Detecting Feedback Vertex Sets of Size k in O*(2.7^k) Time

Jason Li

With Jesper Nederlof (TU Eindhoven)

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Introduction

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Want time FPT in k: f(k)*poly(n)

Goal in FPT setting: minimize function f(k).

poly(n) factor does not matter

Prior Work

Downey and Fellows '92: $f(k) = k^{O(k)}$

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Combines techniques from [BBG’00] and [CNP+11].

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This talk: (3-ε)^k, or how to break 3^k.

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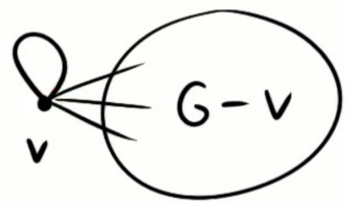

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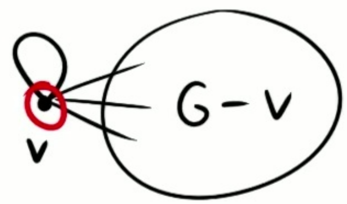

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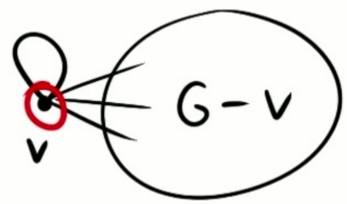

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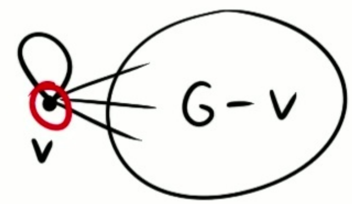

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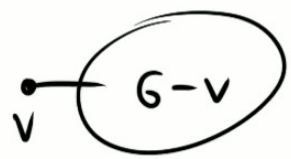

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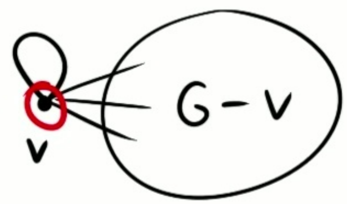

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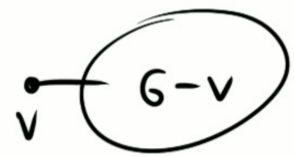

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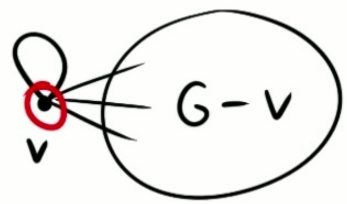

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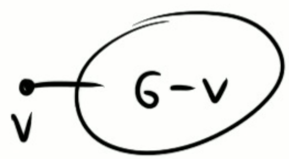

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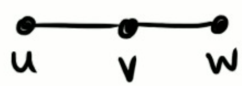

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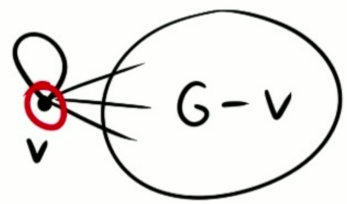

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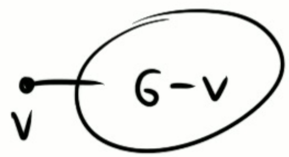

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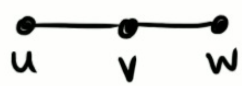

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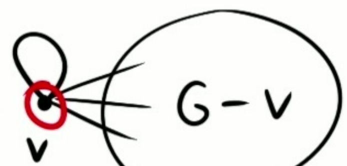

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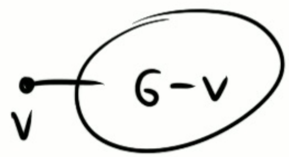

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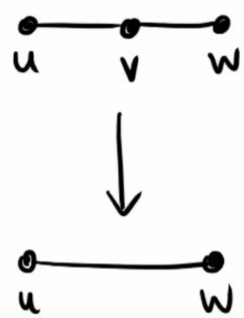

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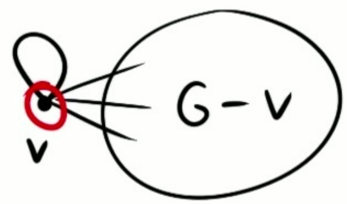

When (1),(2),(3) no longer apply:
- no self-loops
- minimum degree 3

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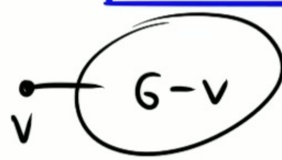

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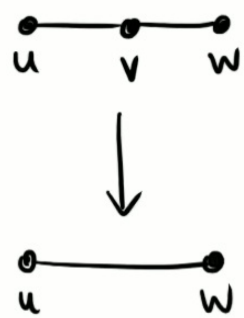

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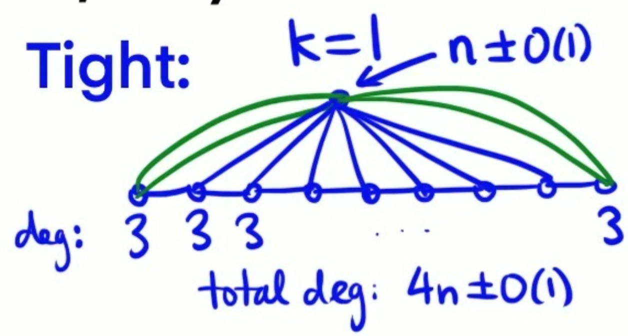

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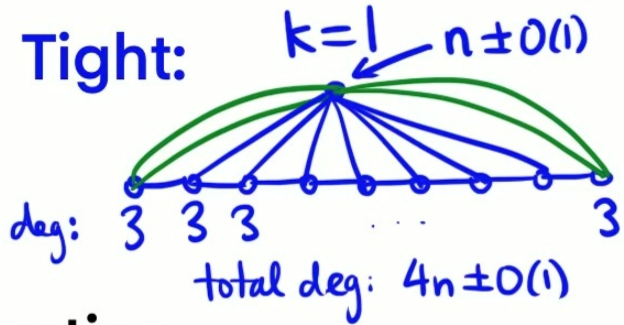

Prob. 1/4 to decrease k by 1 and preserve reduction

=> prob. 1/4^k to go all the way. Repeat 4^k times: O*(4^k) algo.

Our Approach

Dense case: $m >> O(k)$:

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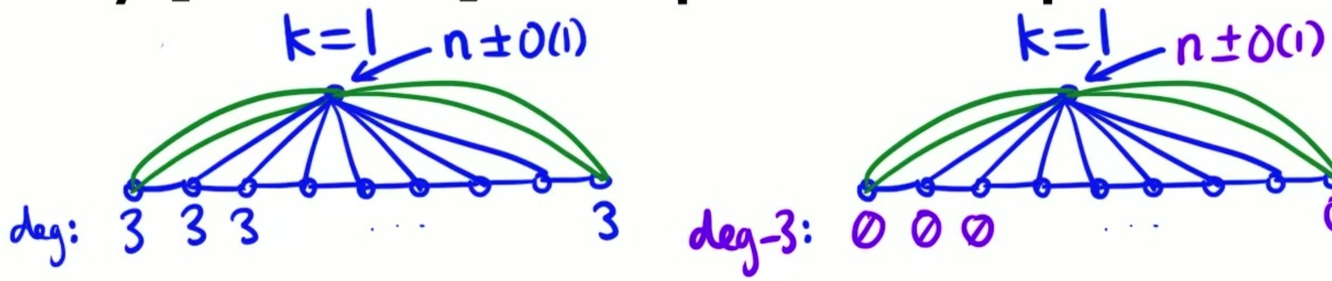

