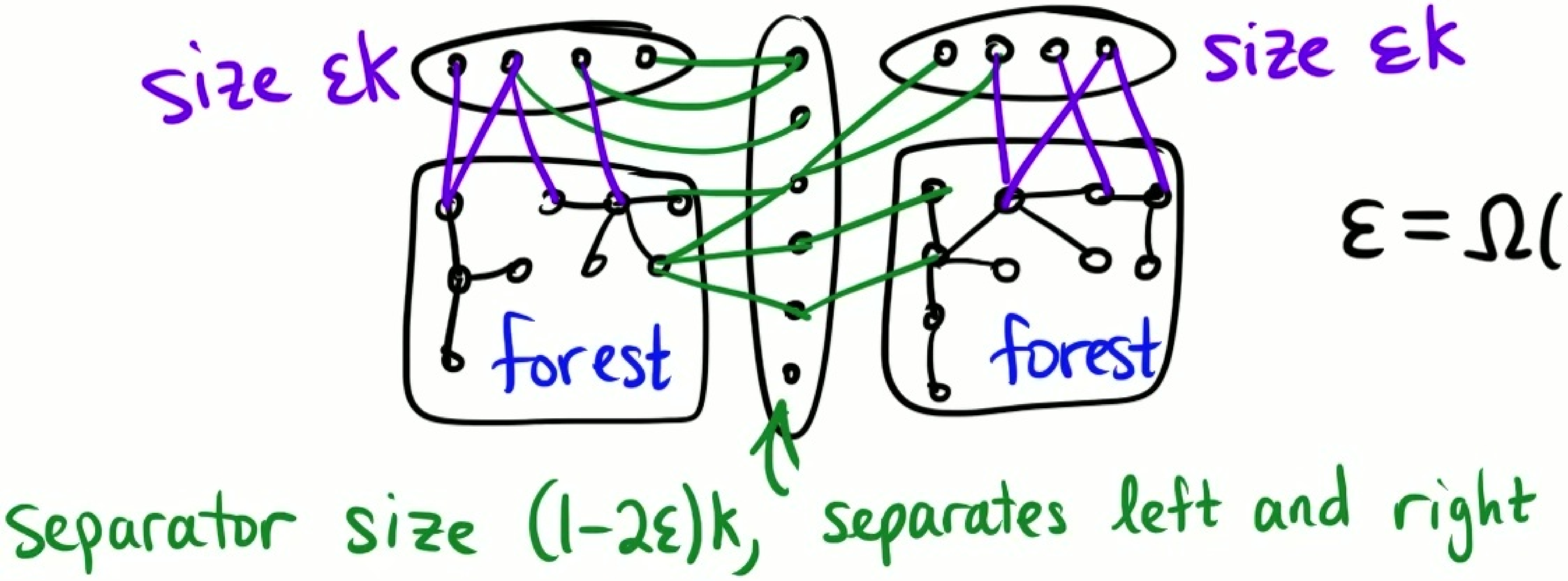


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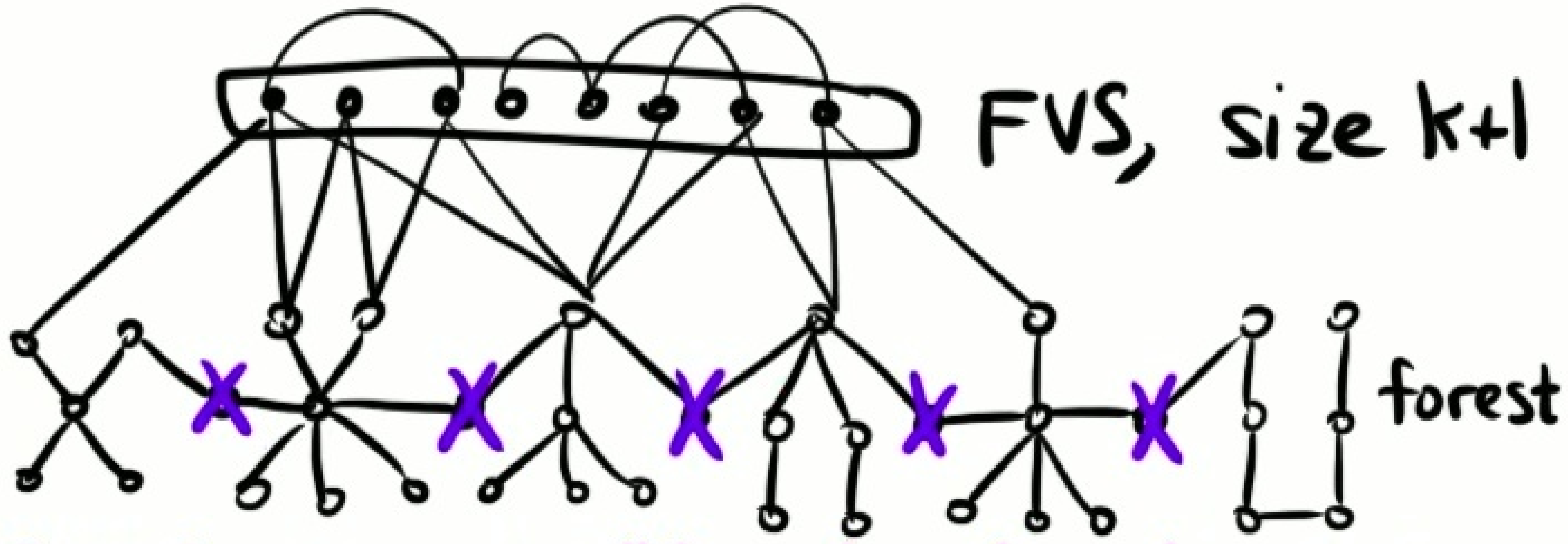


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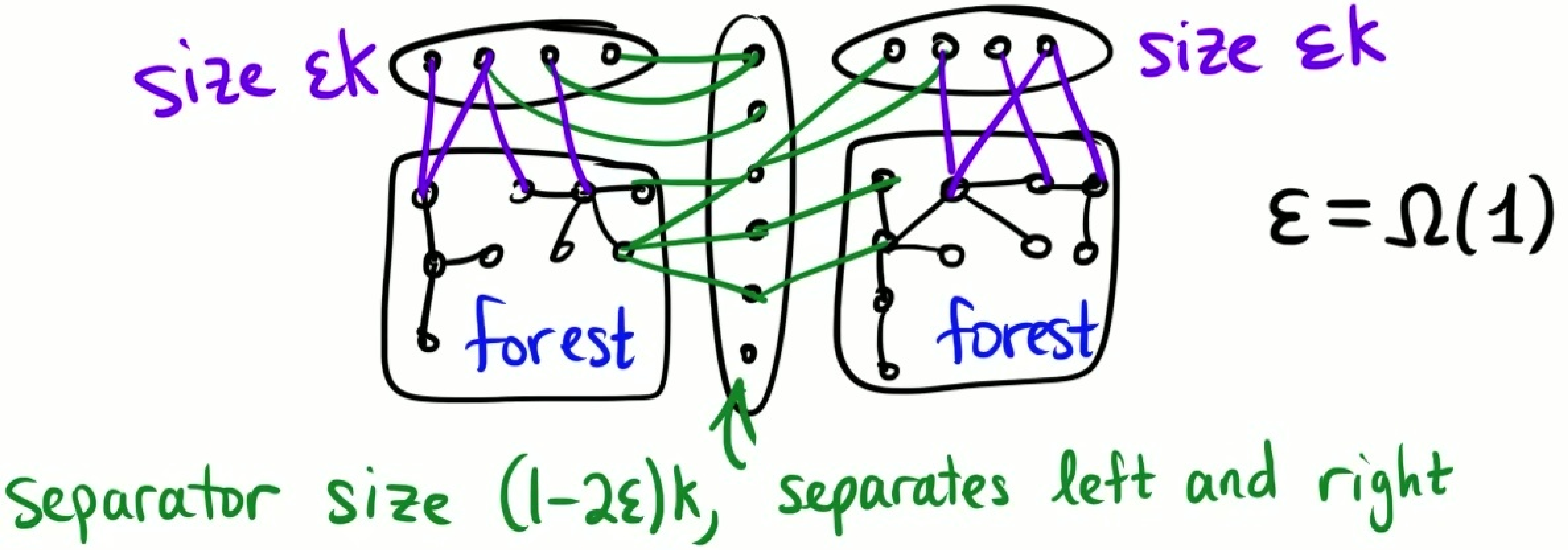


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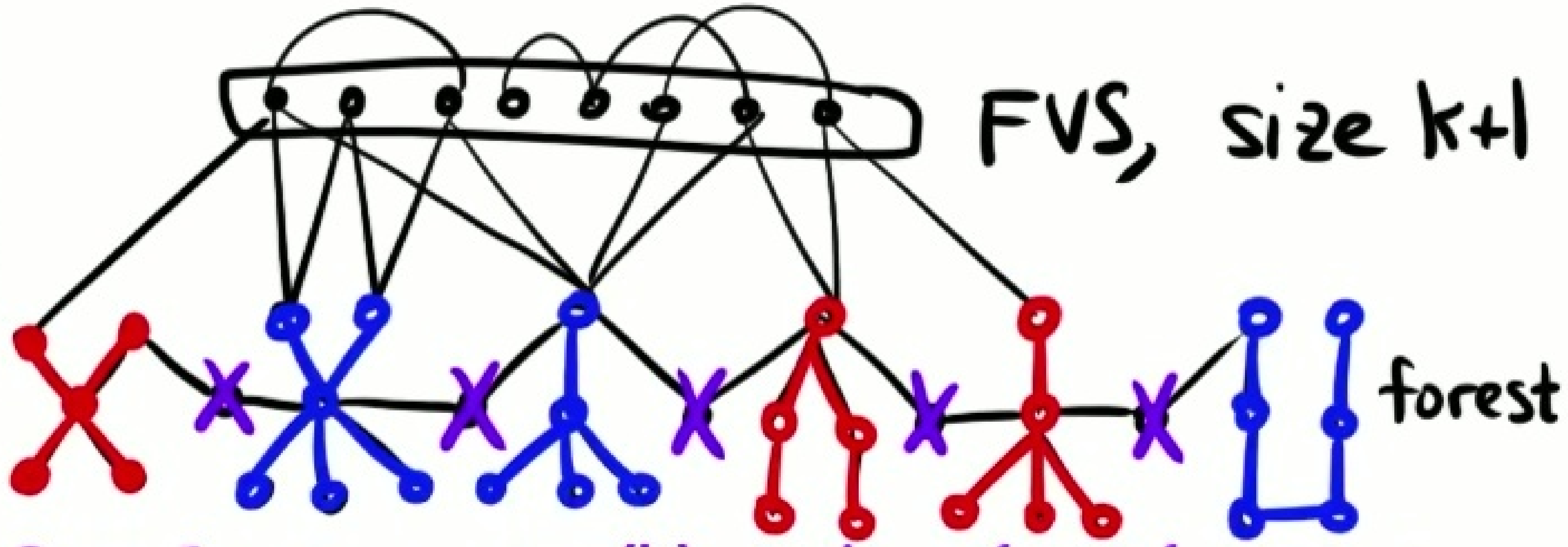


