Deterministic Mincut in Almost Linear Time

Jason Li (CMU)

Work done while visiting Microsoft Research, Redmond

Algorithms Group: Sivakanth Gopi, Janardhan Kulkarni, Jakub Tarnawski, Sam Wong

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or

A Structural Representation of the Cuts of a Graph

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Along the way: structural representation of cuts

Mincut by Sparsification + Tree Packing

Thm [Karger '96]: Suppose given a skeleton graph H s.t.

- H has O(m) edges
- The mincut of H is $n^{o(1)}$
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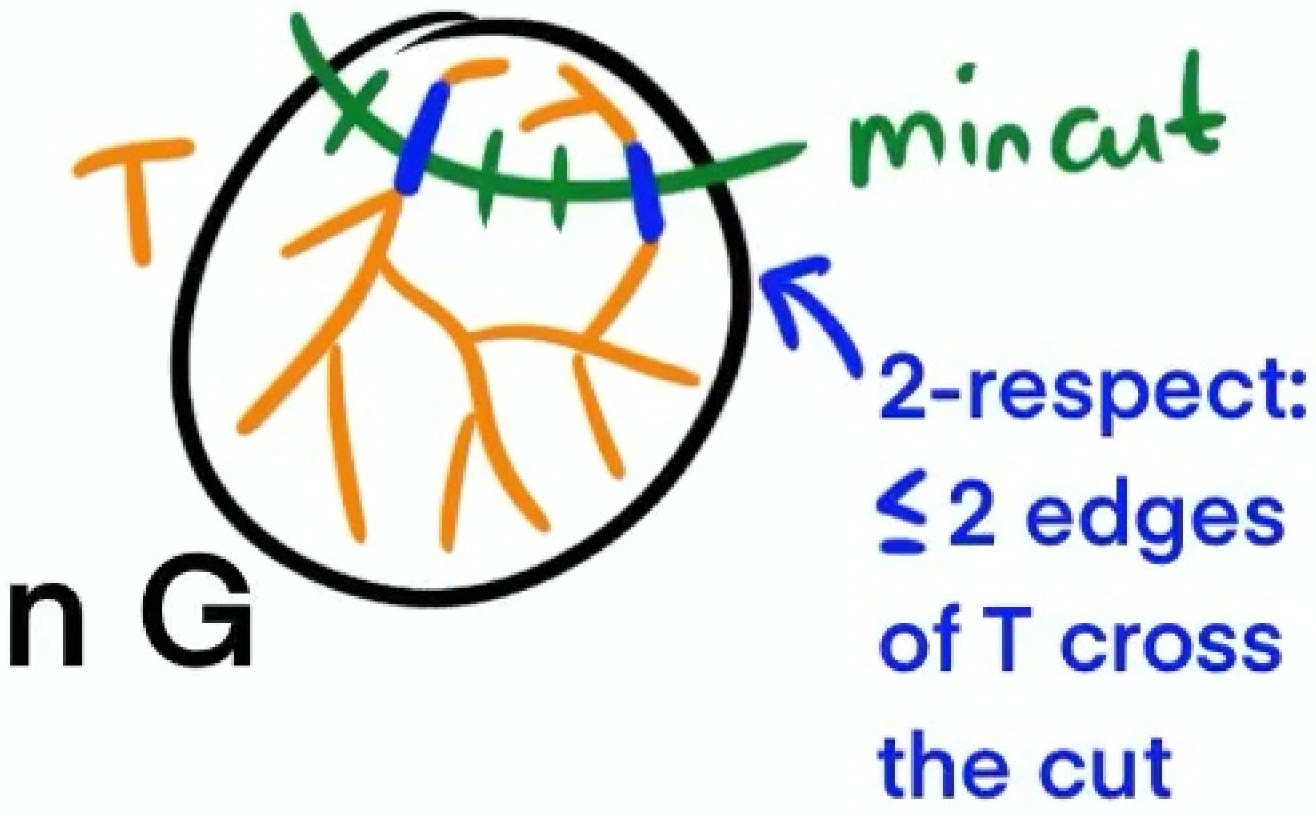

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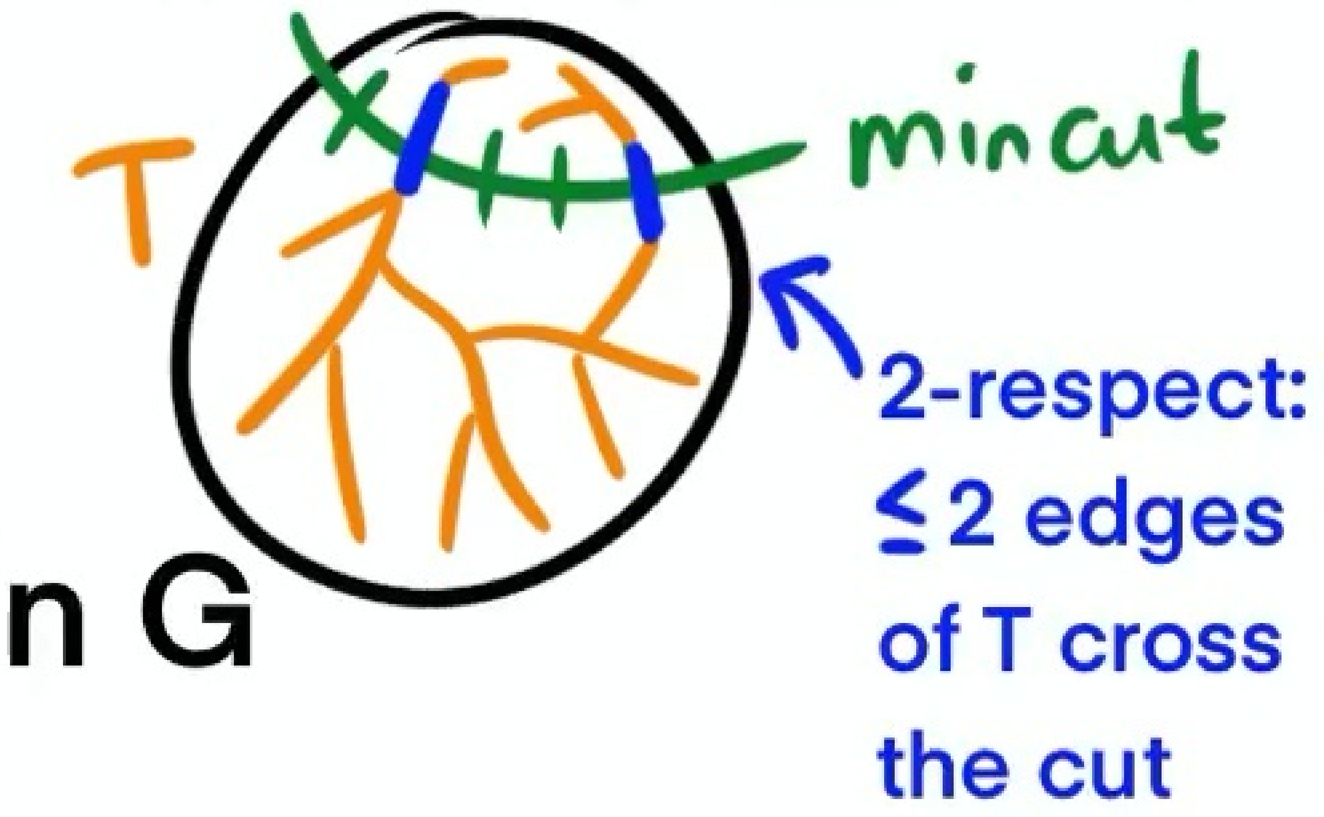

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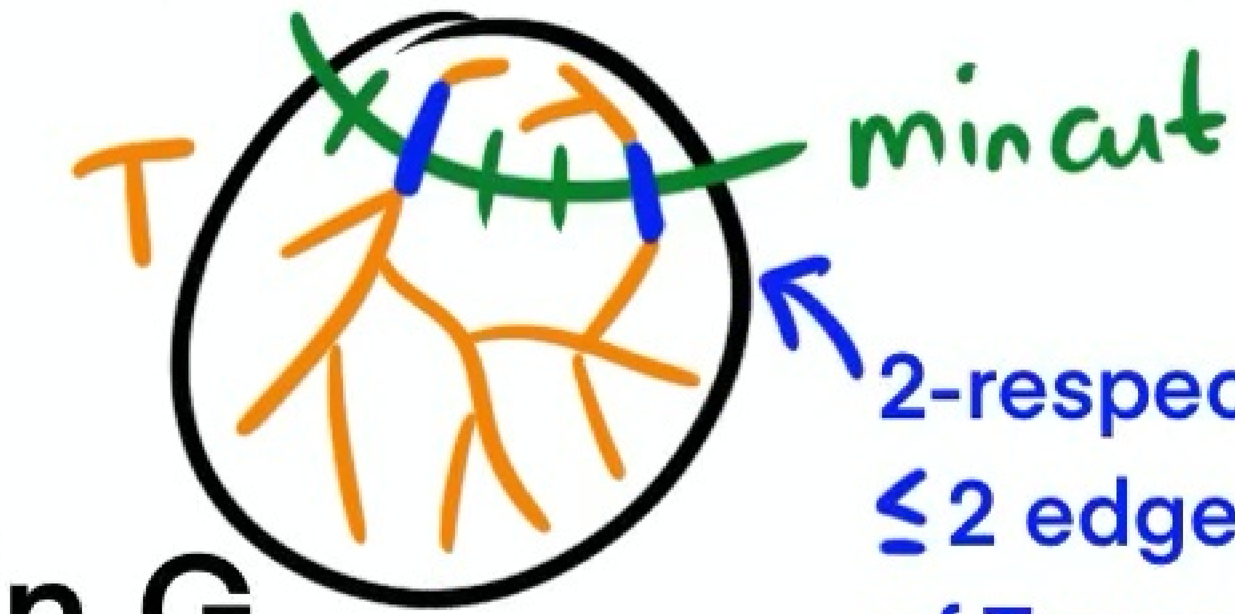