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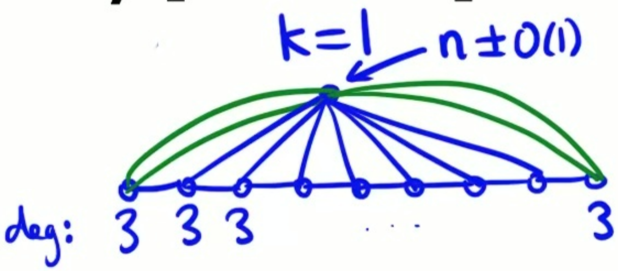

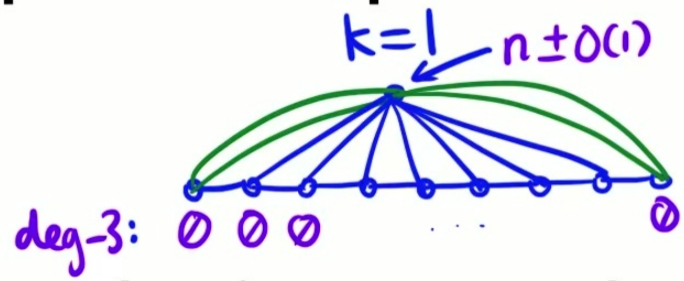

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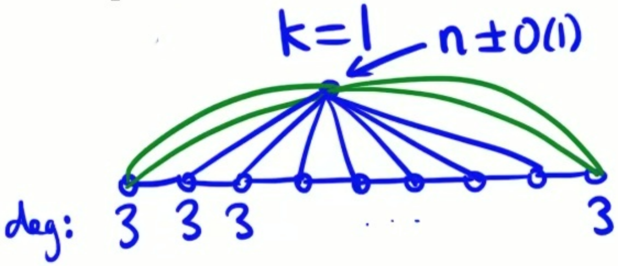

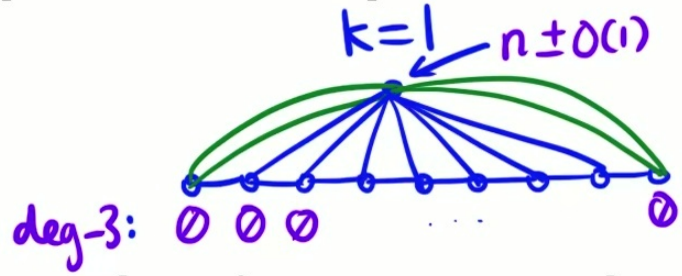

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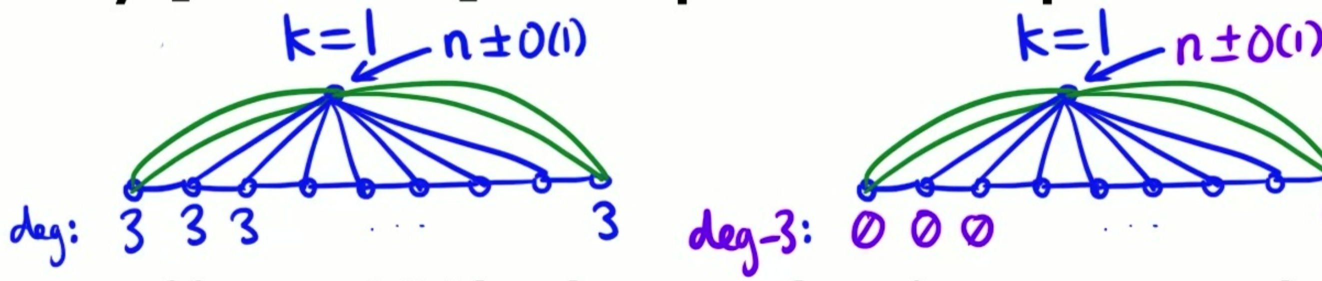

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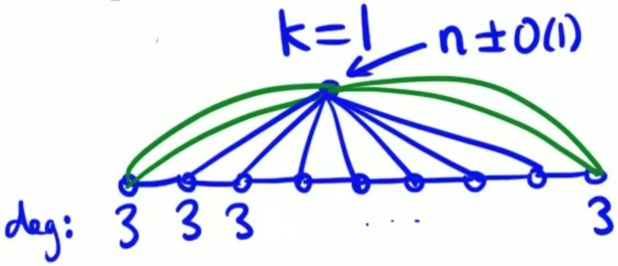

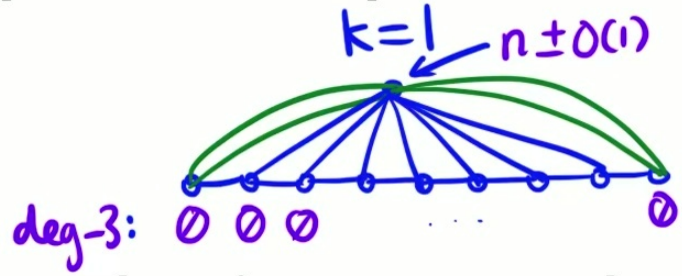

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G has treewidth **(1-Ω(1))k** ⇒ time $3^{(1-\Omega(1))k} = (3-\varepsilon)^k$.

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Lemma: Let F be a FVS of graph G. Then,

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$$\begin{aligned}\sum_{v\notin F}(\deg(v)-3) &= \deg(V\setminus F)-3|V\setminus F| \\ &\leq \deg(F)+2(|V\setminus F|-1)-3|V\setminus F| \\ &\leq \deg(F)-3|V\setminus F|\end{aligned}$$

$$n\geq 4k: \quad |V\setminus F| \geq 3k = 3|F|$$

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$\le$

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$$\underline{\sum_{v\notin F}(\deg(v)-3)} = \deg(V\setminus F) - 3|V\setminus F|$$
$$\le \deg(F) + 2(|V\setminus F|-1) - 3|V\setminus F|$$
$$\le \deg(F) - 3|V\setminus F|$$

$n \ge 4k$: $|V\setminus F| \ge 3k = 3|F|$

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choose $v \in F$ w.p. $\ge 1/2$

If $m \ge 20n$ and F has size k:

$$\underbrace{\deg(V)}_{2m} - \deg(F) \le \deg(F) + 2(\underbrace{|V\setminus F|}_{n-k} - 1)$$

$$\deg(F) \ge \frac{1}{2}(2m - 2(n-k-1)) \ge m - n + k$$

$$\sum_{v\in F}(\deg(v)-3) = \deg(v) - 3|F| \ge m - 4n \ge 0.8m$$

$$\sum_{v\in V}(\deg(v)-3) \le \deg(V) = 2m$$

Dense case: m >> O(k)

Lemma: Let F be a FVS of graph G. Then,

$$\deg(V\setminus F) \leq \deg(F) + 2(|V\setminus F|-1), \quad \text{where } \deg(S) := \sum_{v\in S}\deg(v)$$

Proof:

- G-F is forest => at most $|V\setminus F|-1$ edges with both endpoints in G-F. Each contributes +2 to $\deg(V\setminus F)$, for a total of $\leq 2(|V\setminus F|-1)$
- Every edge between F and $V\setminus F$ contributes +1 to both $\deg(F)$ and $\deg(V\setminus F)$

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$$\sum_{v\in F}(\deg(v)-3) = \deg(v) - 3|F| \geq m - 4n \geq \underline{0.8m}$$

$$\sum_{v\in V}(\deg(v)-3) \leq \deg(V) = \underline{2m} \qquad \frac{0.8m}{2m} \Rightarrow \text{success w.p. } \geq 0.4$$

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Proof:

- G-F is f … G-F.
  Each c …
- Every e … and deg(V\F)

So if success prob ≤ 1/2.99, then must have $m \le 20n \le 80k \Rightarrow m \le O(k)$

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$\frac{0.8m}{2m} \Rightarrow$ success w.p. ≥ 0.4

Iterative Compression

Original problem: given graph G, find FVS size k, or determine none exist.