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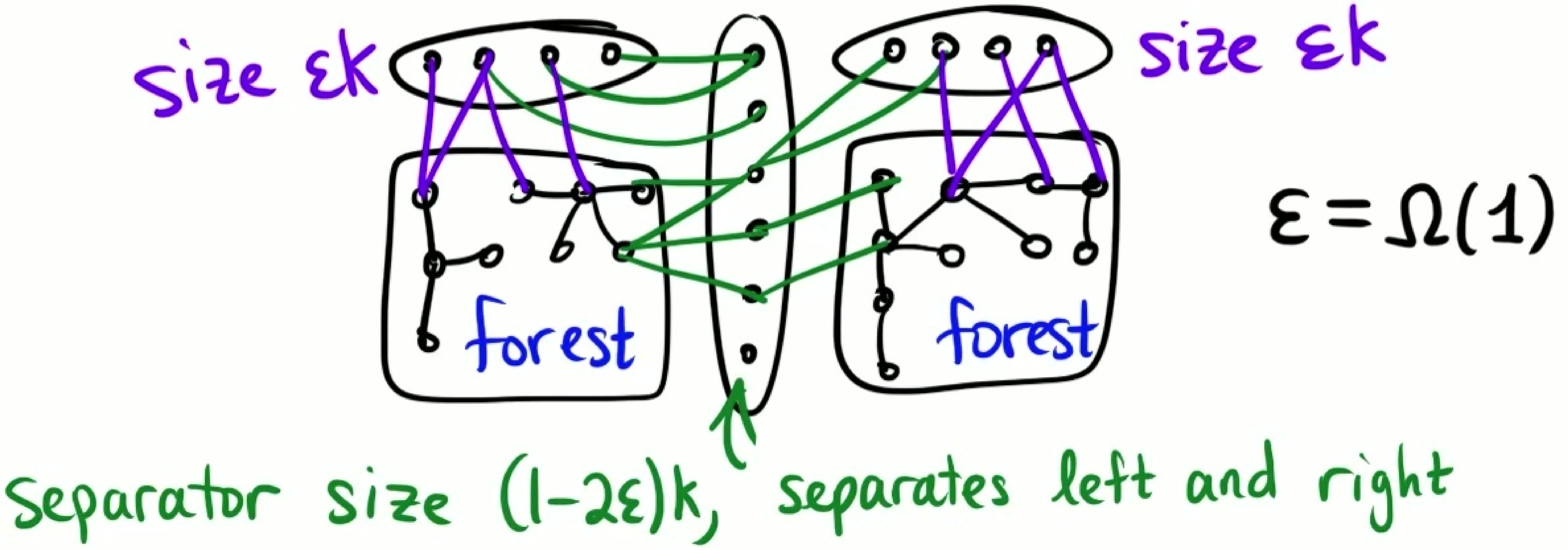


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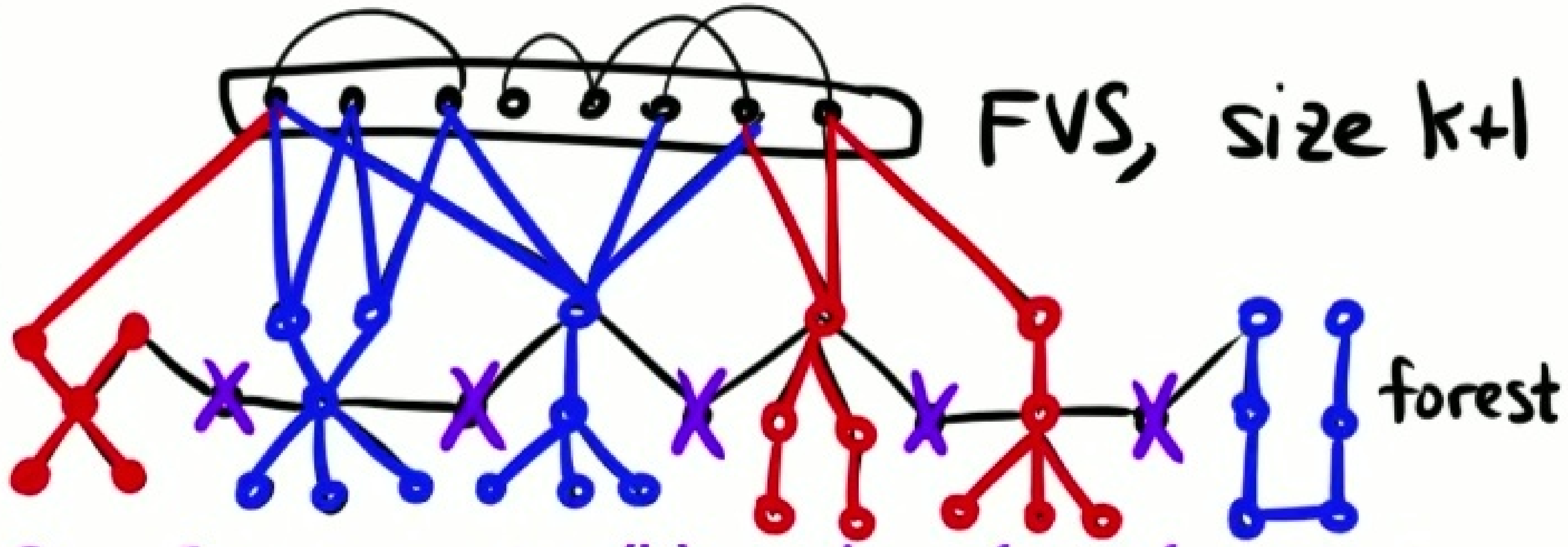


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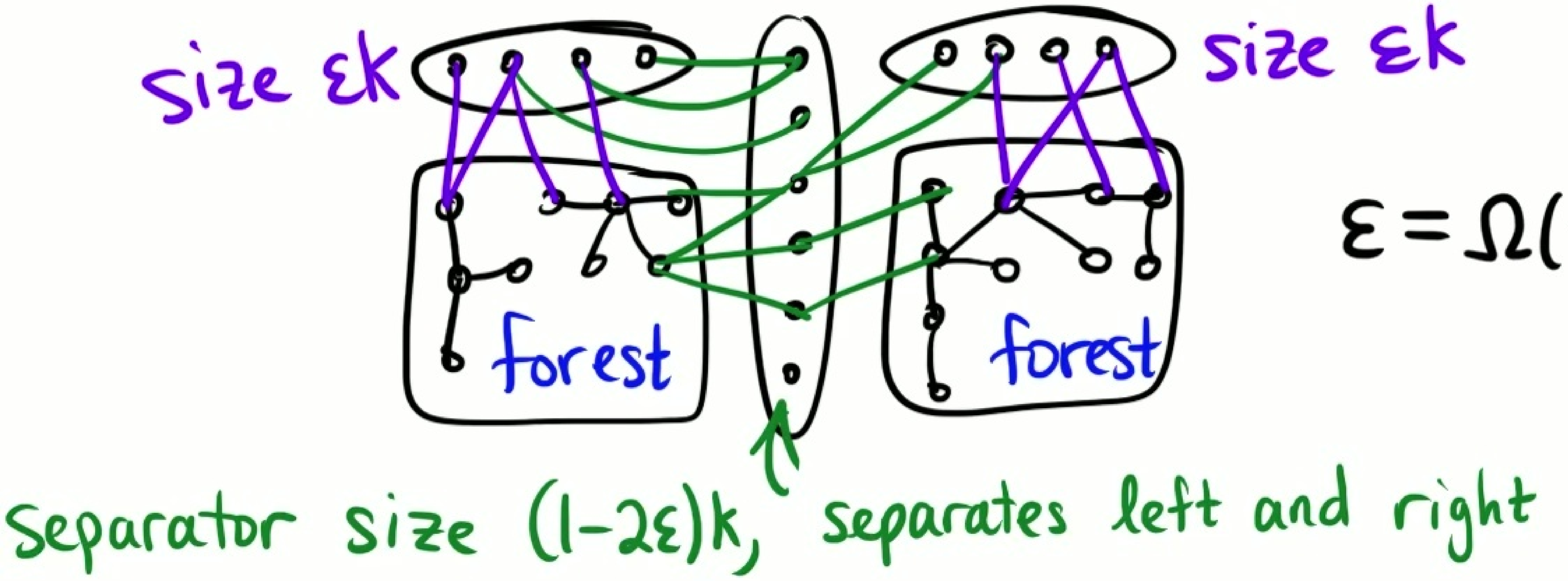


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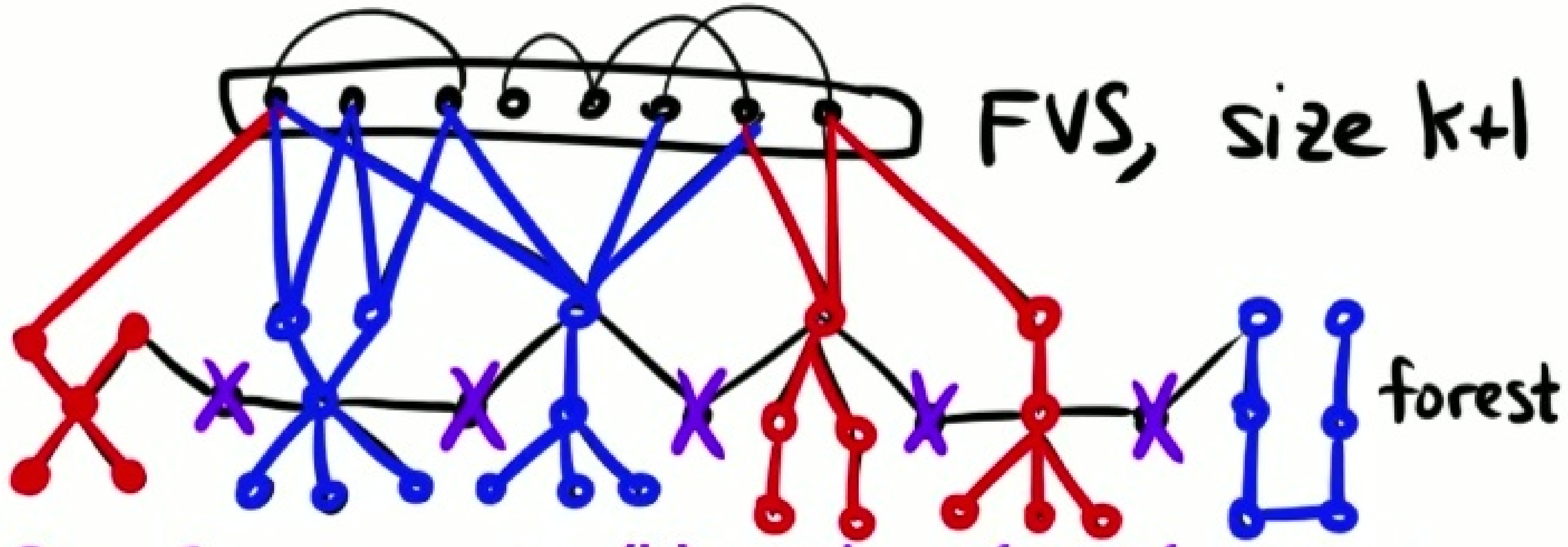


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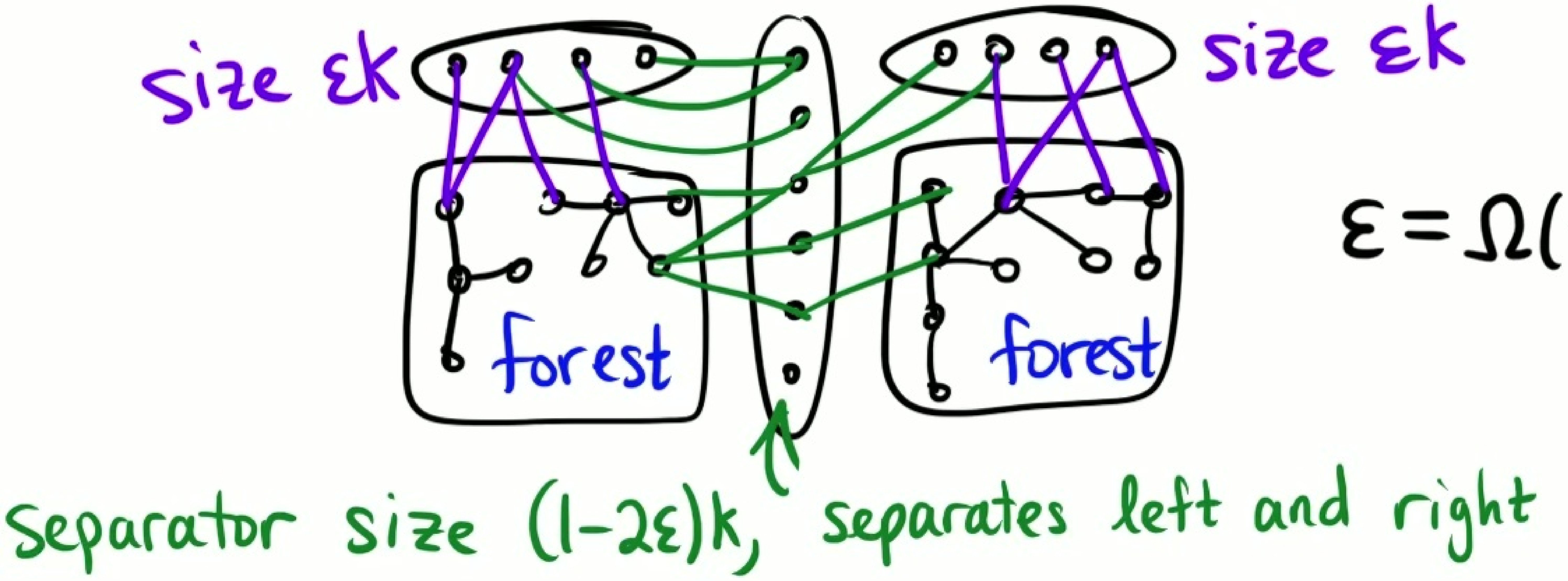