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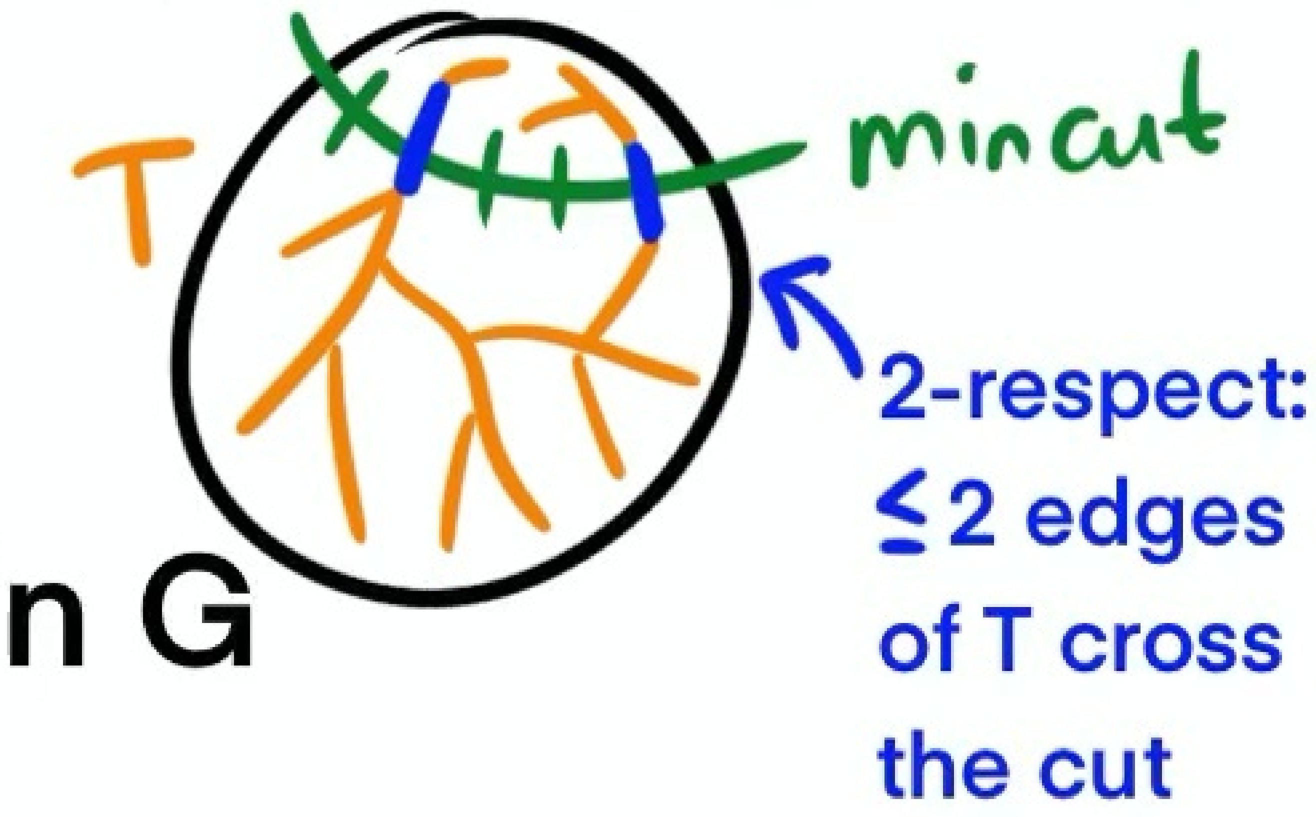

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(1+ε) approximate cut sparsifier suffices: $\exists W$ s.t. $\forall S$: $W \cdot |\partial_H S| \approx (1 \pm \varepsilon)|\partial_G S|$

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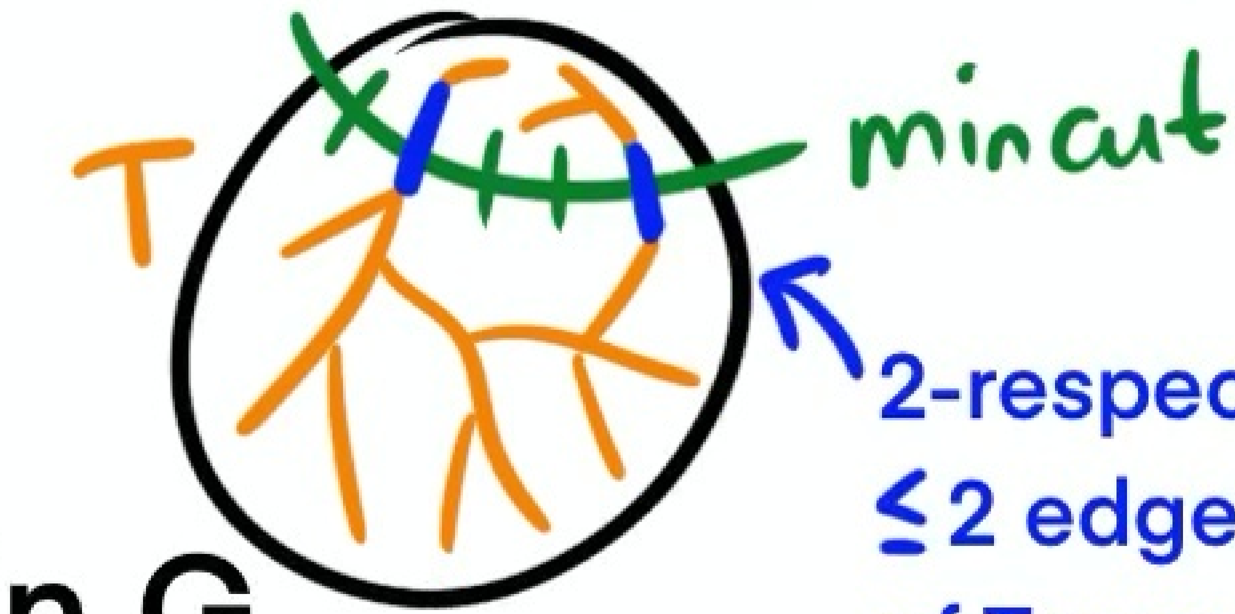

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Sample each edge in G with prob $p := \frac{100 \log n}{\varepsilon^2 \lambda}$. Let H = sampled edges

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- union bound over α: $\sum_{\alpha \geq 1} \frac{1}{n^{\alpha}} = O\left(\frac{1}{n}\right)$.

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Derandomization: structural representation of target objects

Structural Representation of Cuts

Spectral approach: H is a $(1+\varepsilon)$-approximate cut sparsifier of G if

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Laplacian matrix of G

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This work: combinatorial representation via expander decomposition

Structural Representation: Roadmap

1. Expander case: why are expanders easy?

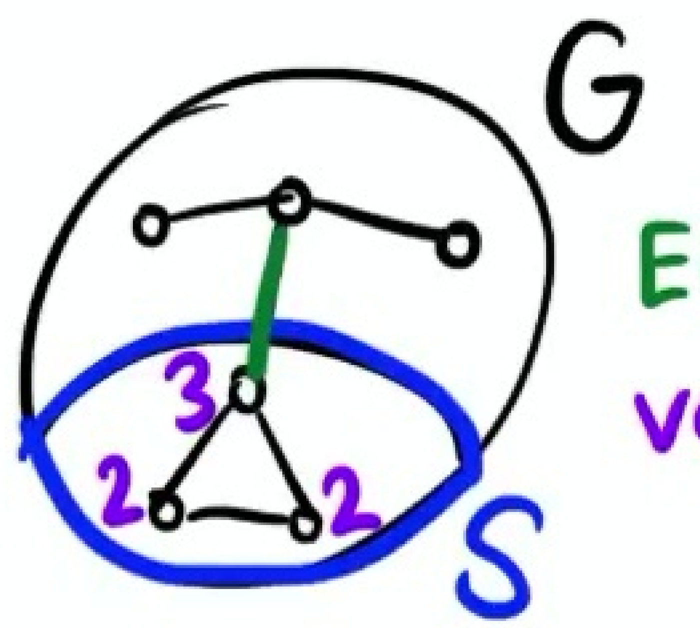
2. “Expander of expanders”: how to generalize?

3. Expander decomposition and additional challenges

Expanders

Conductance of a graph: $\Phi(G) = \min\limits_{\substack{S \subseteq V \\ \mathrm{vol}(S) \le \mathrm{vol}(V \setminus S)}} \dfrac{|E(S, V\setminus S)|}{\mathrm{vol}(S)}$

"volume" of S: sum of degrees in S



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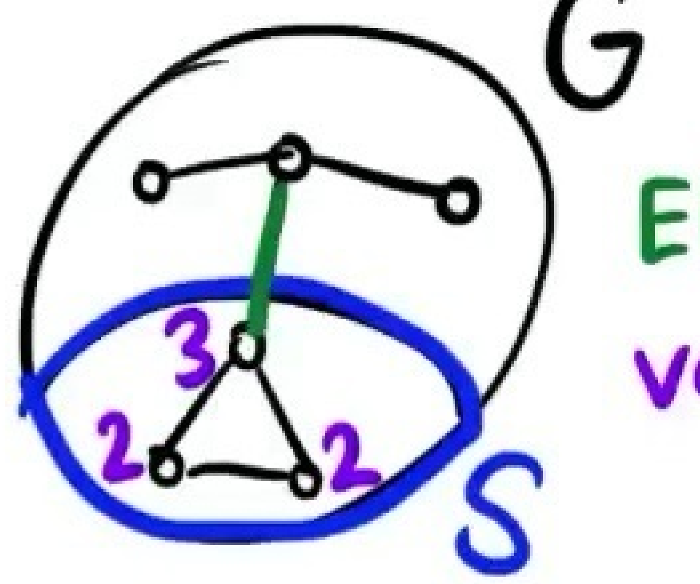