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- Get size (k+1) FVS as input for free

Sparse case

Lemma: Given a graph with $m \leq 100k$, and given a FVS of size $k+1$,

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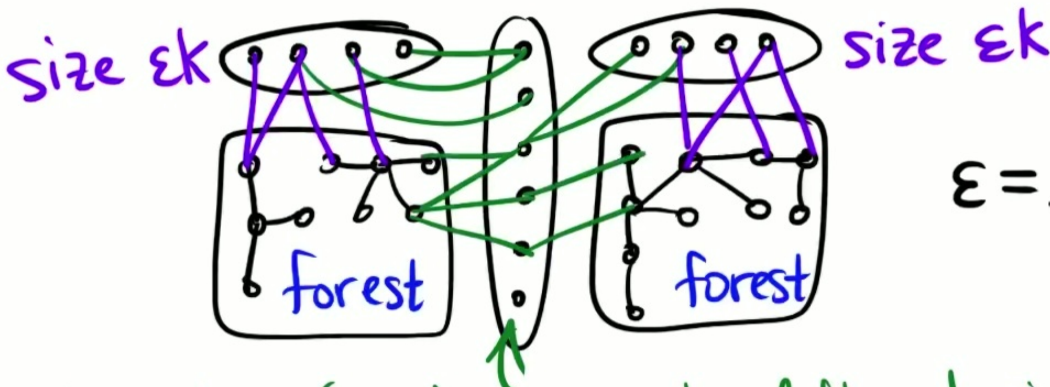

$\varepsilon = \Omega(1)$

separator size $(1-2\varepsilon)k$, separates left and right

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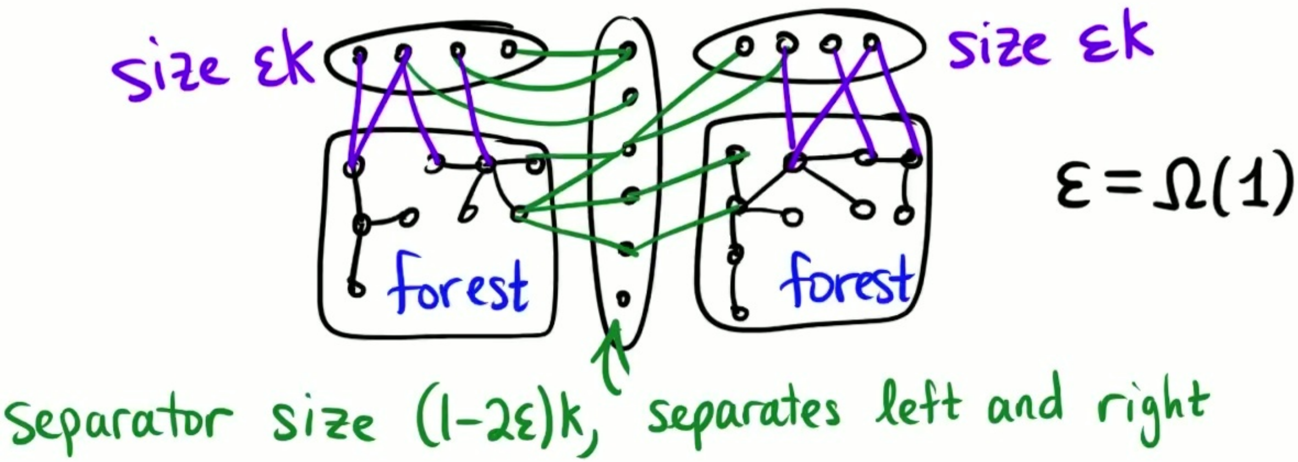

Claim: a graph with this decomposition has treewidth $(1-\Omega(1))k$

Proof:

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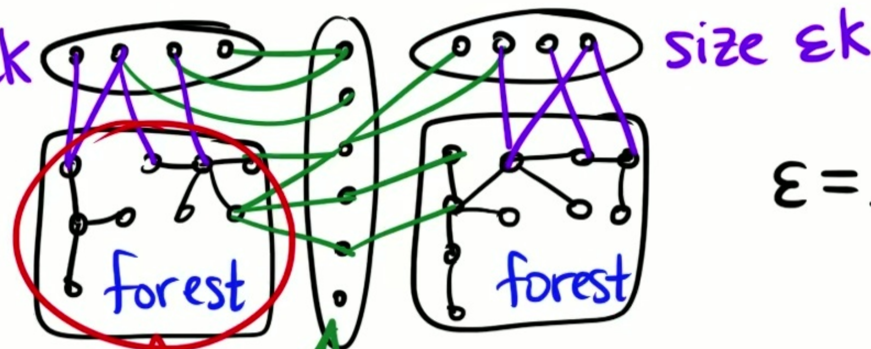

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tw( ) = 1 (forest)

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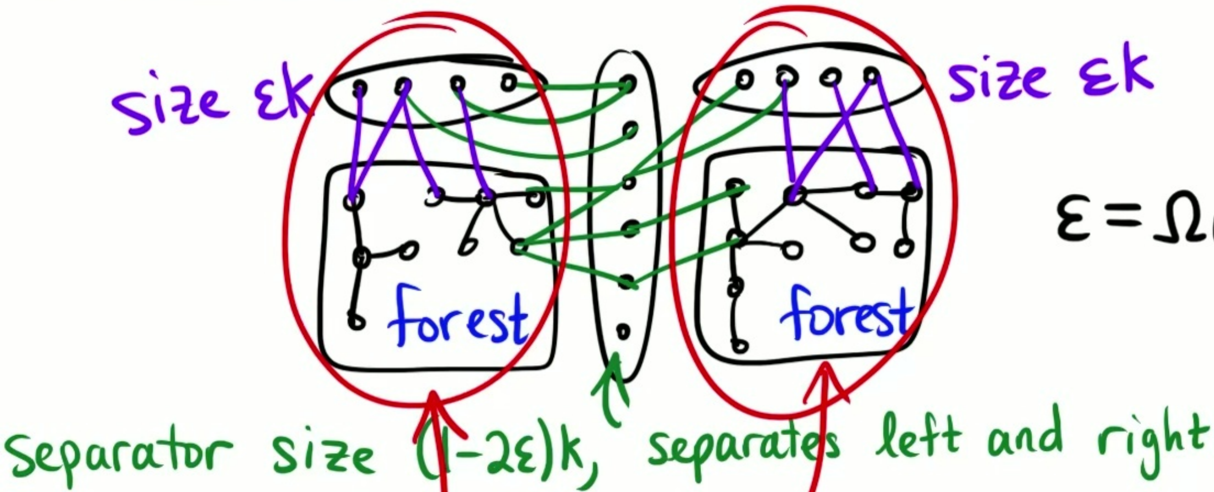