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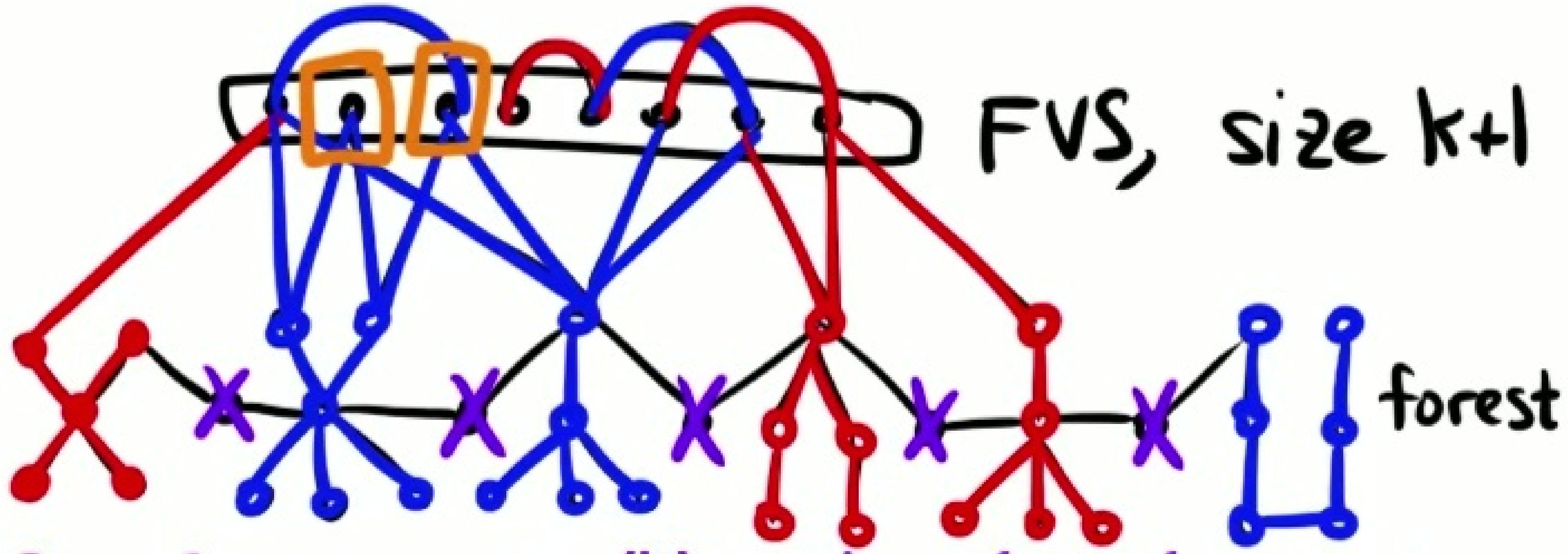


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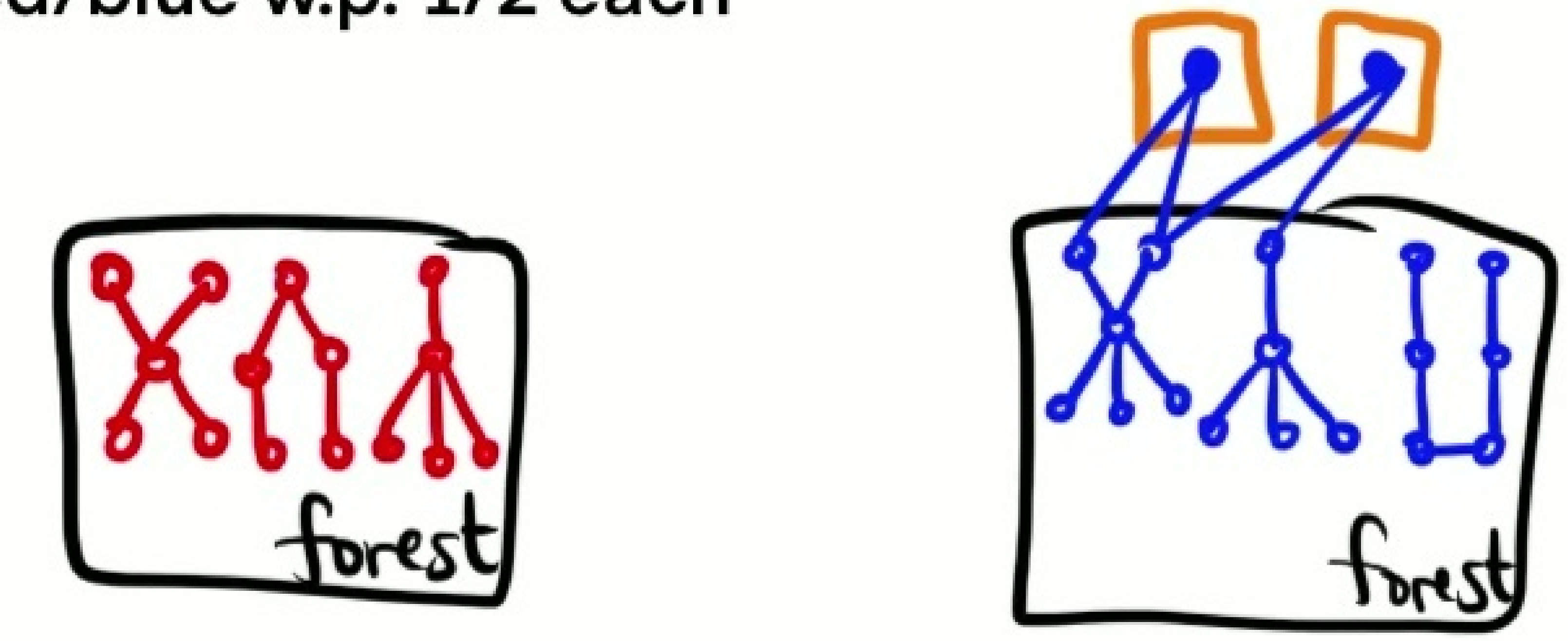

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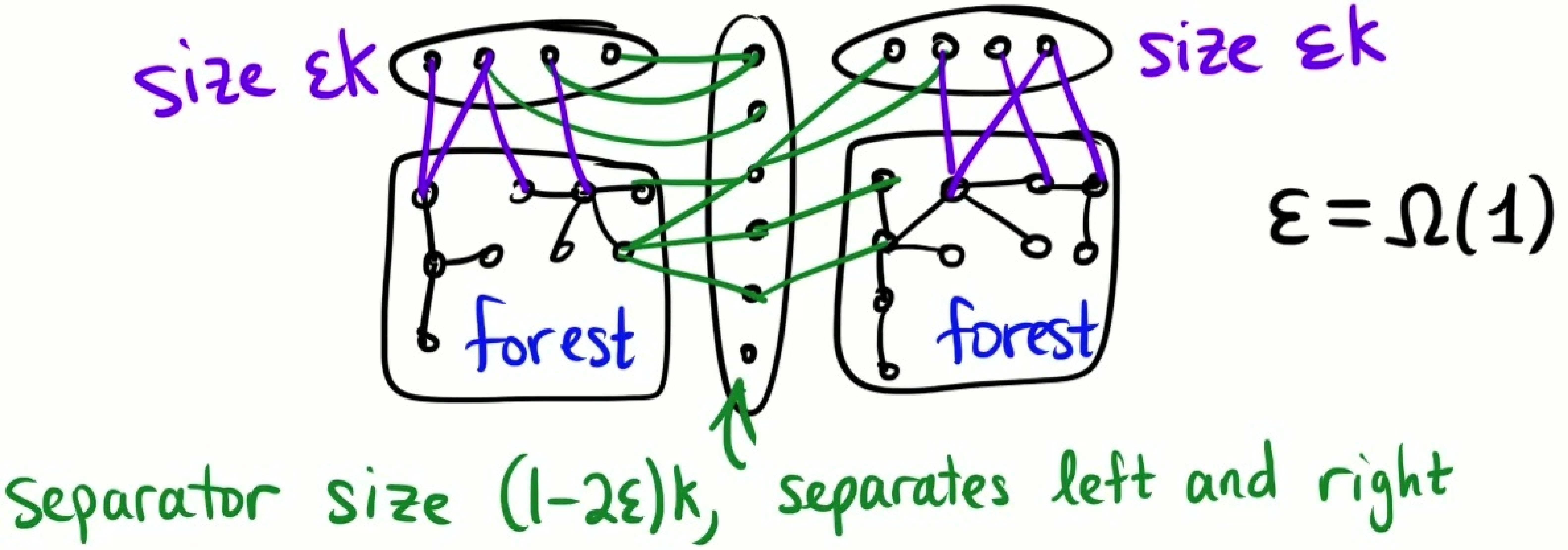


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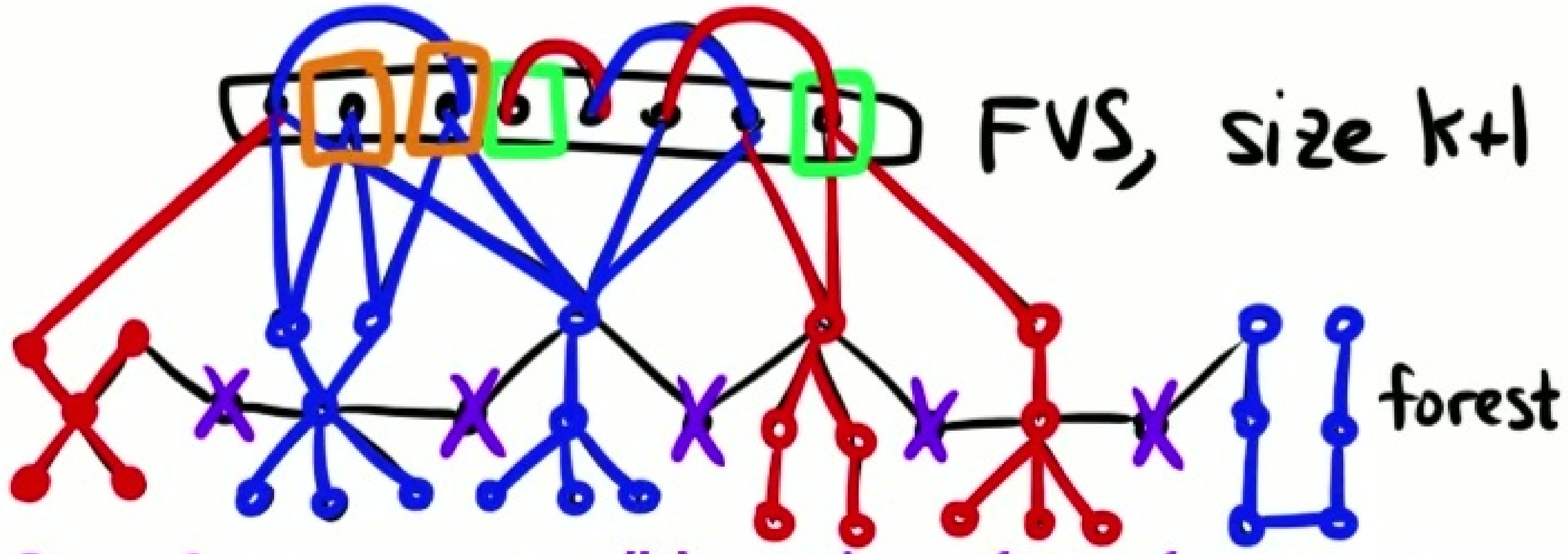