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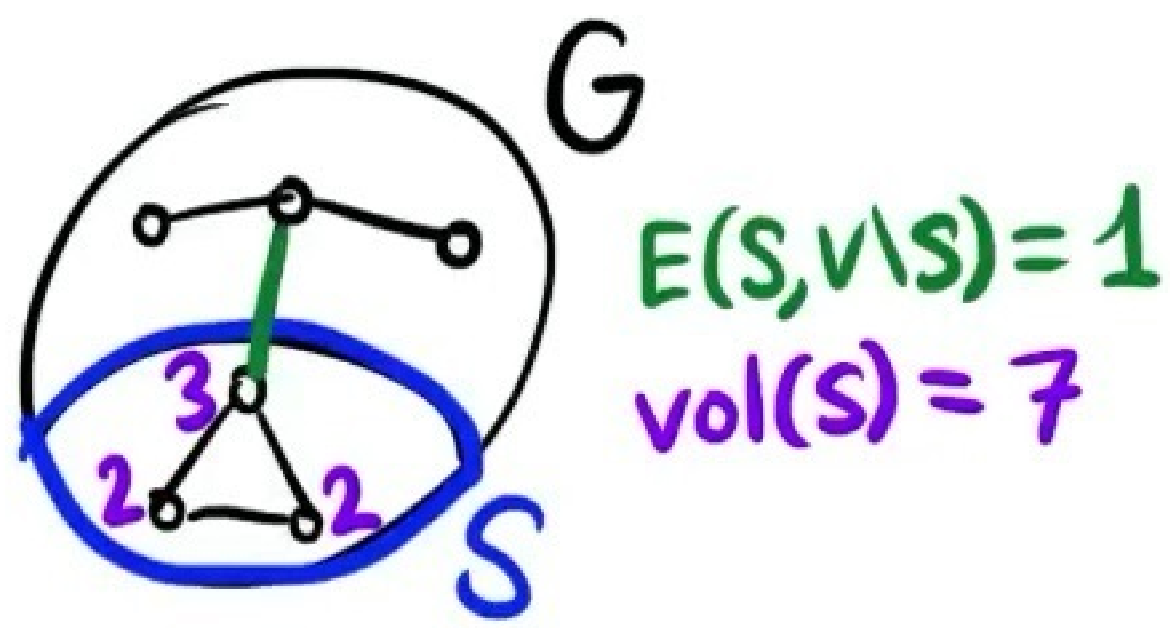

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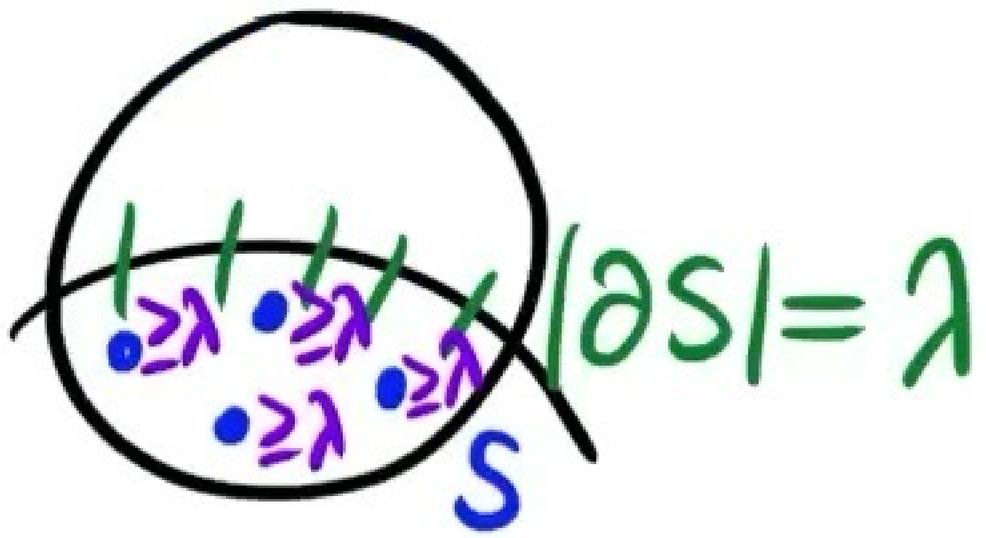

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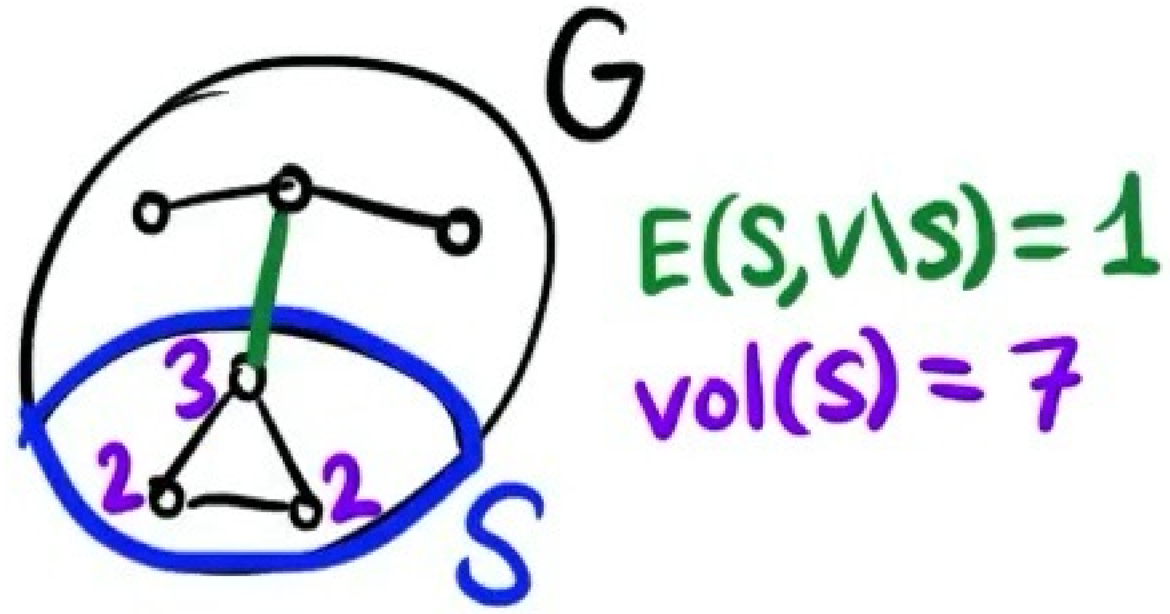

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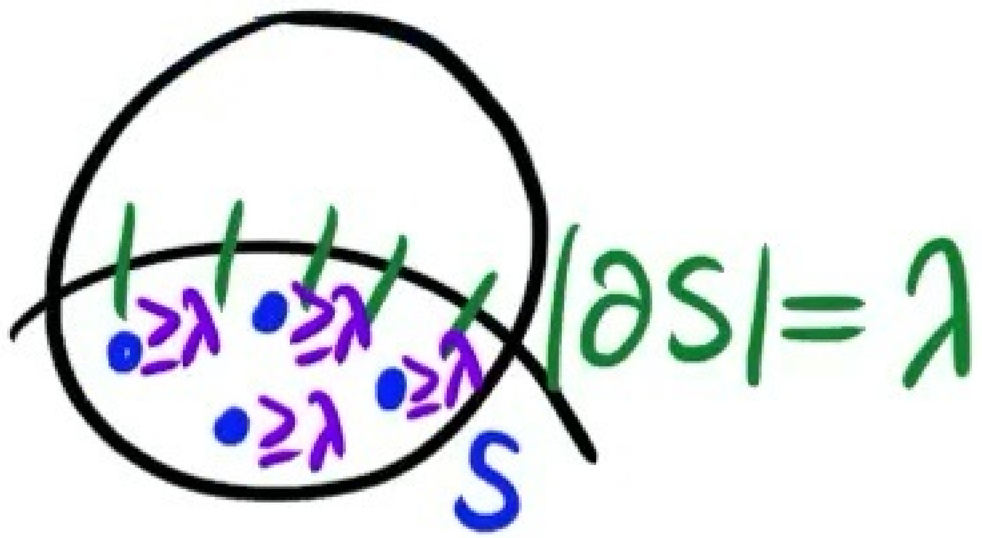

$\phi \leq \Phi(G) \leq \frac{|E(S, V\setminus S)|}{\mathrm{vol}(S)} \leq \frac{\alpha\lambda}{\lambda|S|} \iff |S| \leq \alpha/\phi$

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First goal: ensure that $|\partial_H S| \approx_{(1+\varepsilon)} p|\partial_G S|$ for all unbal. cuts $\partial S : |S| \leq \frac{\alpha}{\phi}$

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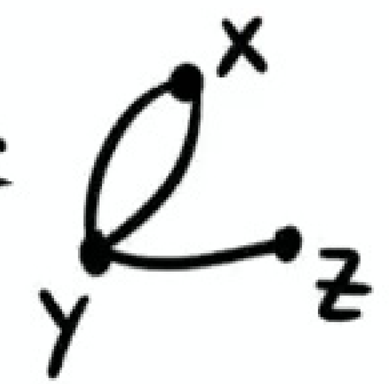
Proof:

Graph Laplacian: algebraic representation of cuts

$G =$ 

$$L_G = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} +2 & -2 & 0 \\ -2 & +3 & -1 \\ 0 & -1 & +1 \end{bmatrix} \end{array}$$

(y,y): +deg(y)

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Derandomization: Unbalanced Cuts

First goal: ensure that $|\partial_H S| \approx_{(1+\varepsilon)} p|\partial_G S|$ for all unbal. cuts $\partial S: |S| \le \frac{\alpha}{\phi}$

(includes all α-approximate mincuts for a ϕ-expander)

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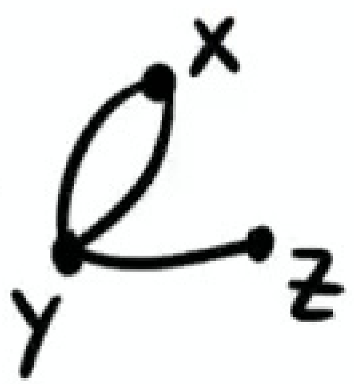
$G =$ 