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$\mathrm{tw}(\quad) \leq 1 + \varepsilon k$ (forest + εk vertices)

$\mathrm{tw}(G) \leq 1 + \varepsilon k + (1-2\varepsilon)k$ (add last $(1-2\varepsilon)k$ vertices)

$= (1 - \Omega(1))k$

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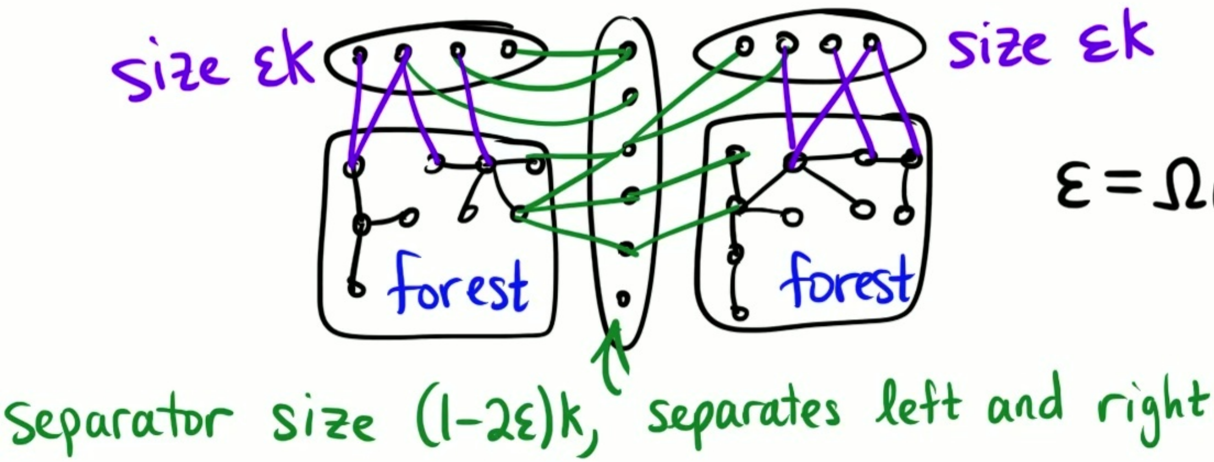

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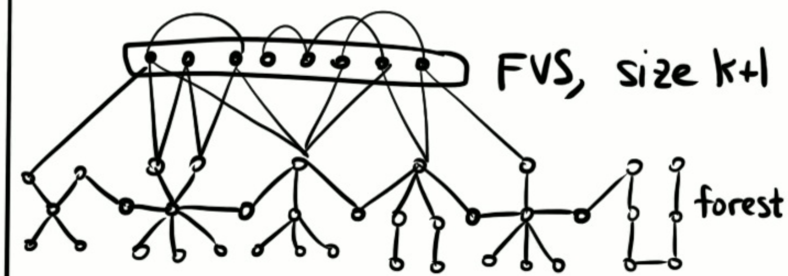

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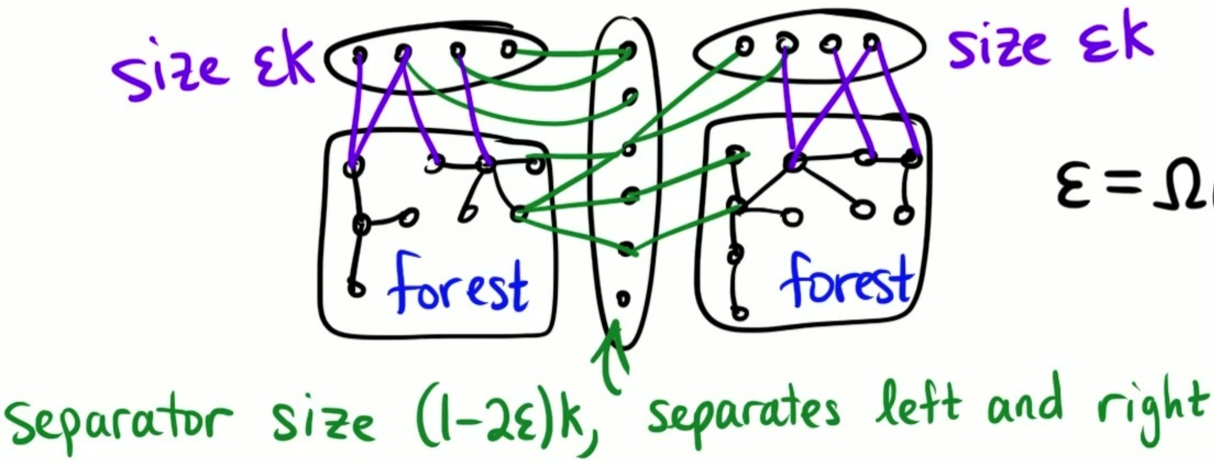

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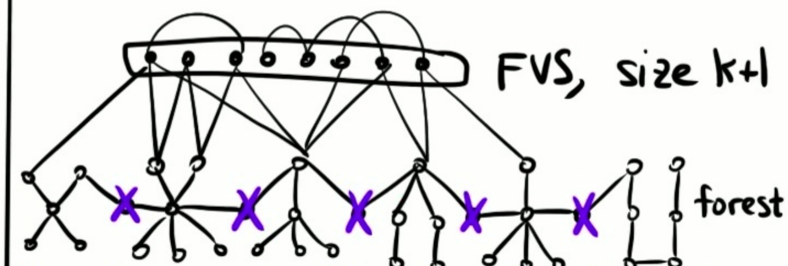

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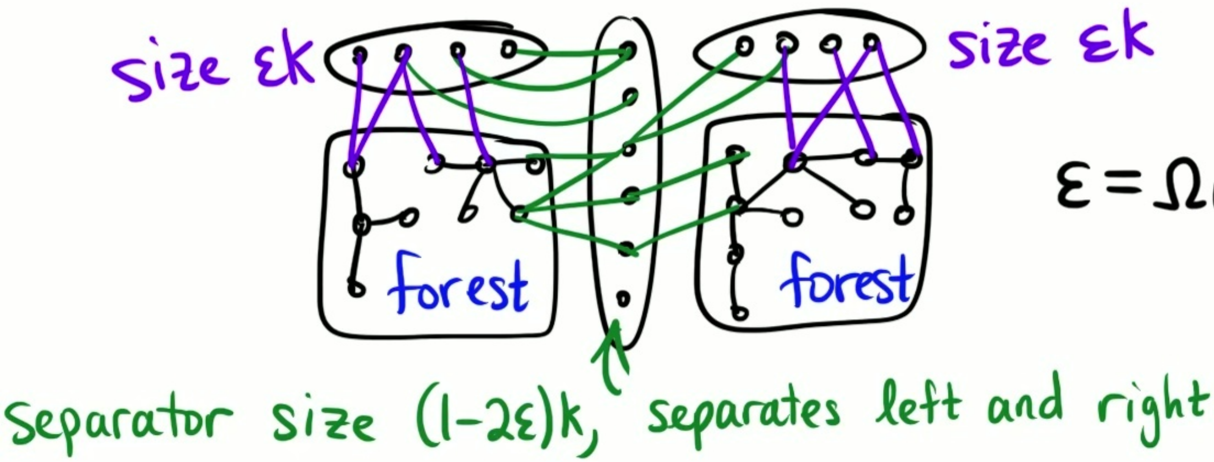