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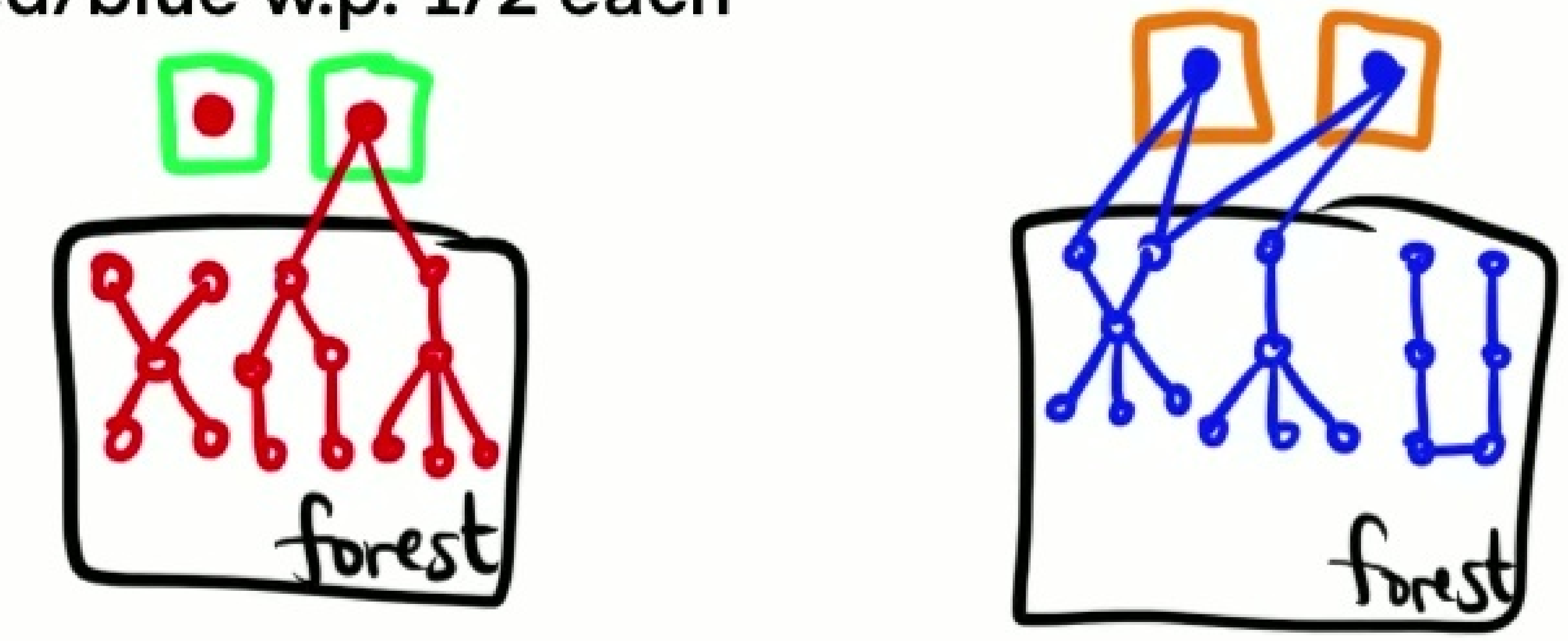

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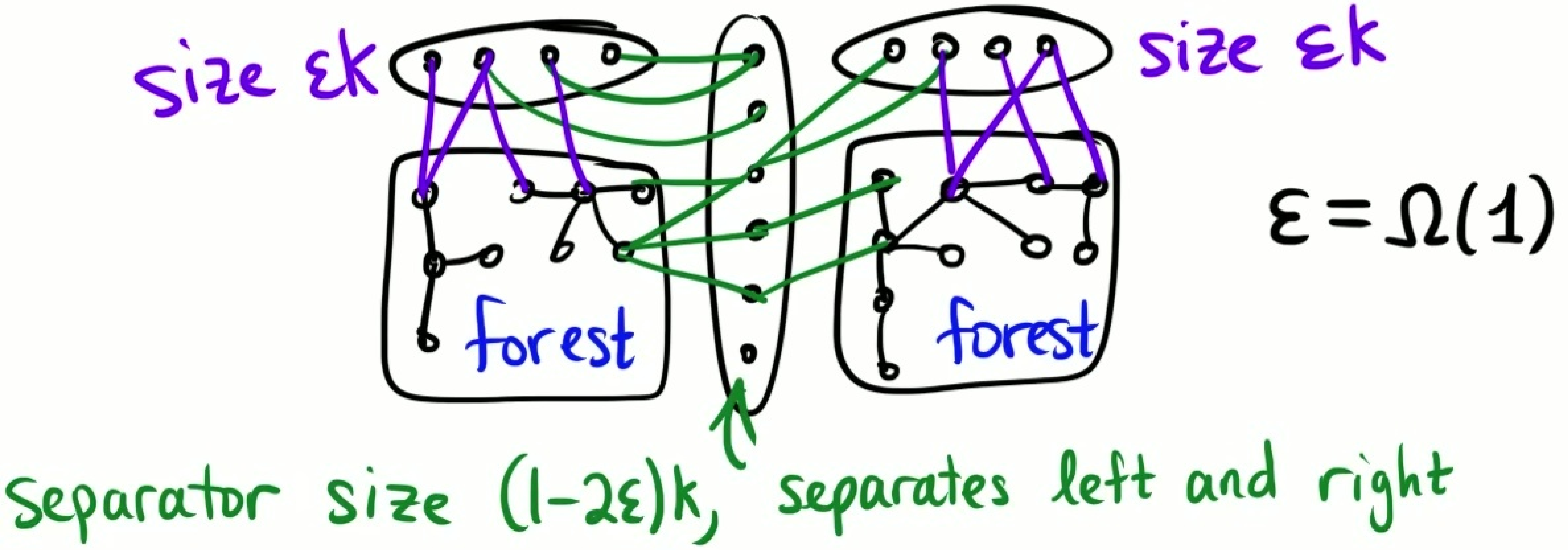


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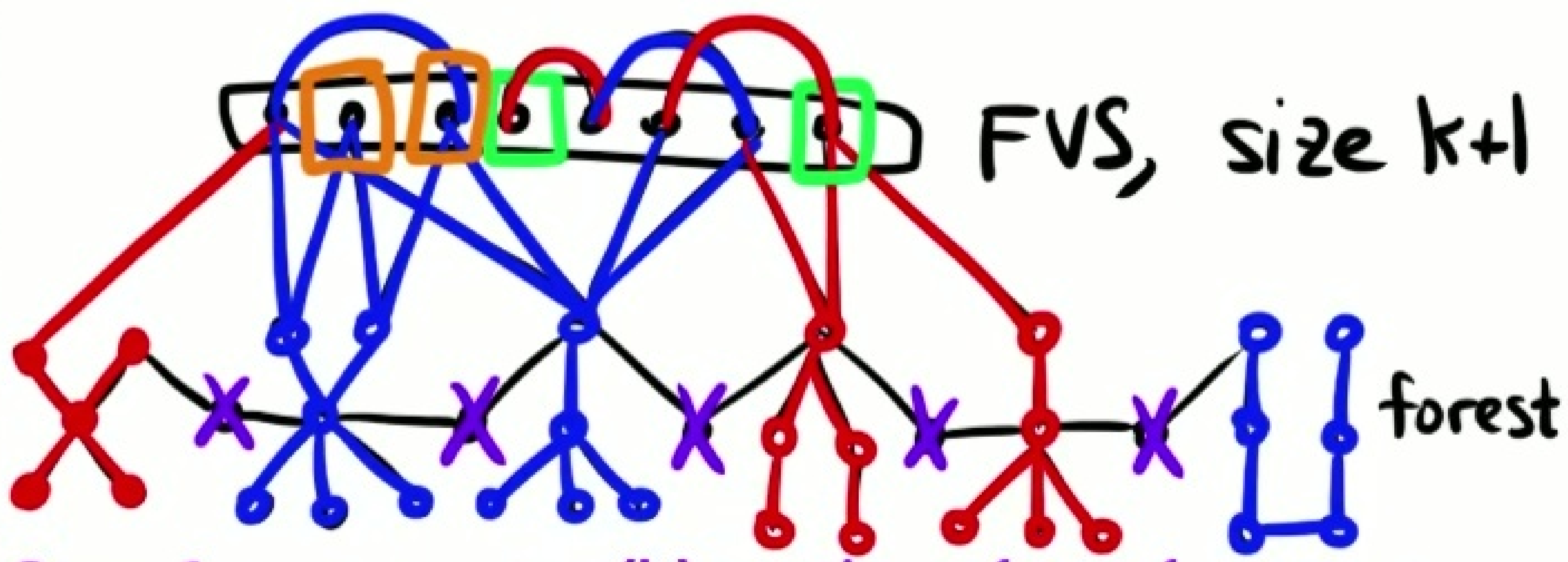


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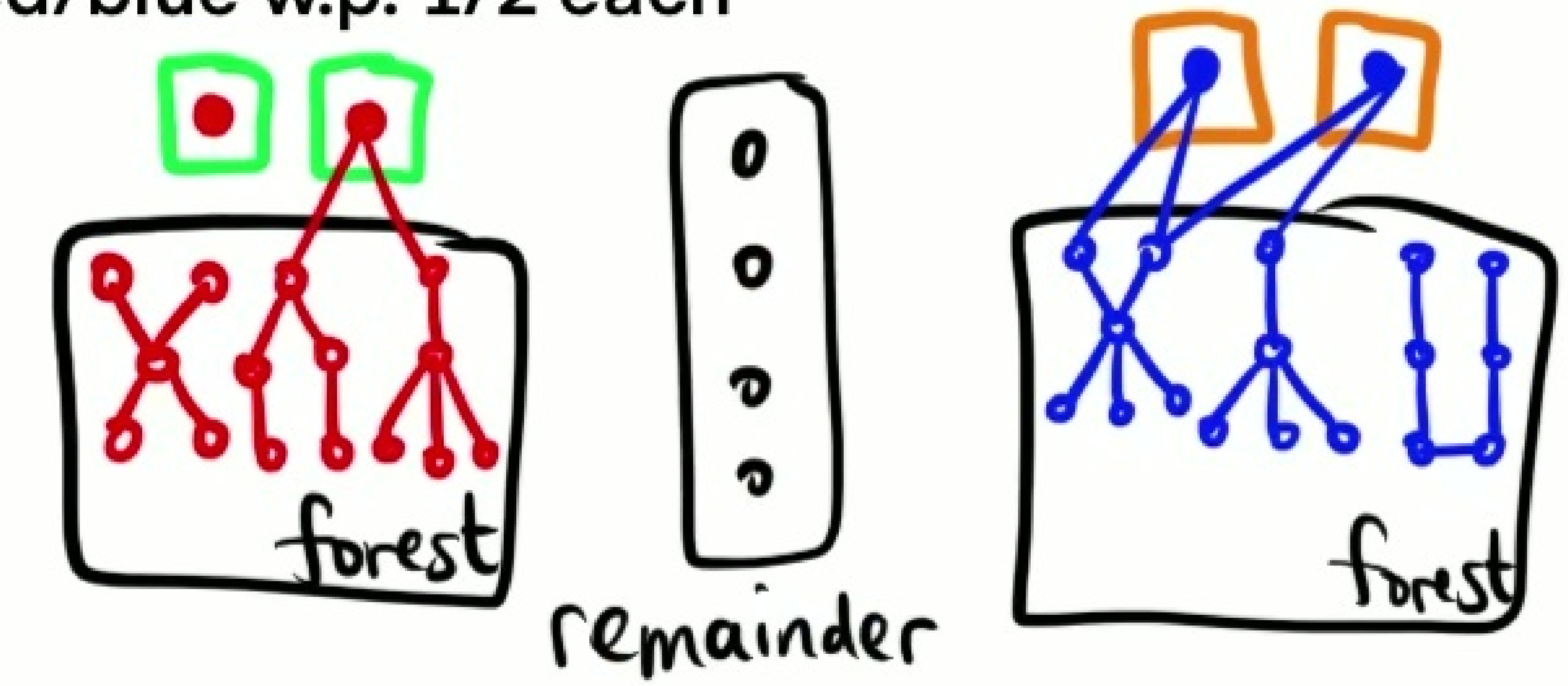