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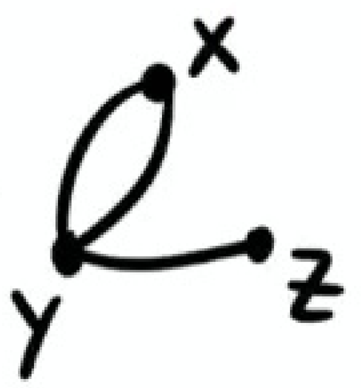
G = 

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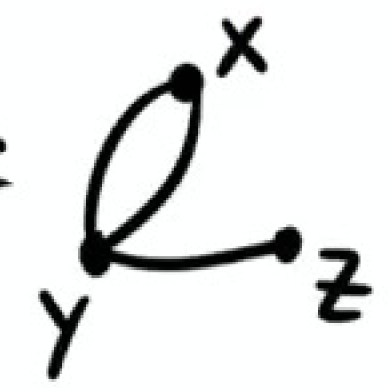
$G =$ 

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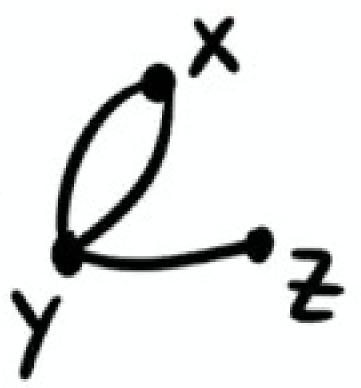
$G =$ 

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Reduced verification to m+n constraints

Efficient algorithm via pessimistic estimators:

- Compute $\widetilde{\Pr}(v \text{ fail})$: Chernoff bound of $\Pr[\deg_H(v) \not\approx p \cdot \deg_G(v)]$

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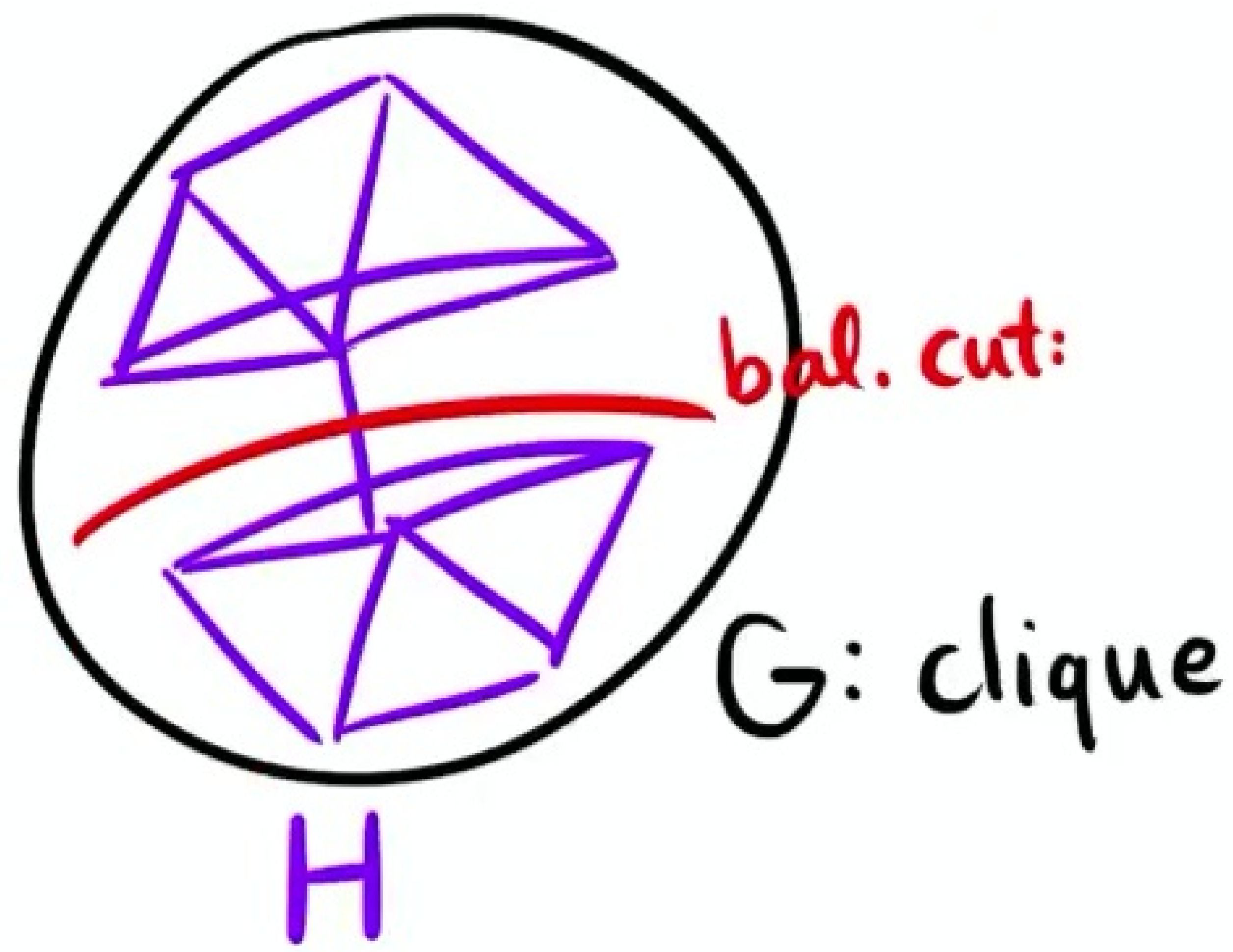
  $\sum_v \widetilde{\Pr}(v \text{ fail}) + \sum_{u,v} \widetilde{\Pr}(u,v \text{ fail}) \ll 1$
- given edge e, update $\widetilde{\Pr}(\cdot)$ as $\widetilde{\text{prob.}}$ conditional on choosing/skipping e

  (only need to update 3 terms)
- choose/skip e depending on which is smaller

Balanced Cuts

H preserves unbalanced cuts,
but not balanced cut!



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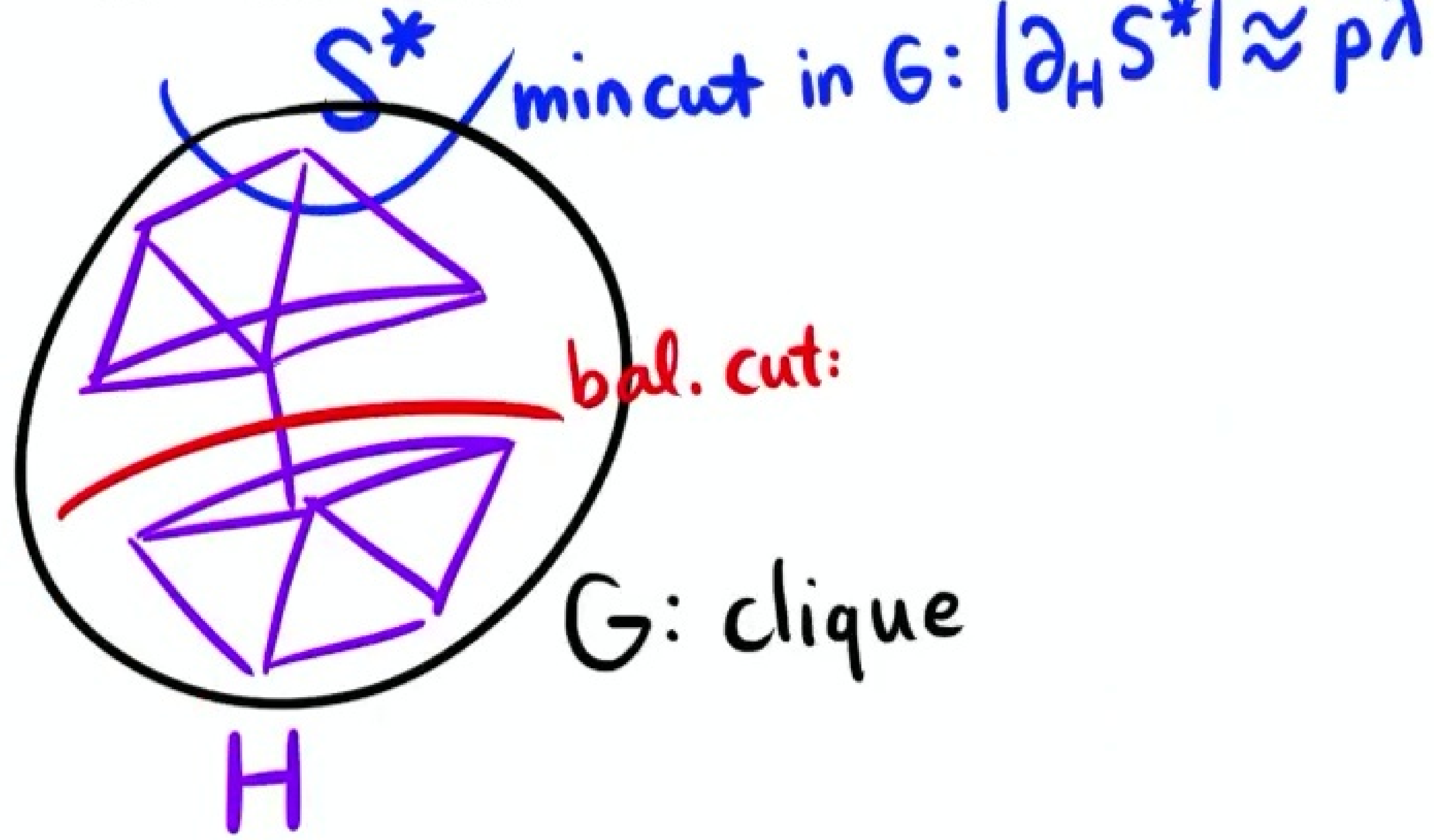