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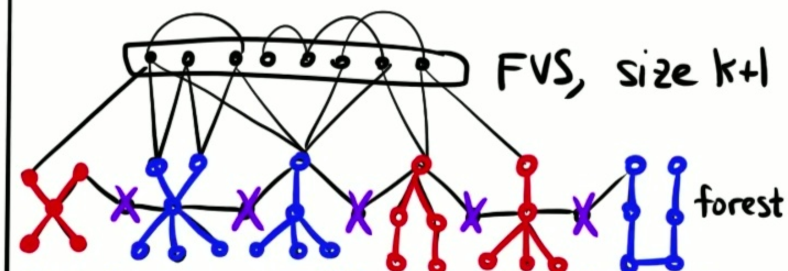

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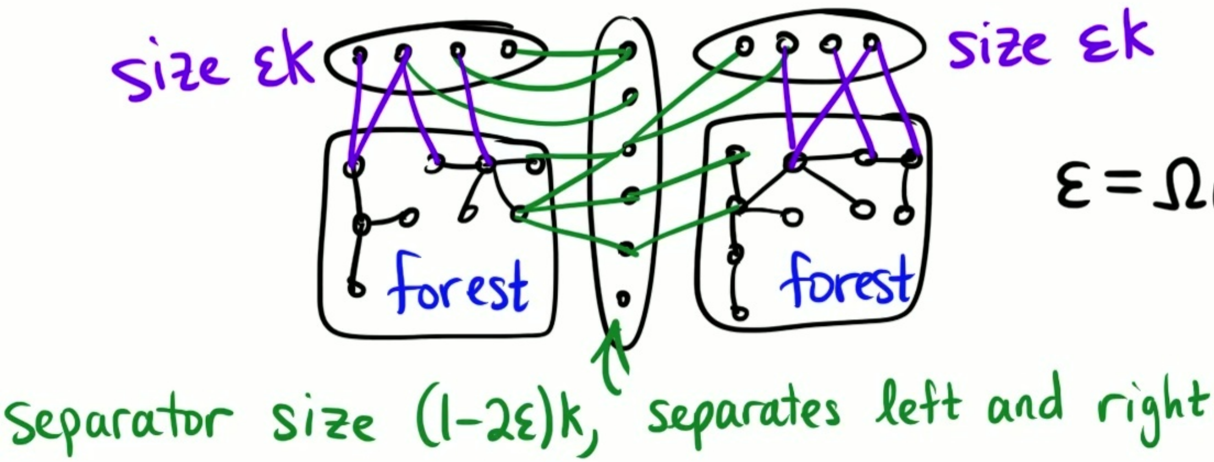

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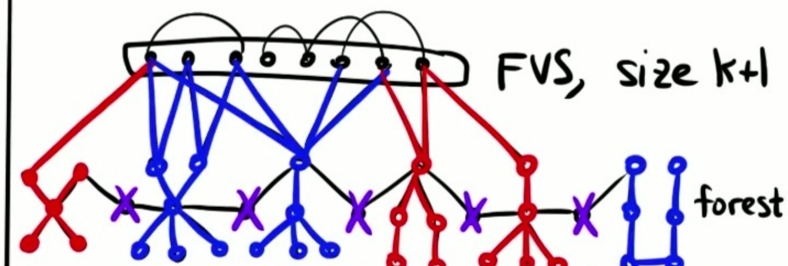

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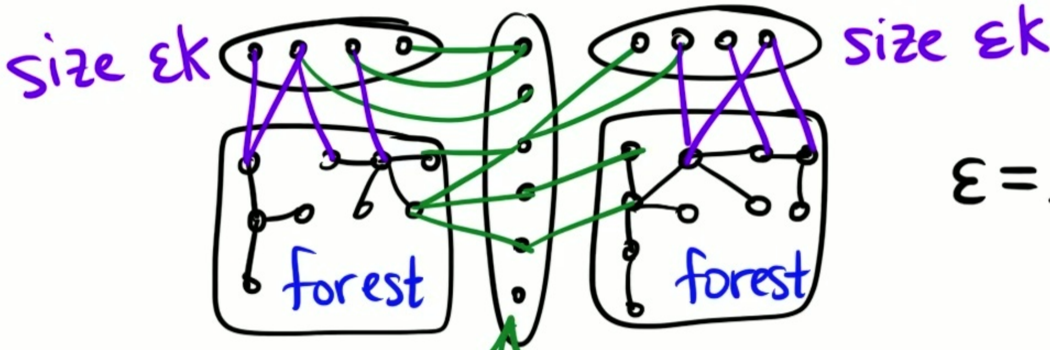

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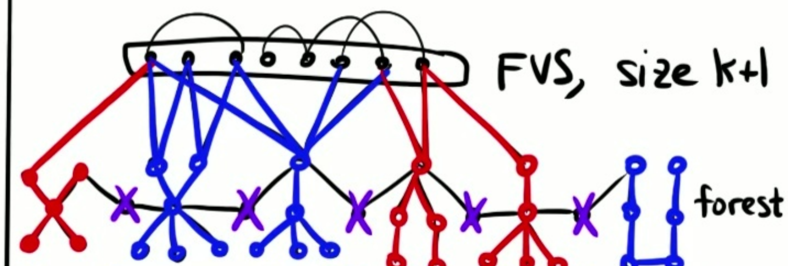

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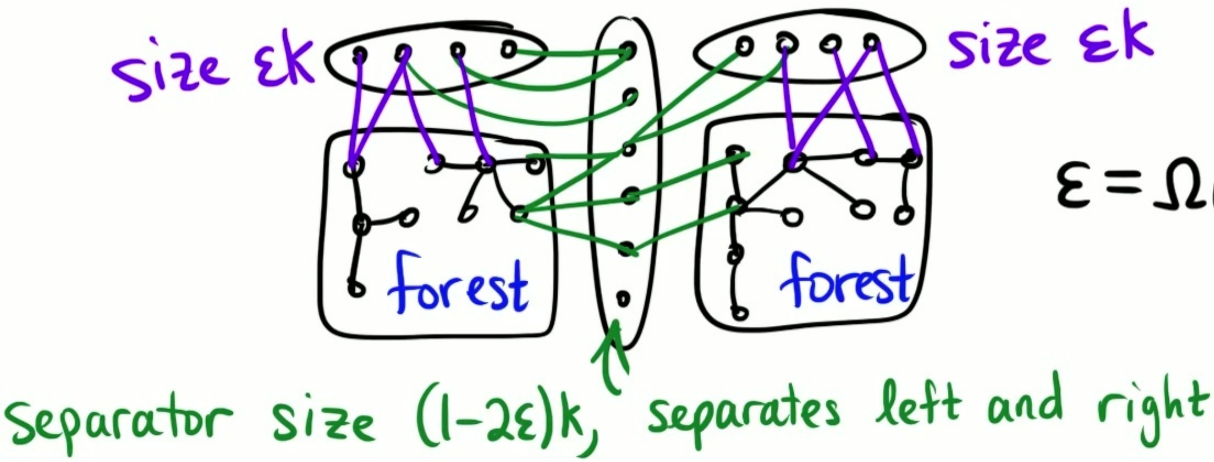

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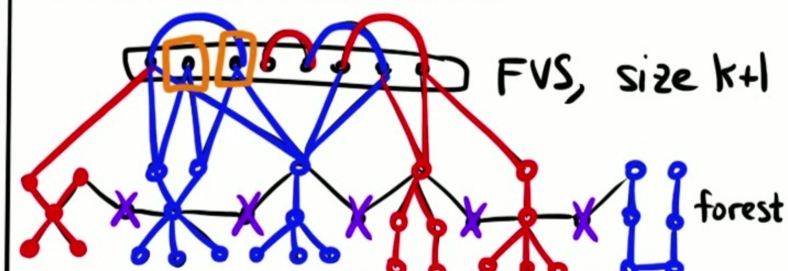