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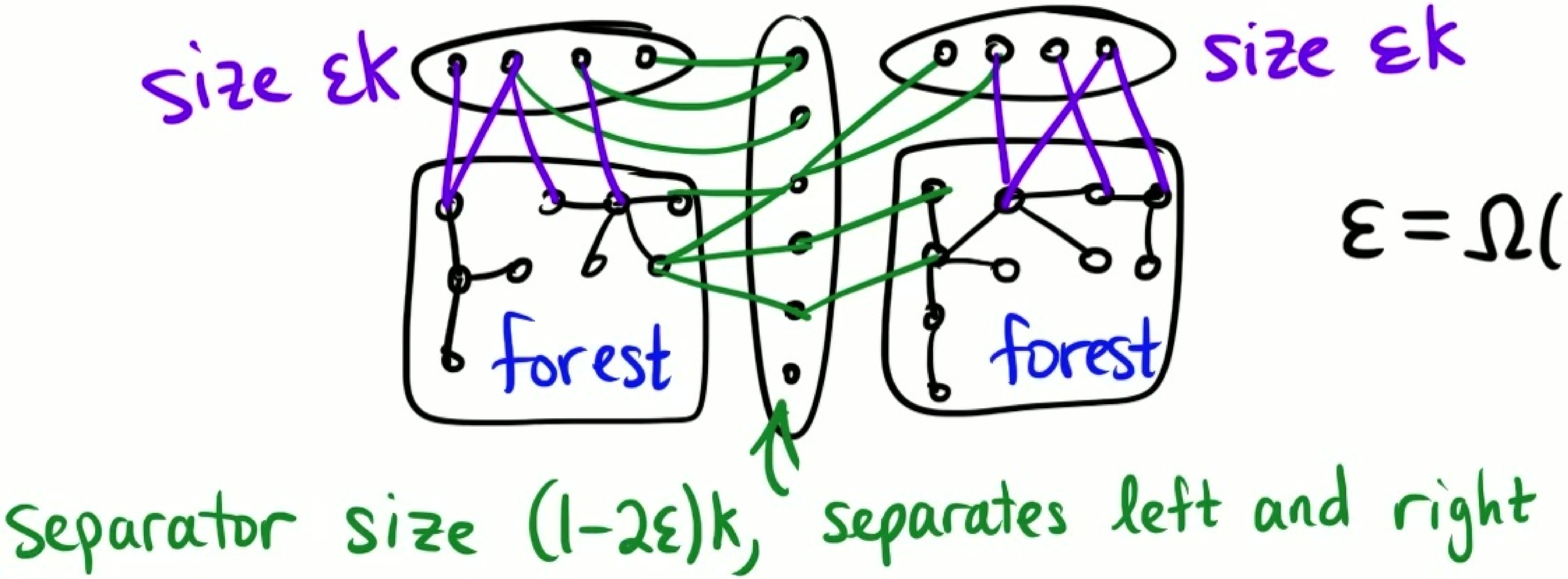


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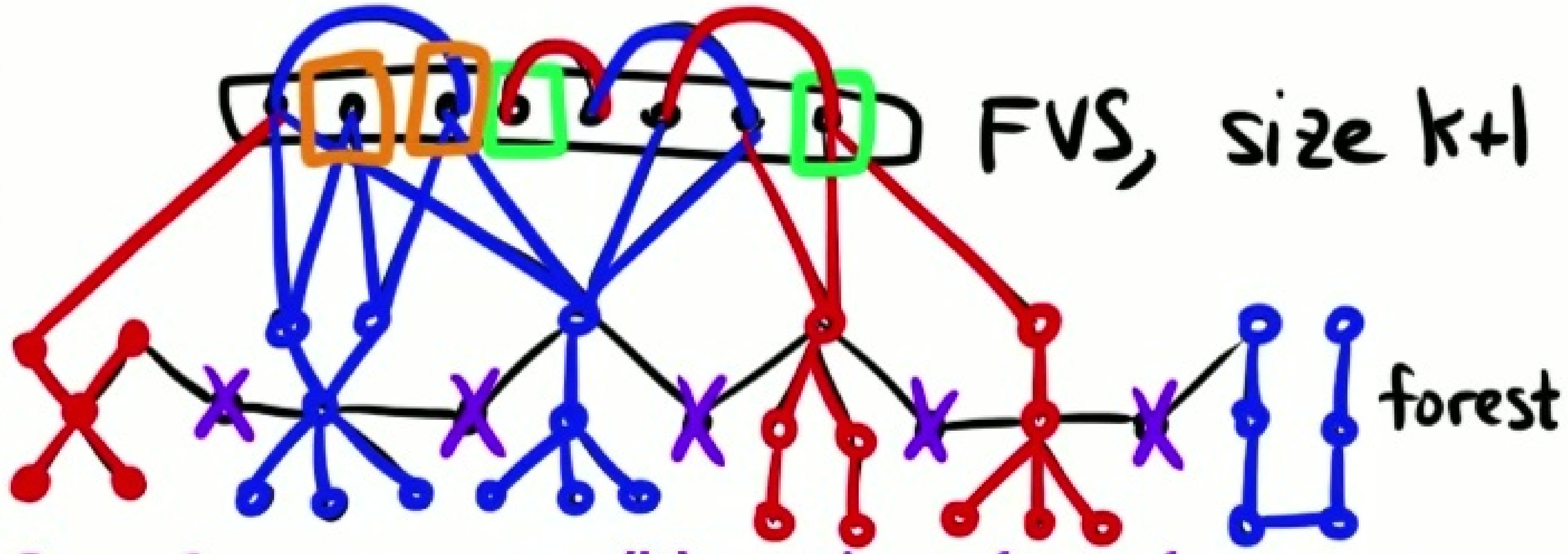


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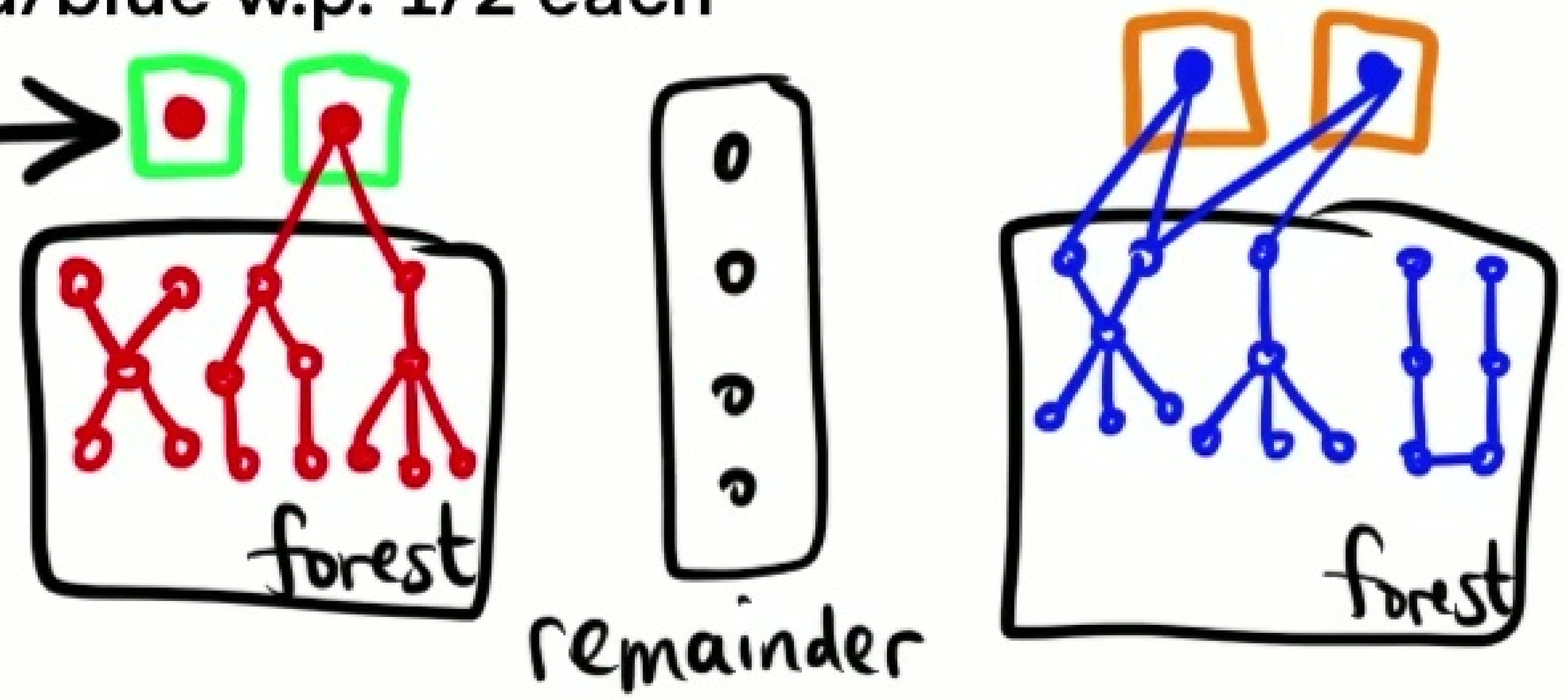

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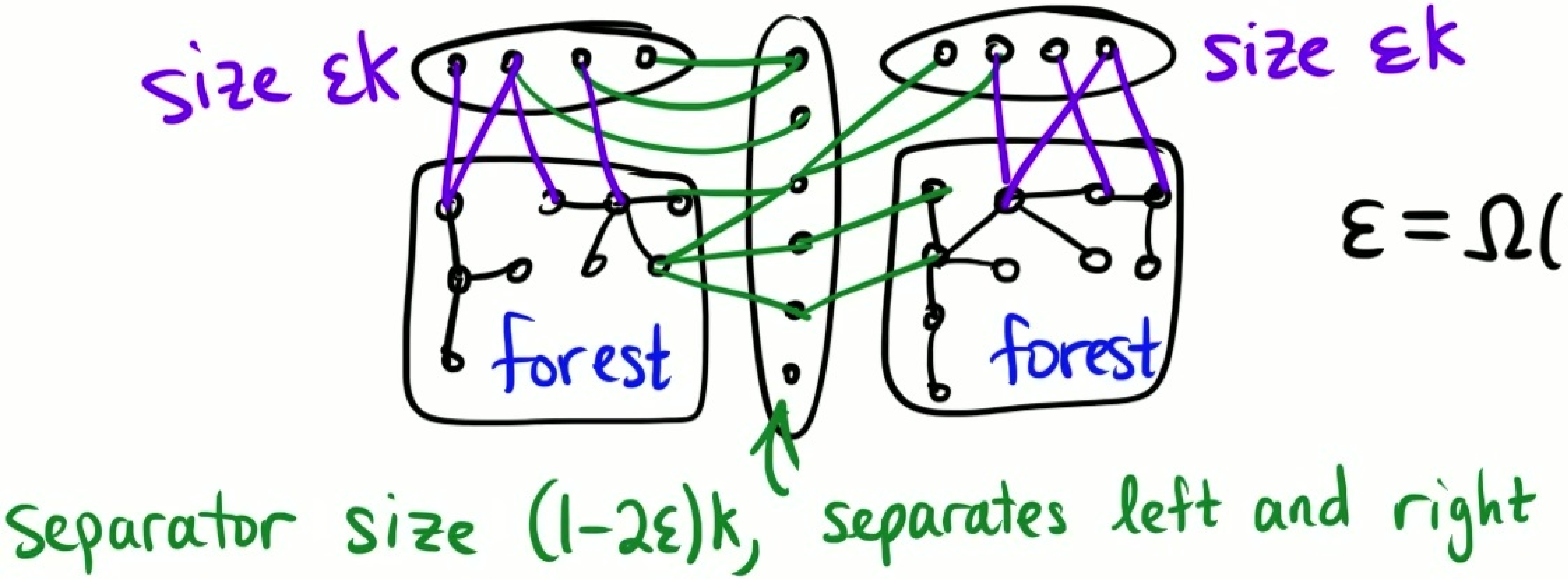