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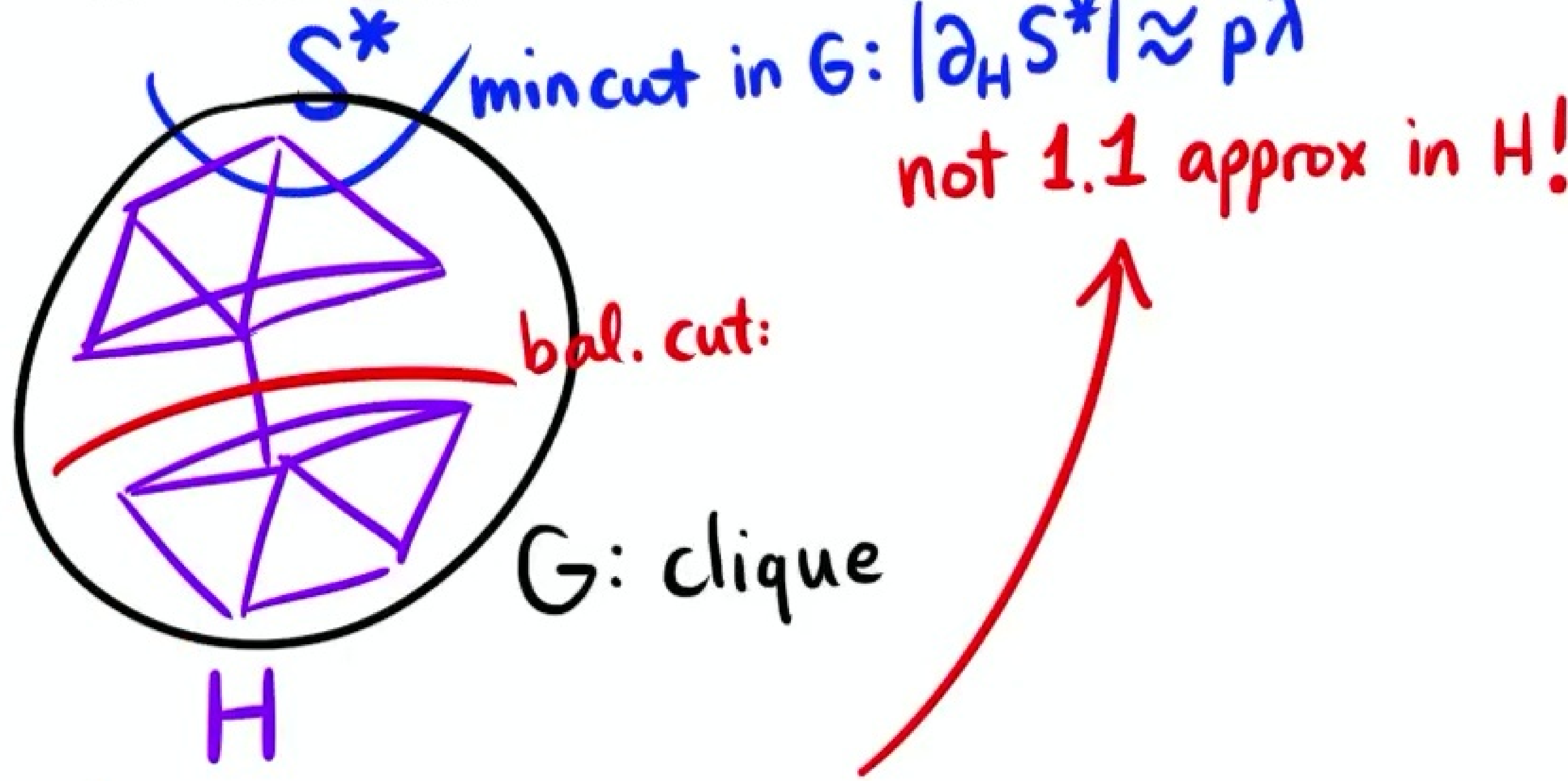

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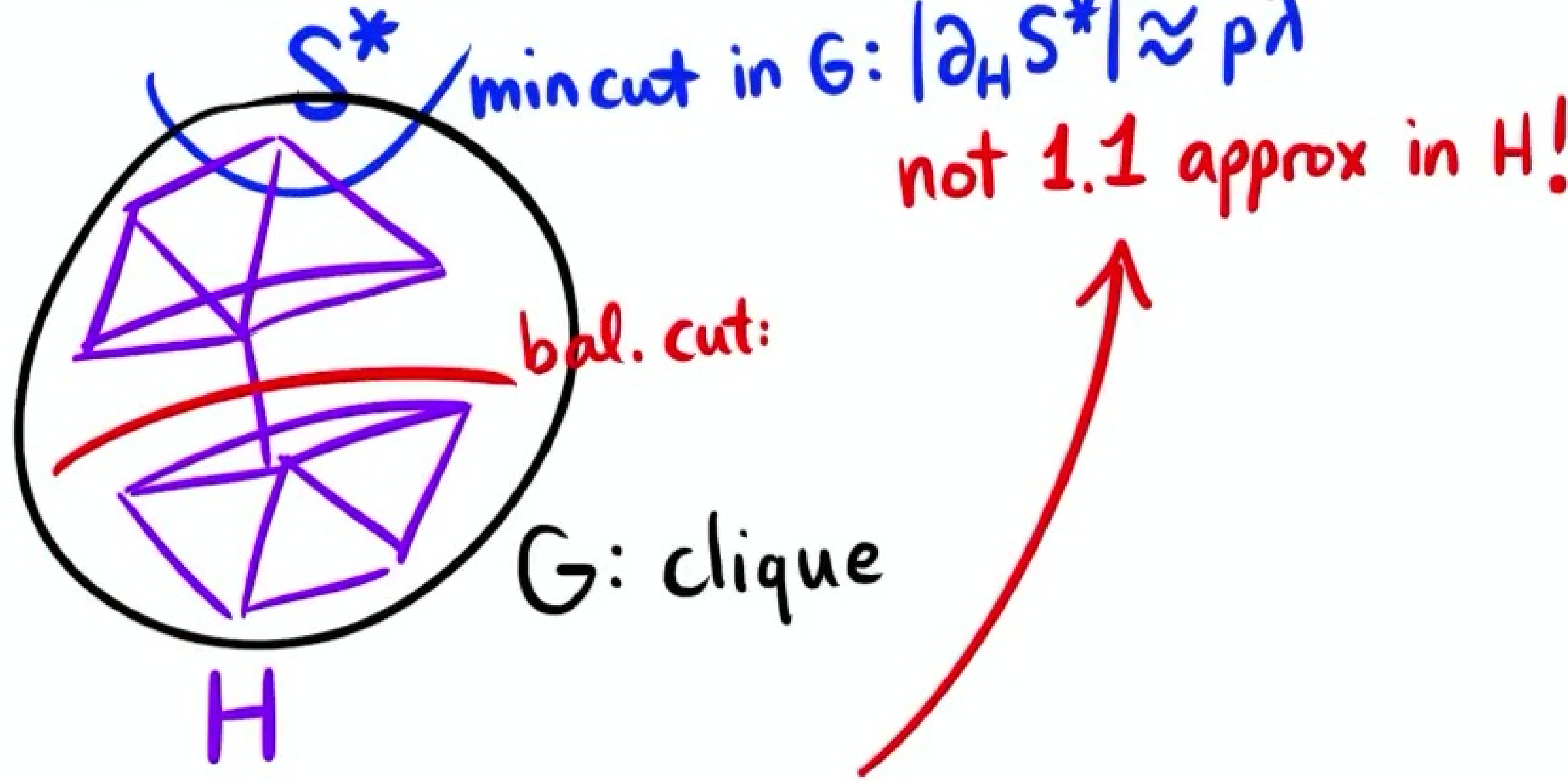

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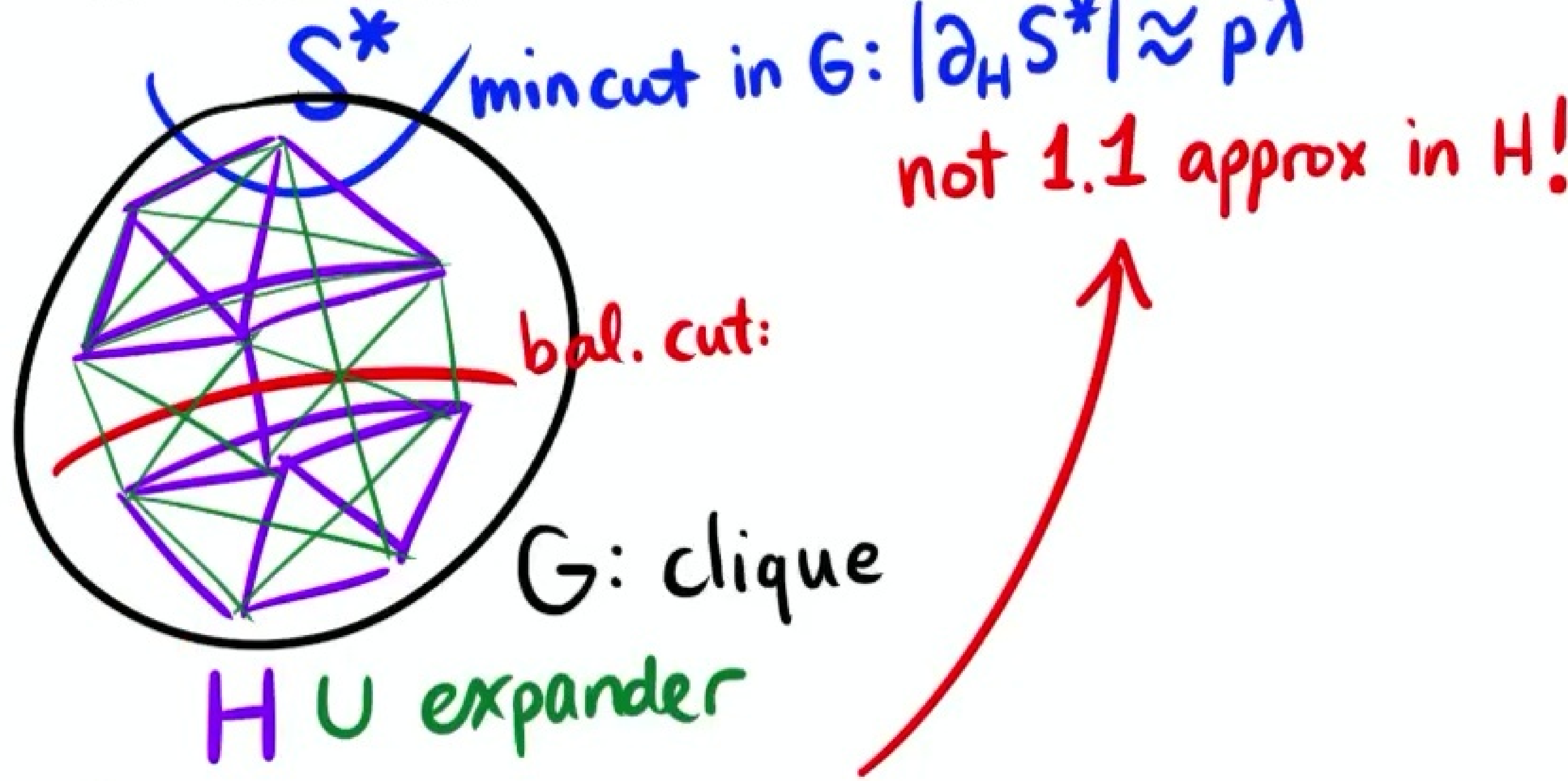