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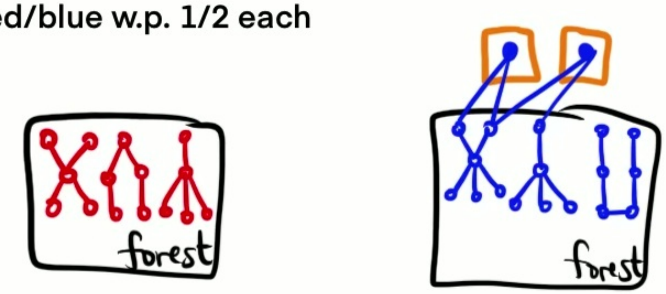

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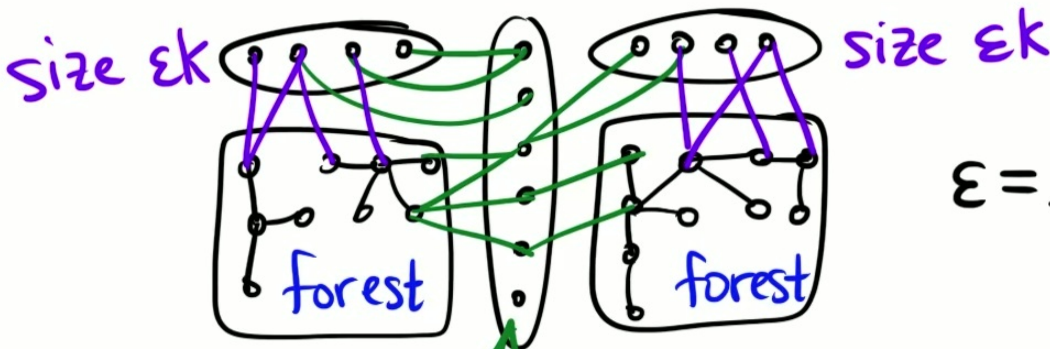

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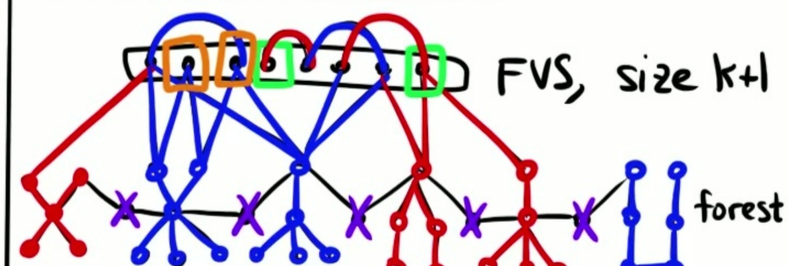

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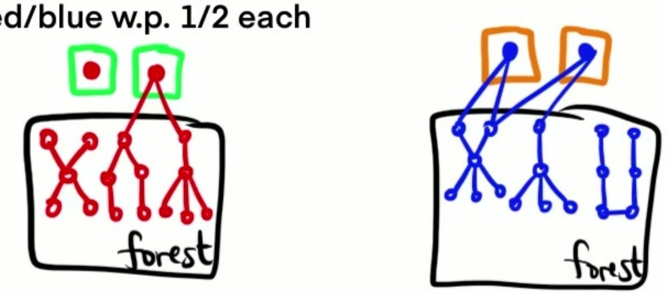

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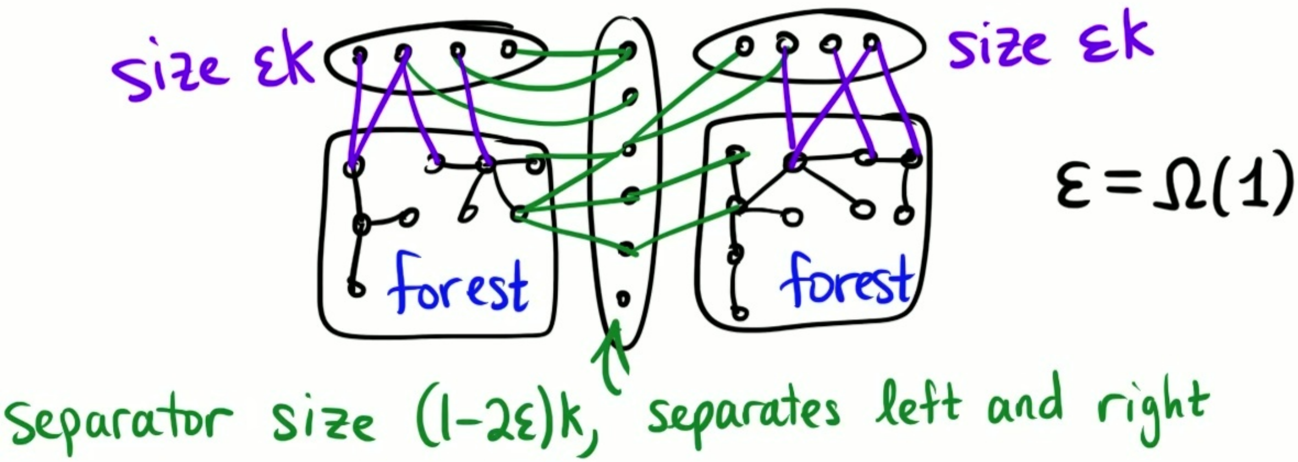

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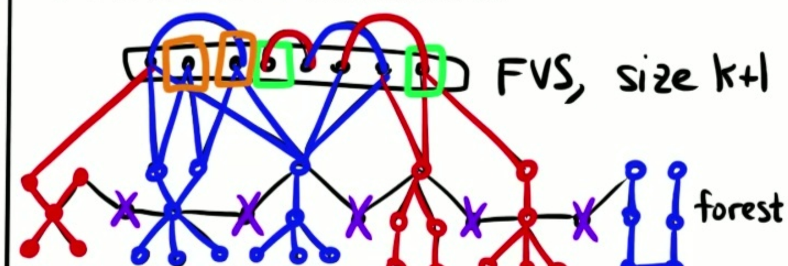

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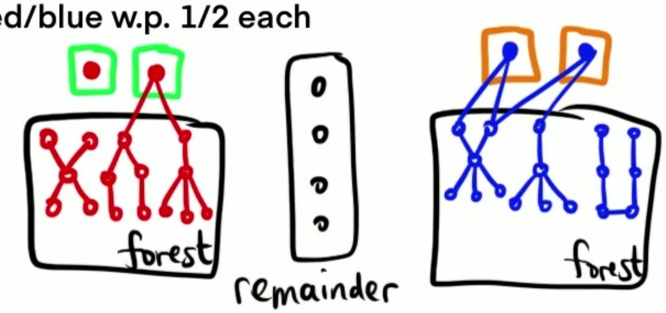