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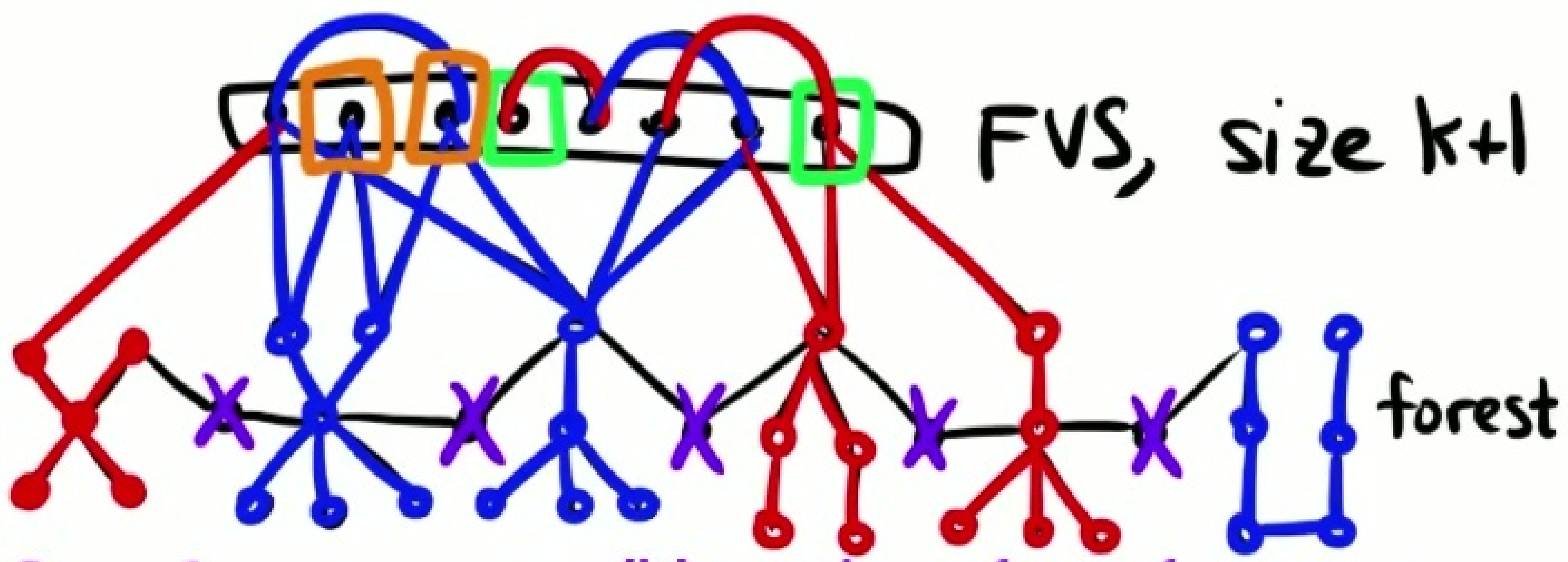


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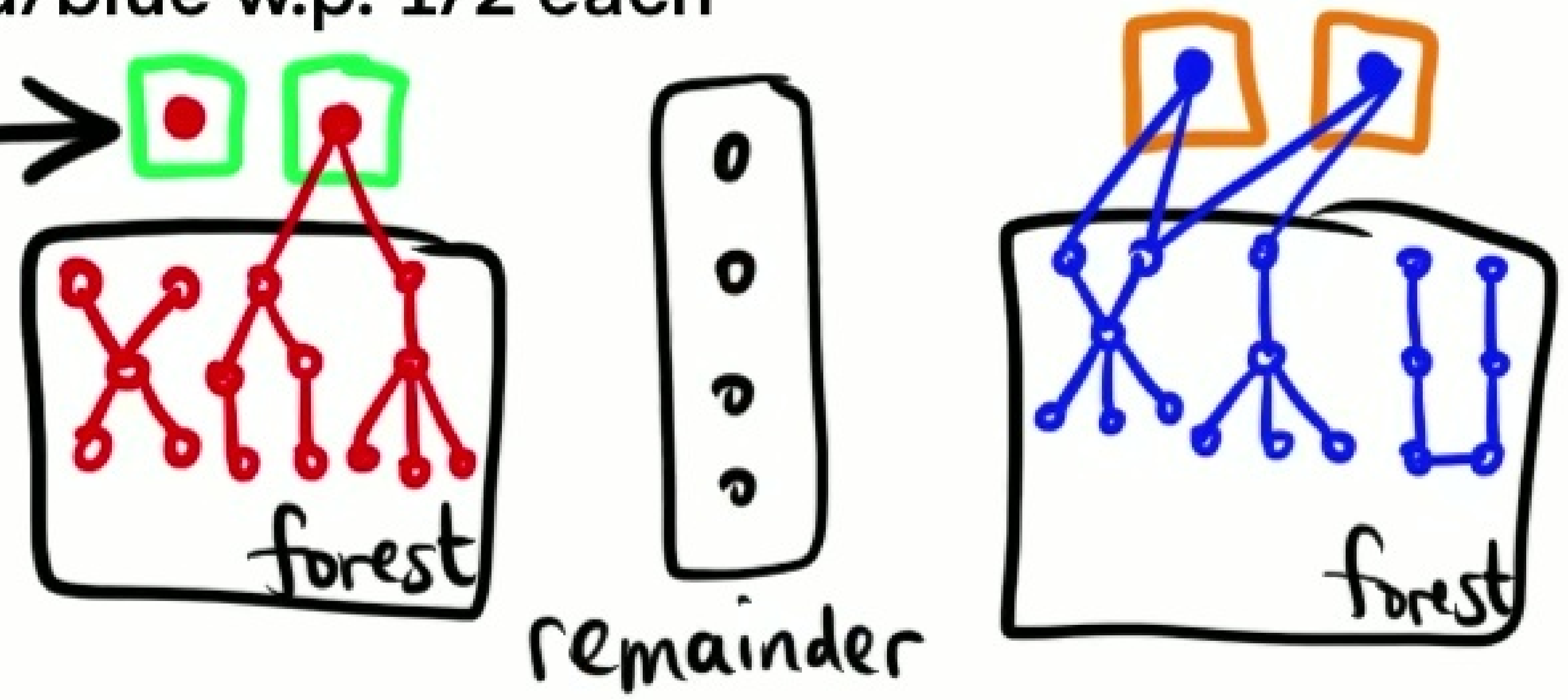

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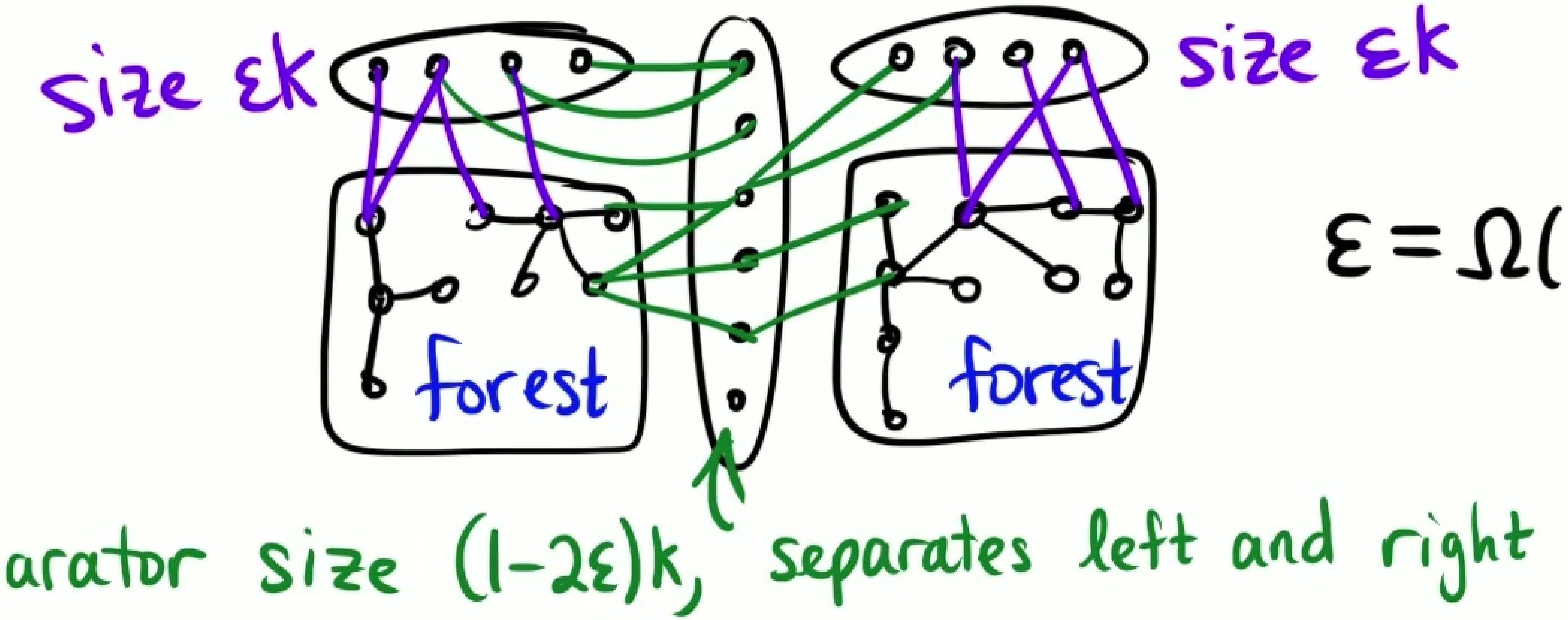


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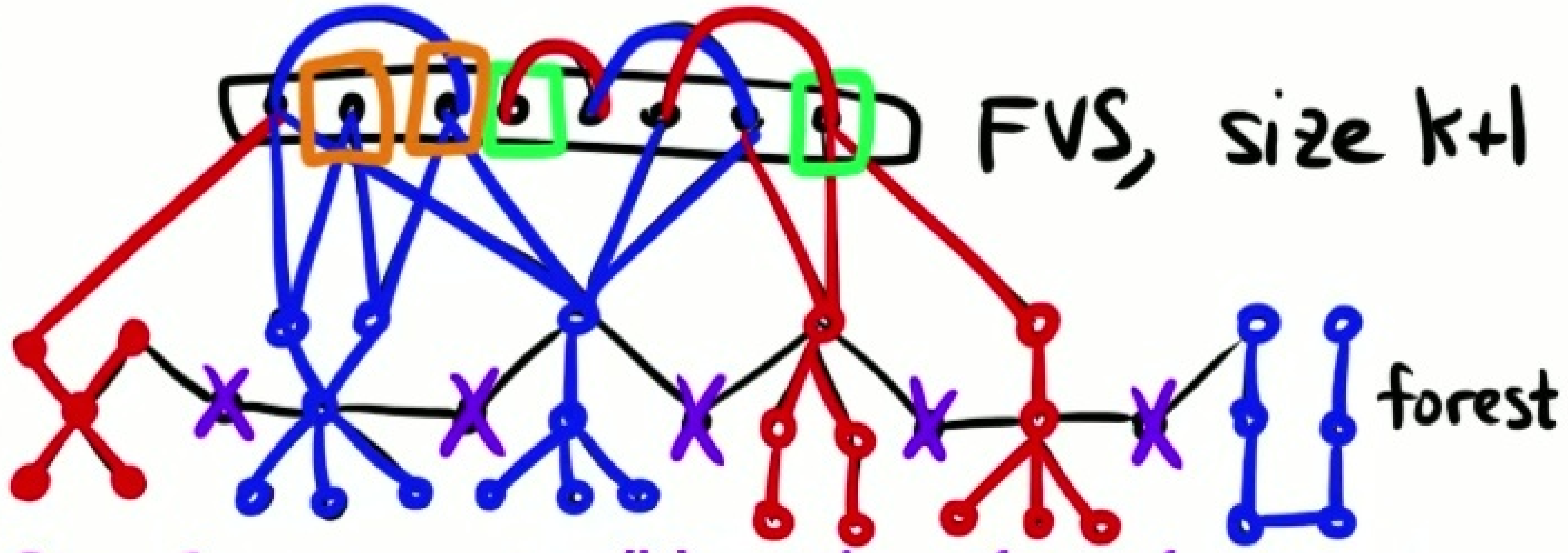