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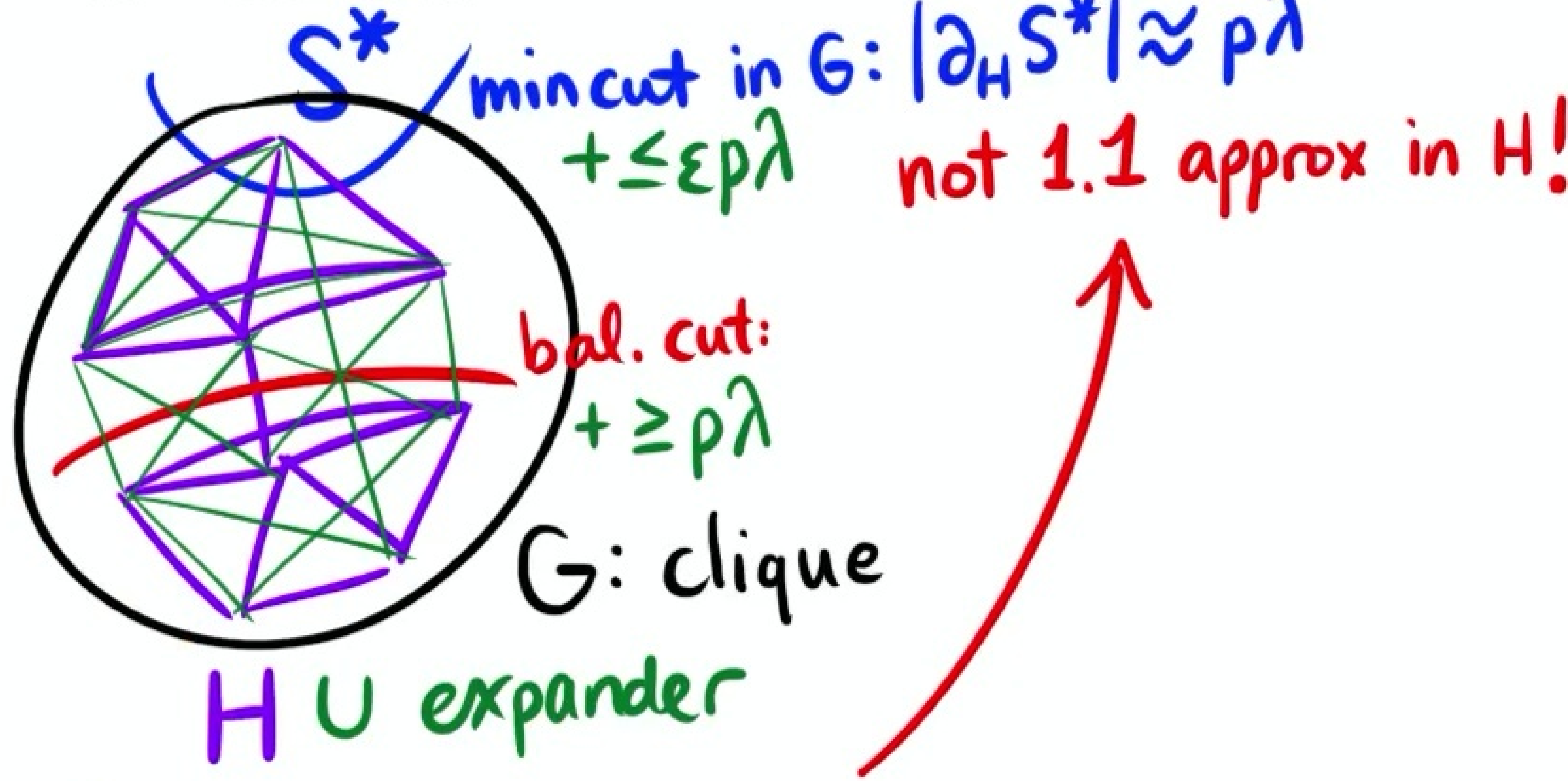
- any balanced cut increases by $\geq \rho\lambda$

Balanced Cuts



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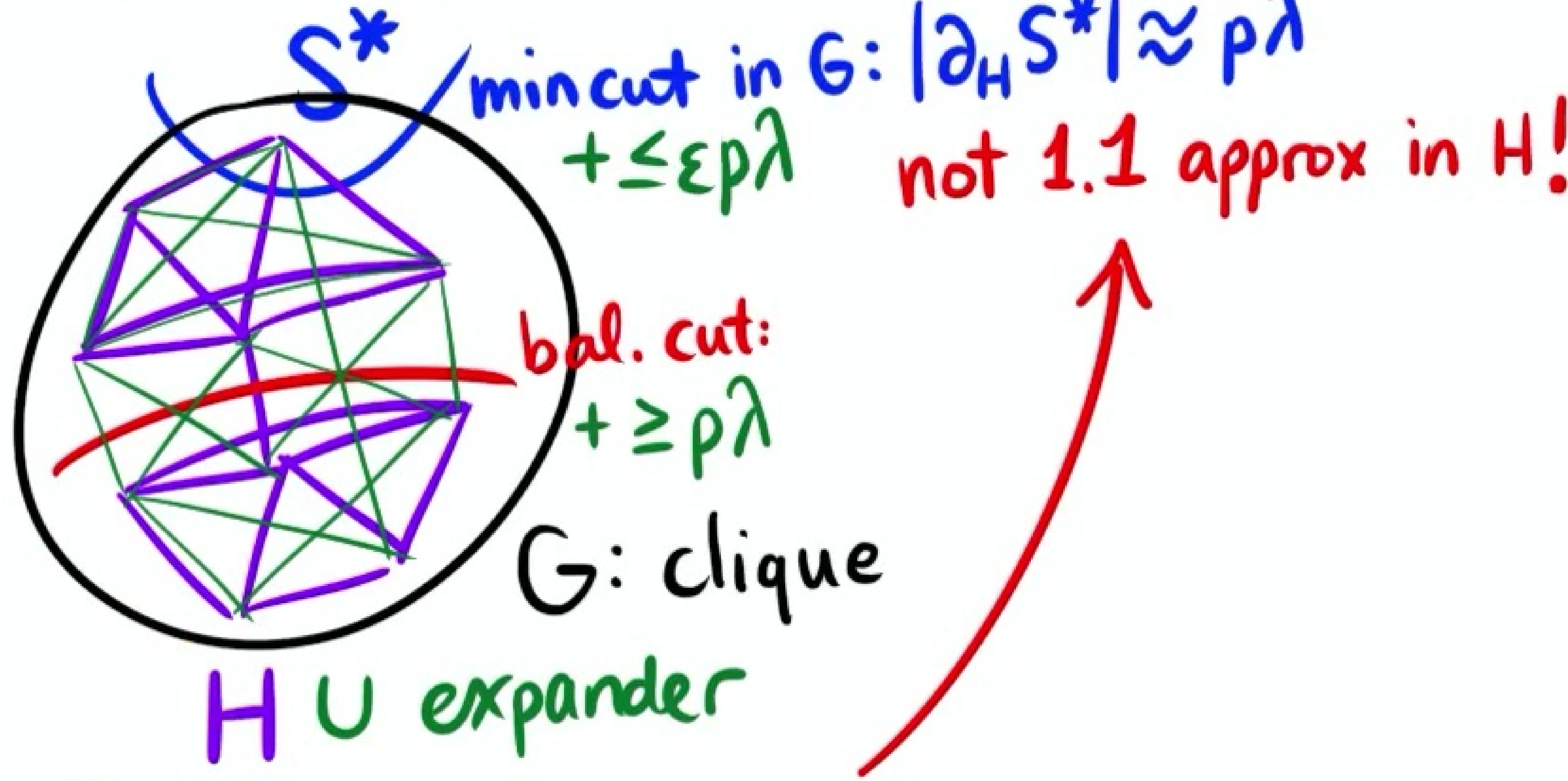