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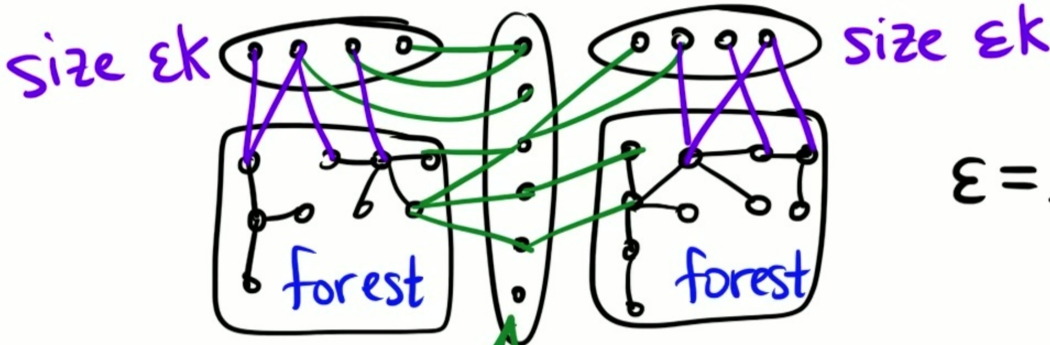

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$\Pr[\text{all incident edges red}] \geq 2^{-\deg(v)}$

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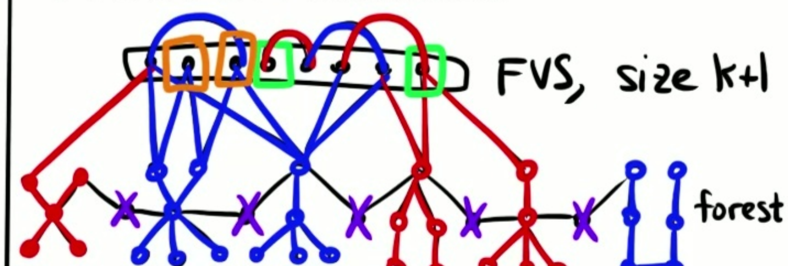

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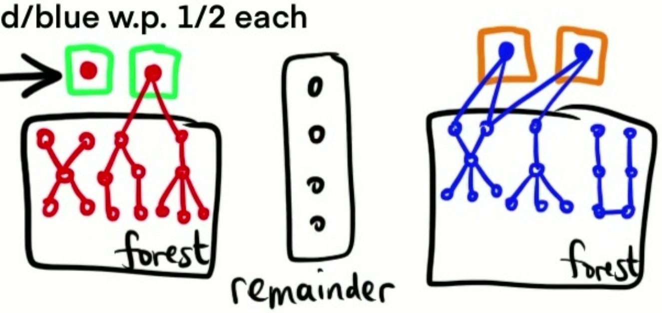

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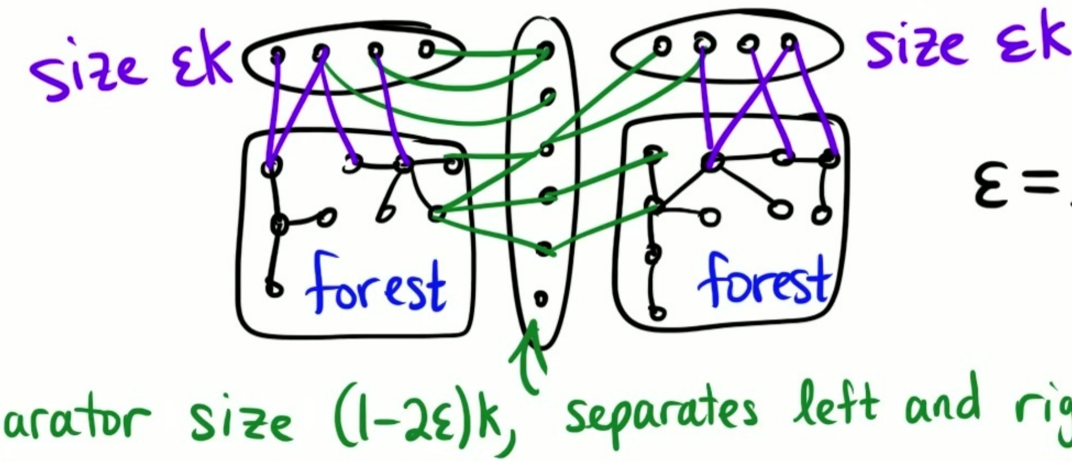

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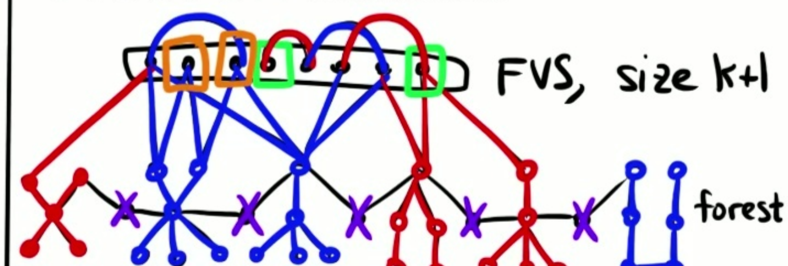

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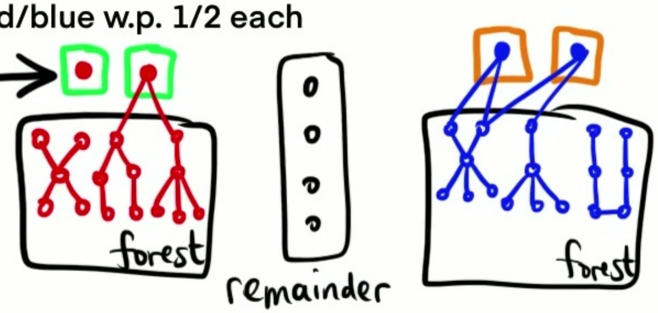

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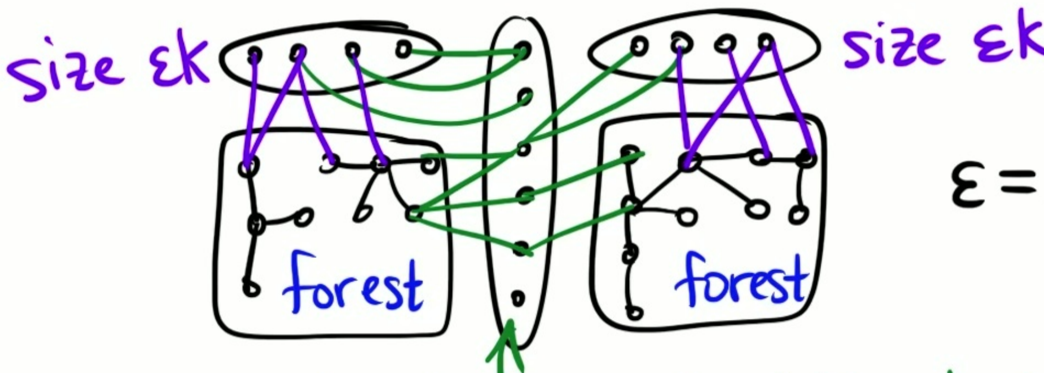

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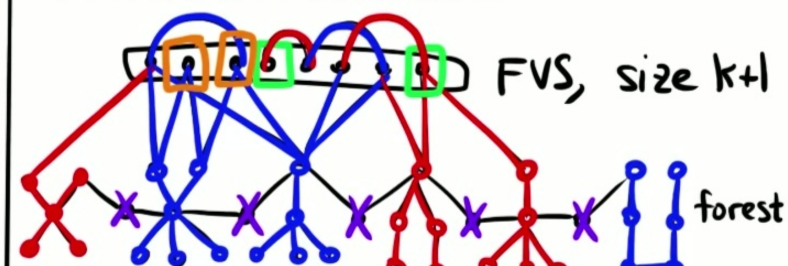