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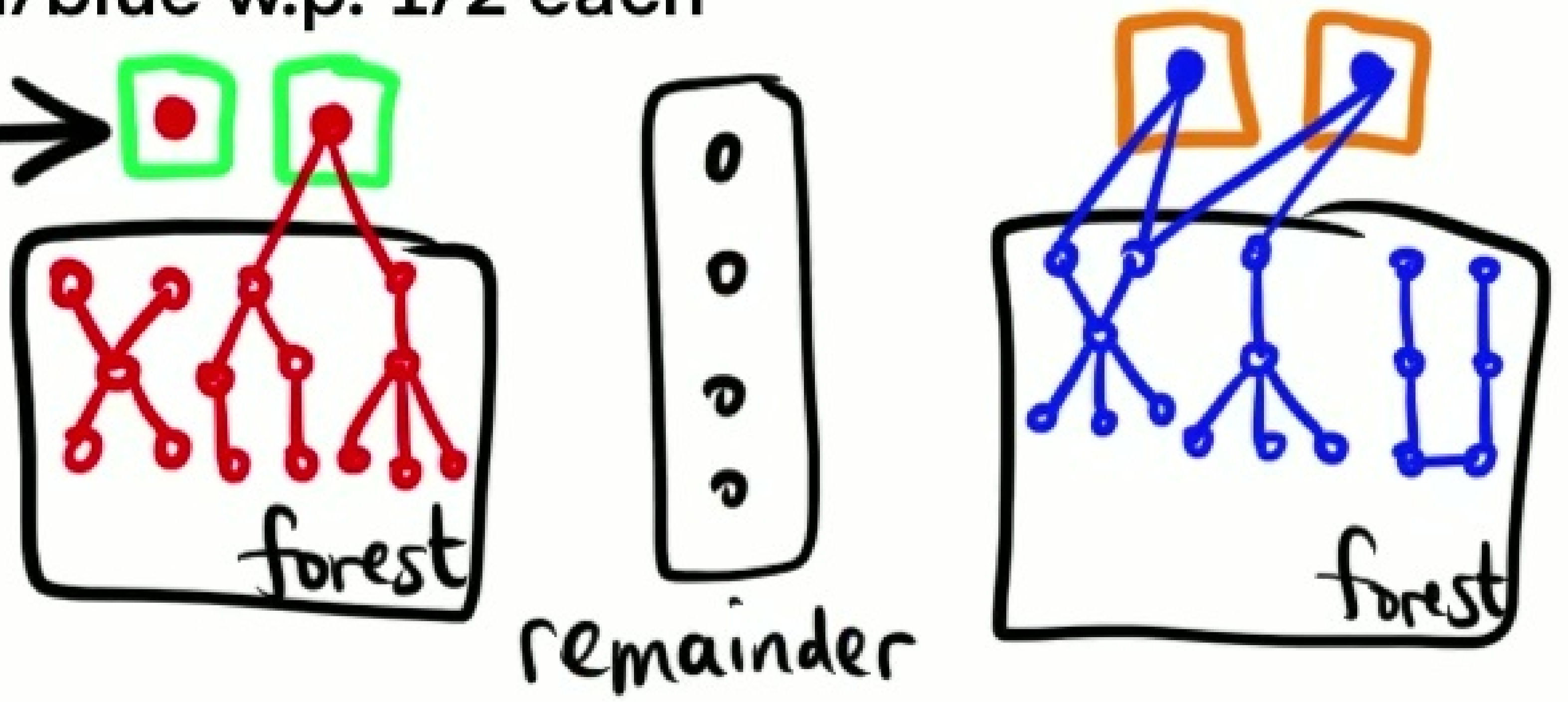

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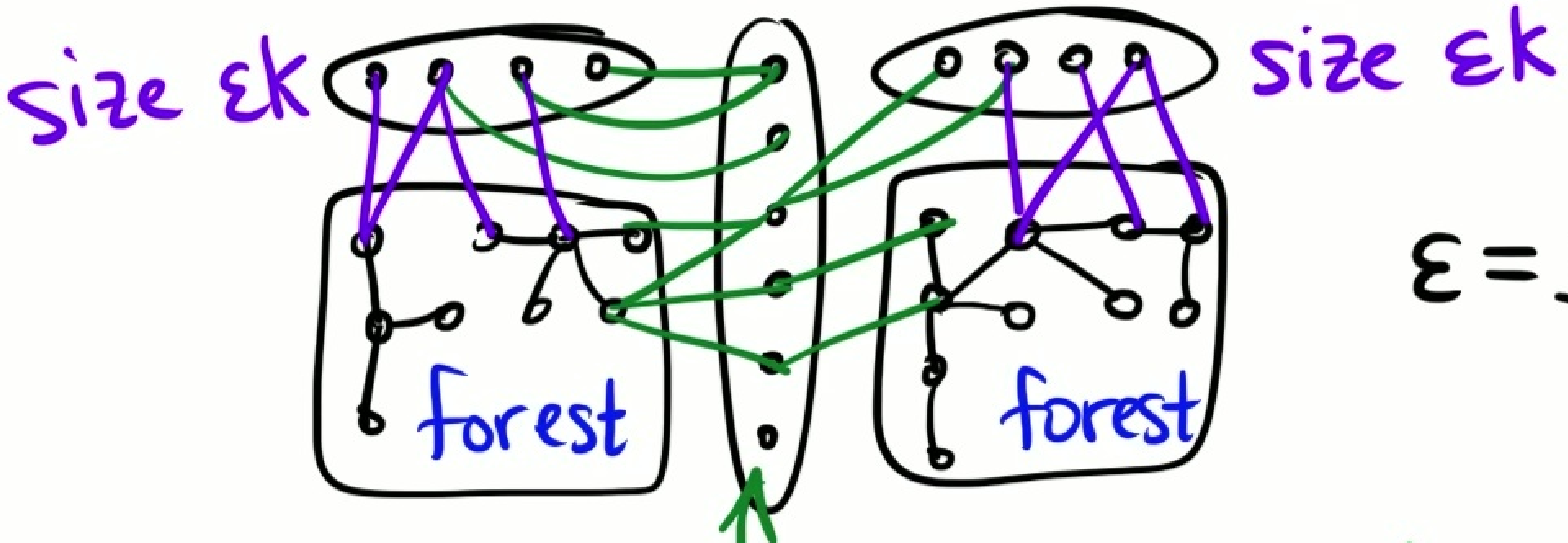


Separator size $(1-2\varepsilon)k$, separates left and right

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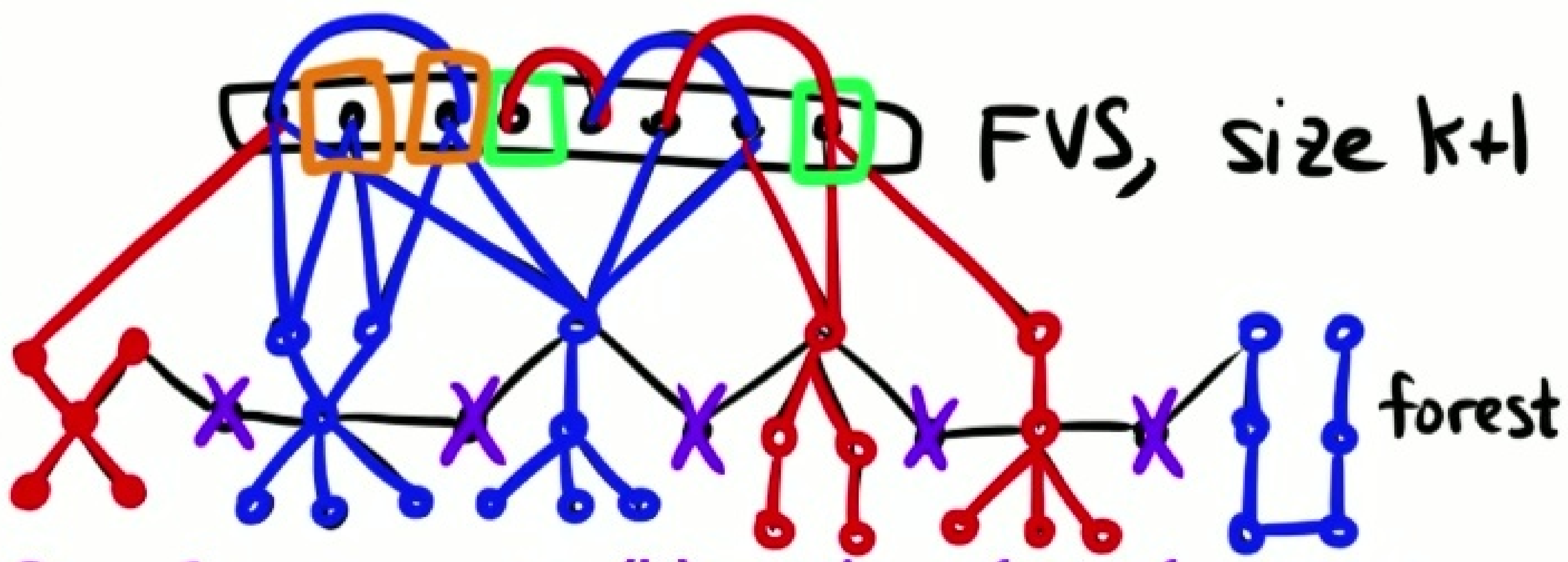