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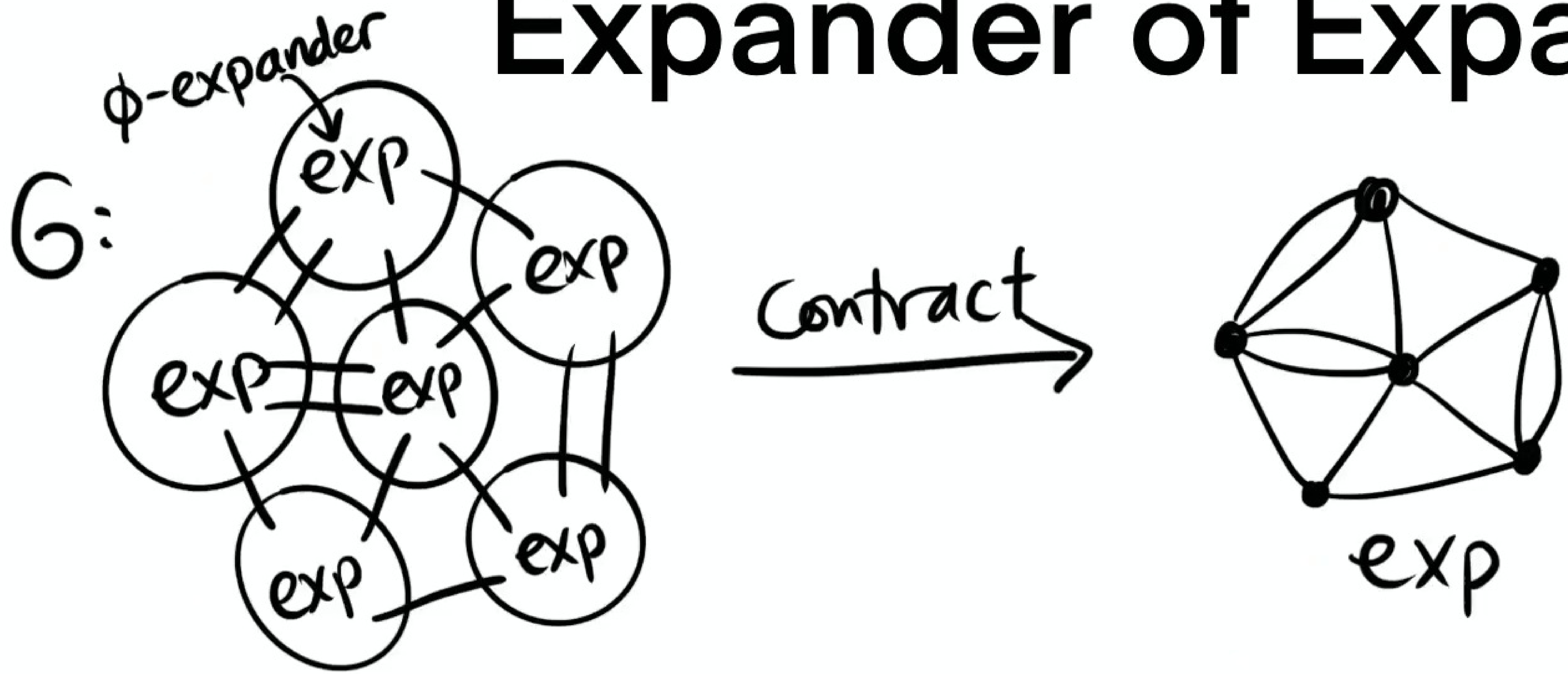
Not a (1+ε)-approximate cut sparsifier, but OK for mincut

Expander: Recap

Preserve all unbalanced cuts up to $(1 \pm \varepsilon)$ by preserving degrees and parallel edges

Force balanced cuts to be large by overlaying an arbitrary expander

Expander of Expanders



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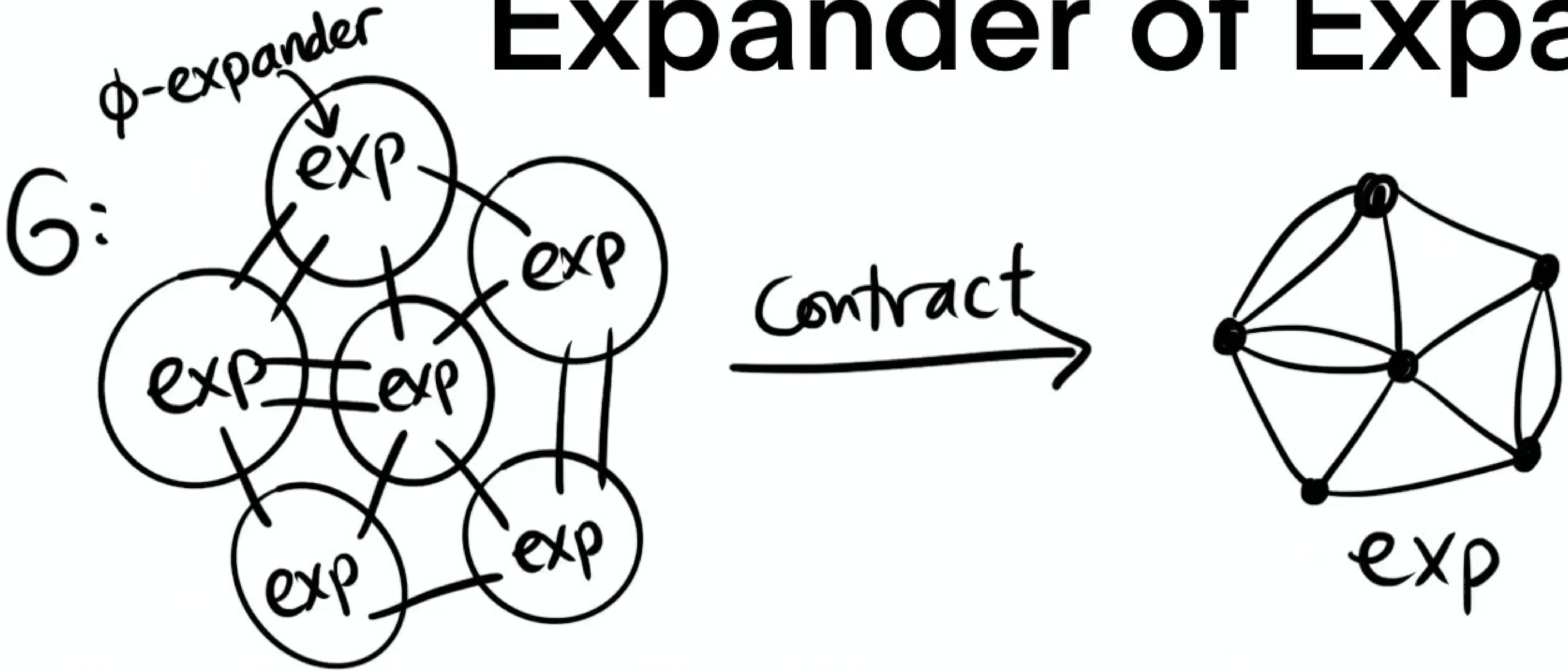

Expander decomposition of G:

partition V into $V_1,\dots,V_k$ s.t.

$G[V_i]$ is an expander for all i

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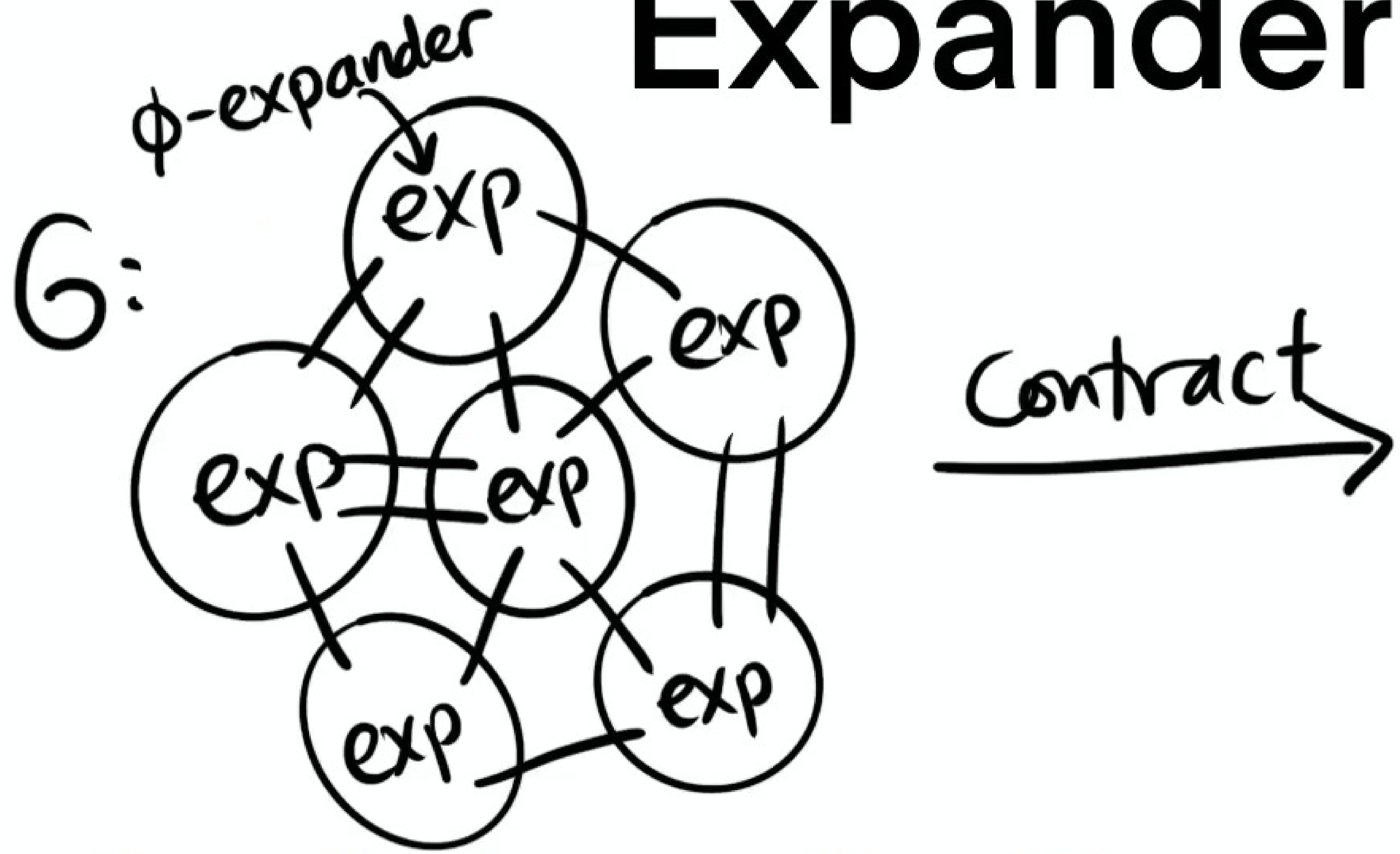