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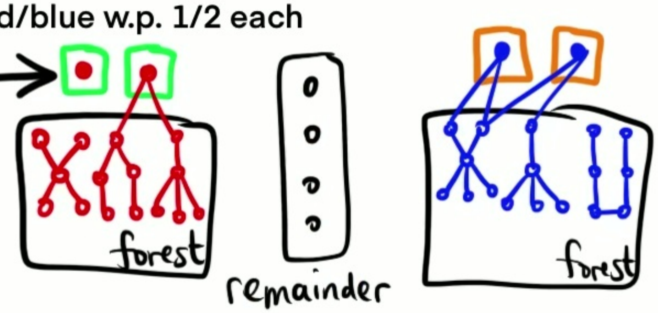

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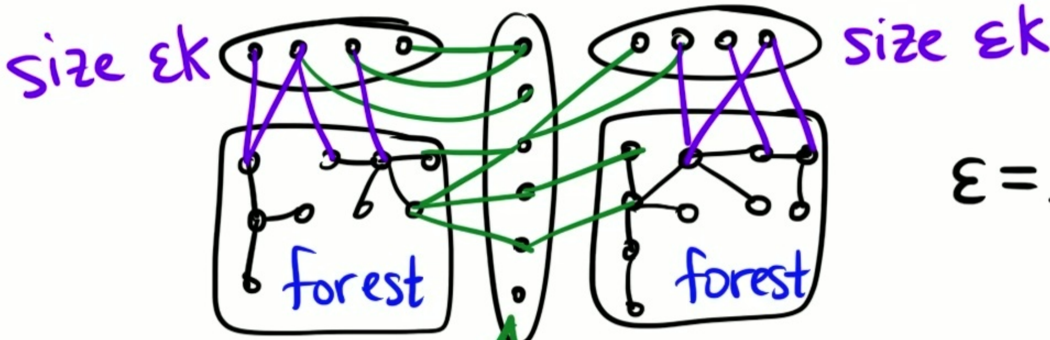

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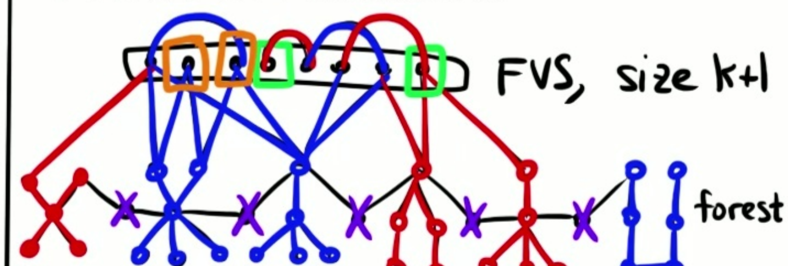

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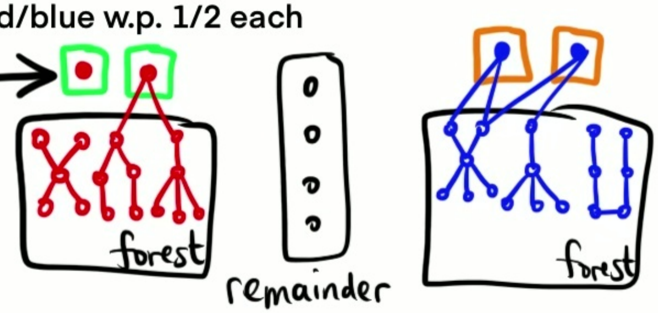

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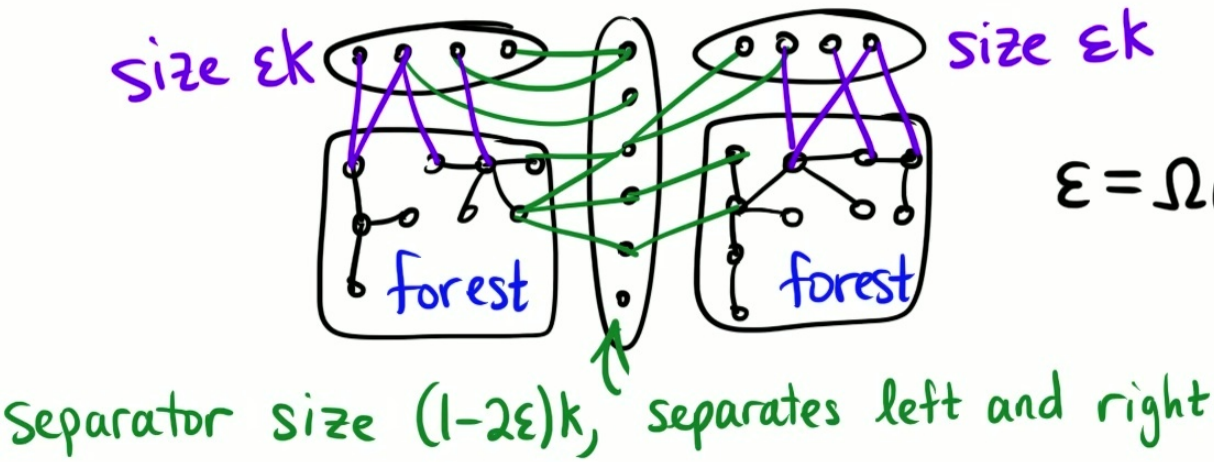

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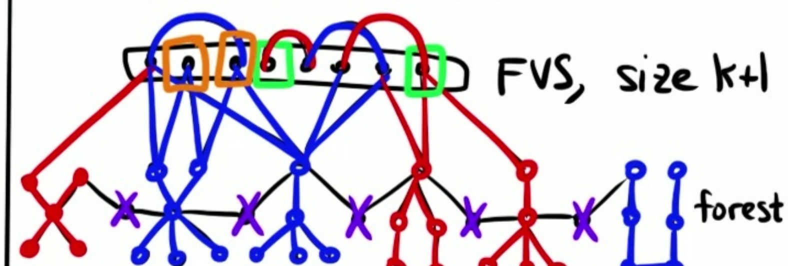

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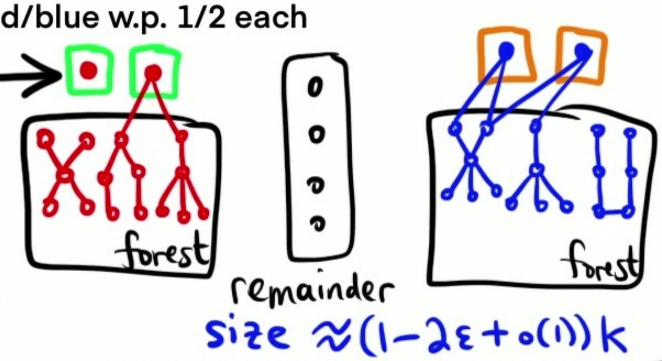

Chernoff bound (since each is "small"): # ≈ # ≈ εk for some $\varepsilon \ge 2^{-200}$

Speedup: O*(2.7^k) time

- Tighten (deg(v)-3) analysis and open 3^tw algorithm [CNP+11]
- [CNP+11] actually solves a counting problem
  - special arithmetic structure: speed up via fast matrix multiplication

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Open problems

- Our main conceptual message: 3^k can be broken (randomized)
  - Faster deterministic algorithm? [BBG'00] is inherently randomized
- 2^k possible?
- SETH lower bound? No 1.00001^k lower bound known!