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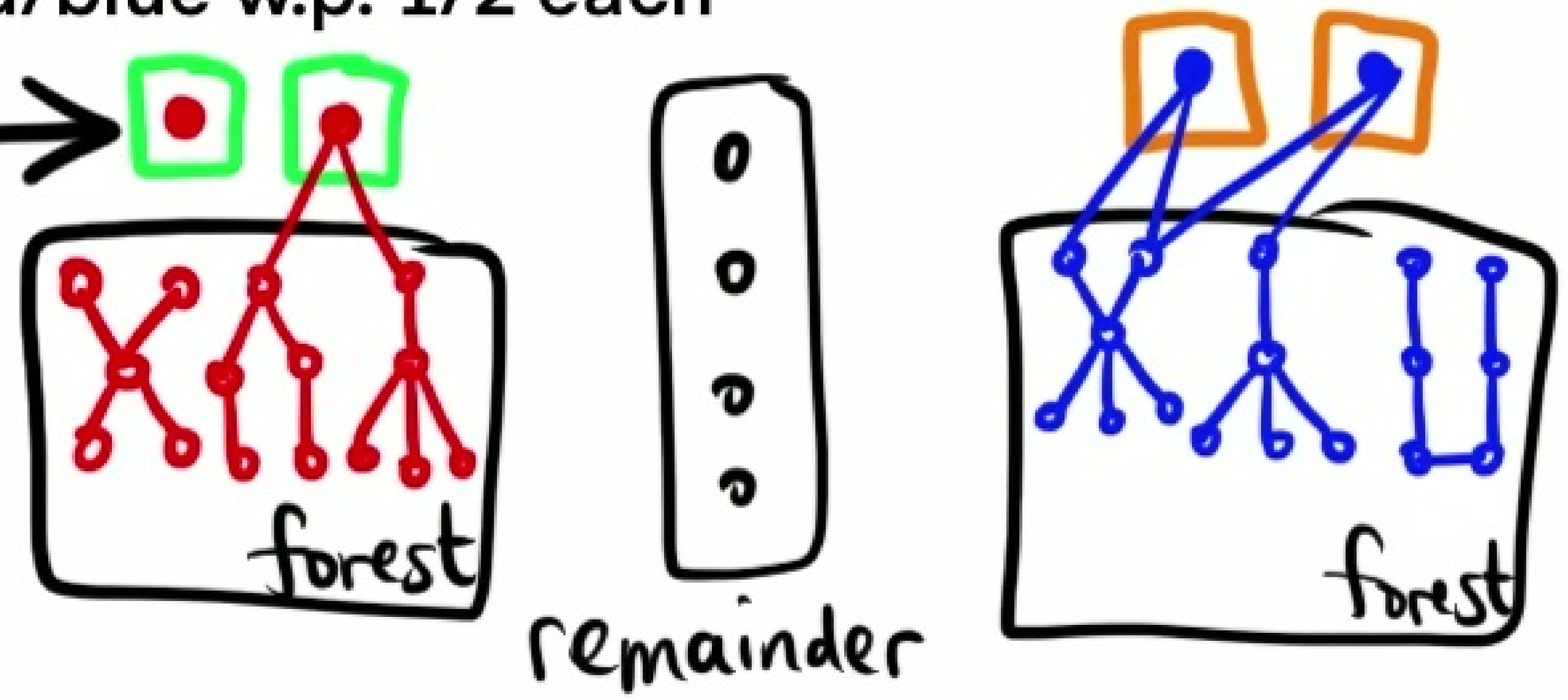


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Chernoff bound (since each [tree] is "small"): #(green box, red dot) $\approx$ #(orange box, blue dot) $\approx \varepsilon k$ for some $\varepsilon \ge 2^{-200}$

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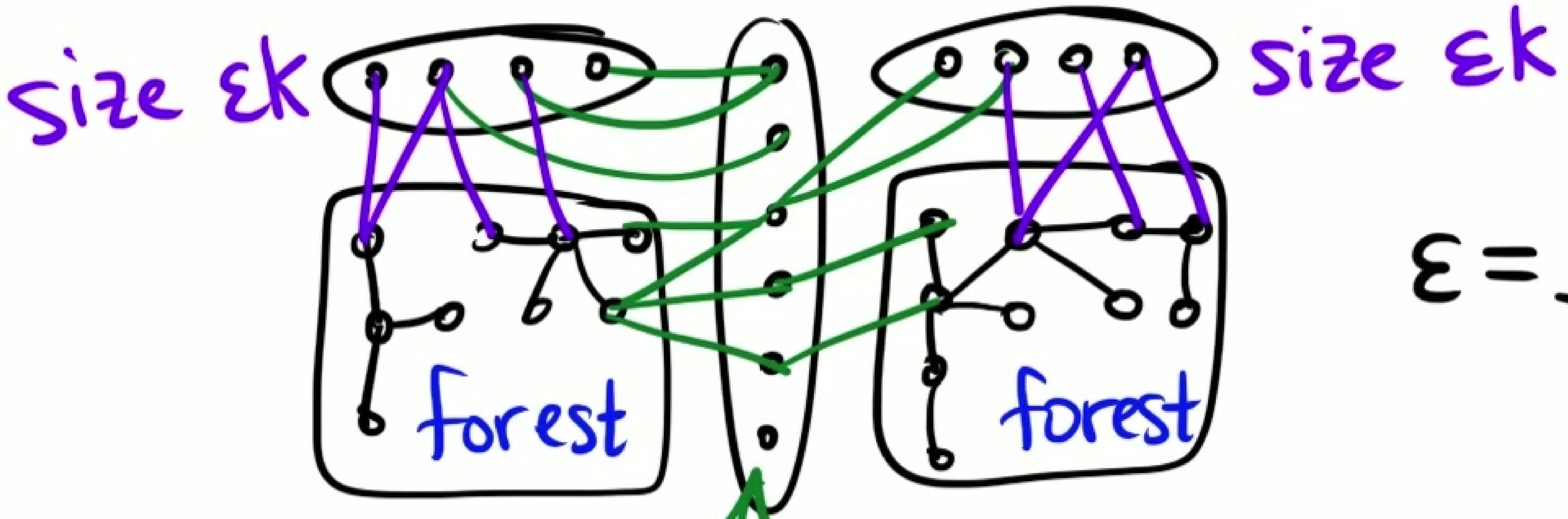


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Jensen: $\ge |FVS| \cdot 2^{-(\text{avg deg})} \ge 2^{-200} k.$ ($|FVS| = k+1$; avg deg $\le \frac{2m}{k+1} \le 200$)

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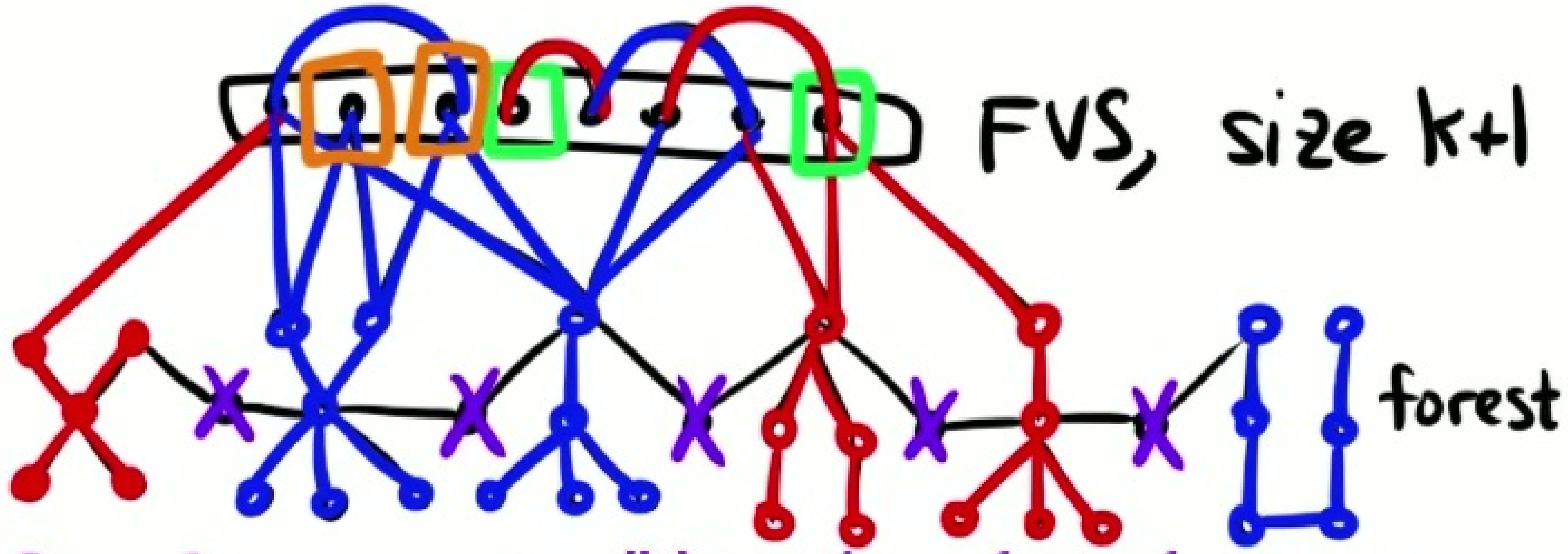


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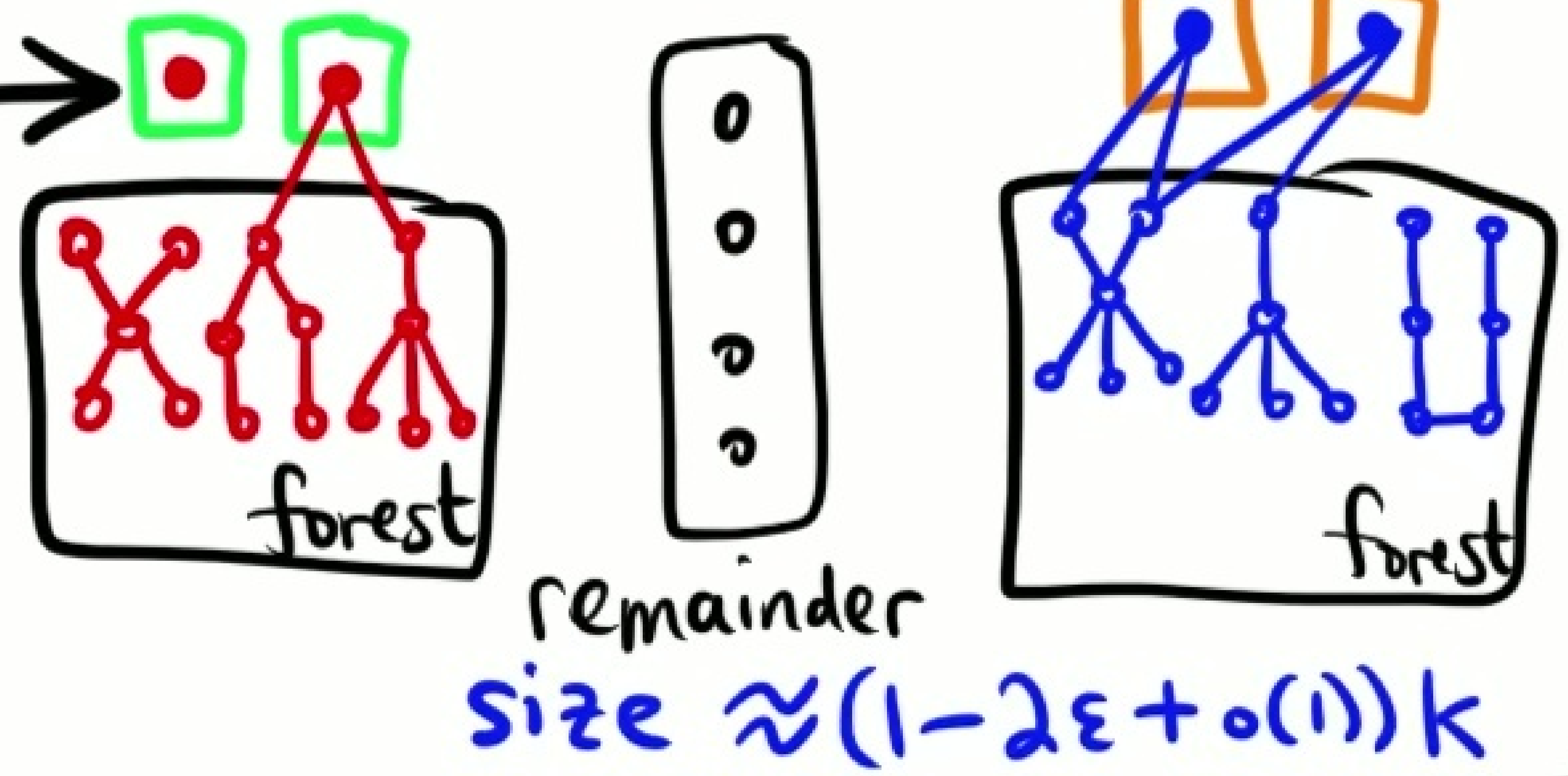

# Speedup: O*(2.7^k) time

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## Open problems

- Our main conceptual message: 3^k can be broken (randomized)
  - Faster deterministic algorithm? [BBG'00] is inherently randomized
- 2^k possible?
- SETH lower bound? No 1.00001^k lower bound known!