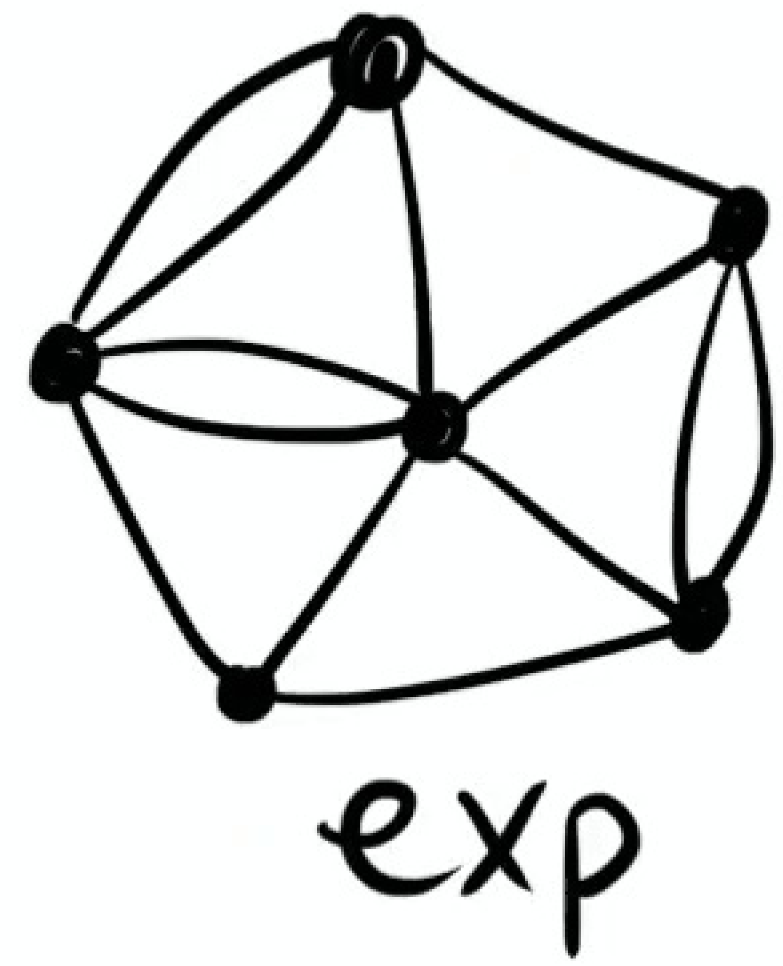

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How to define unbalanced?

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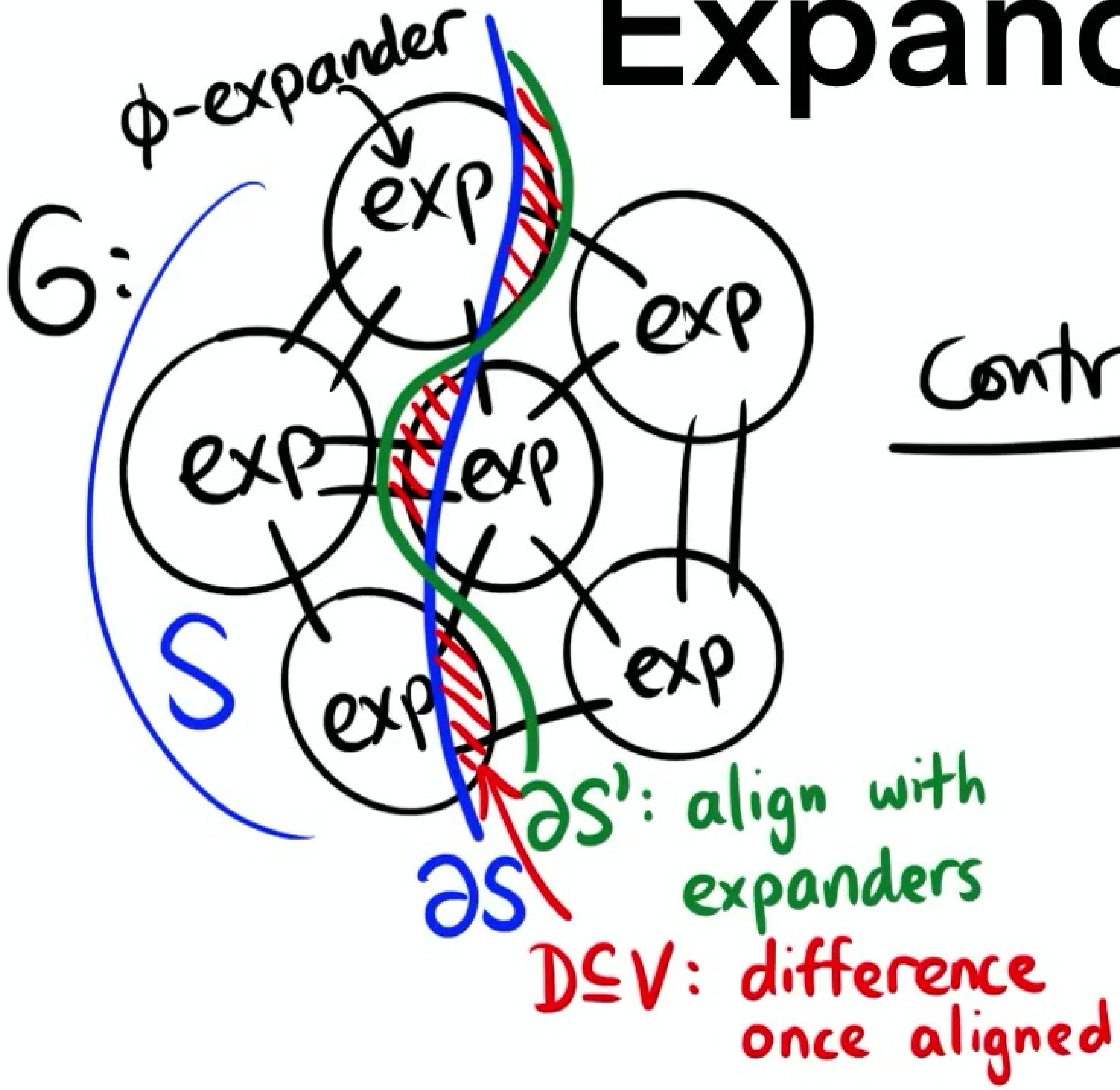

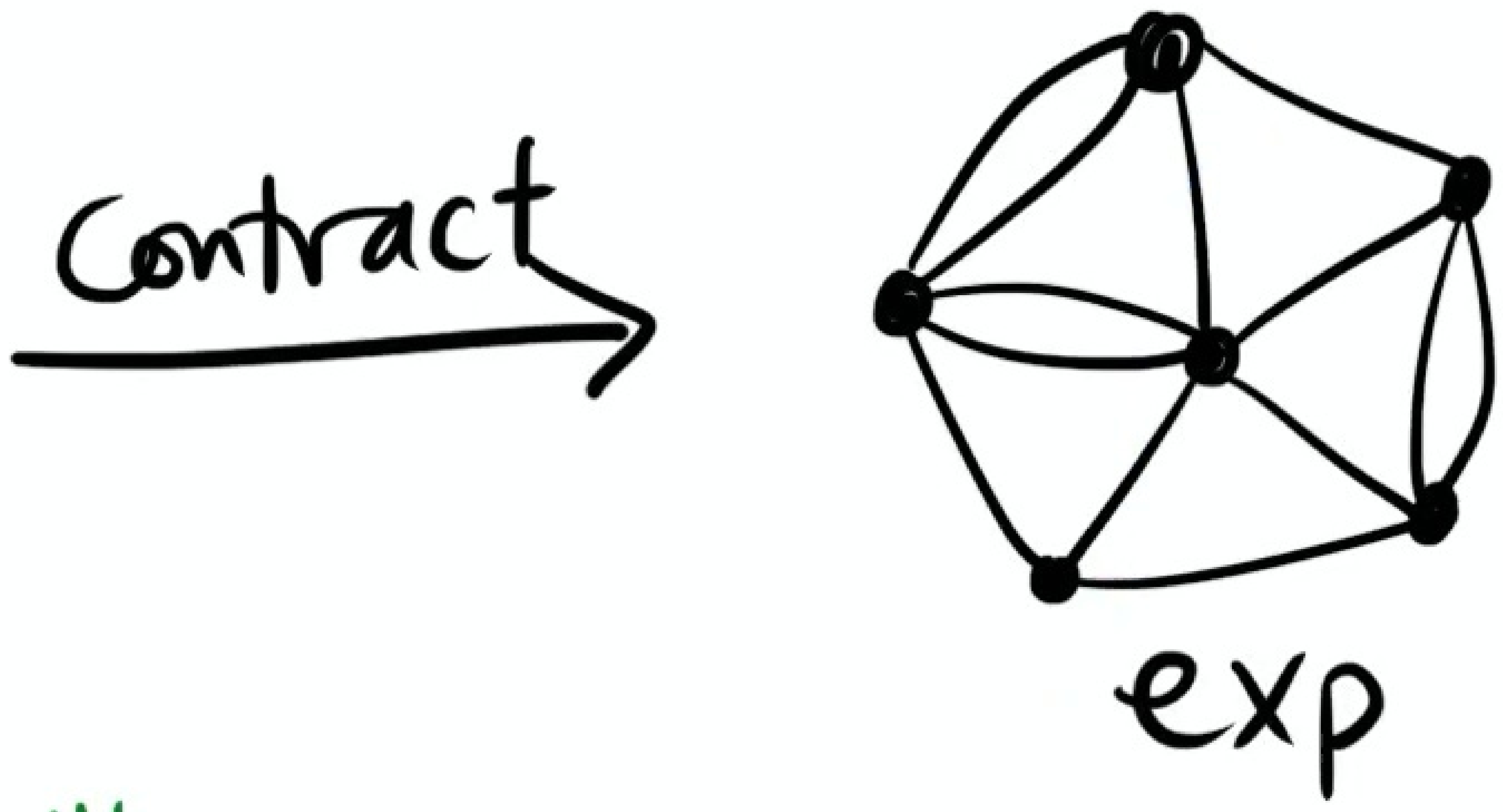

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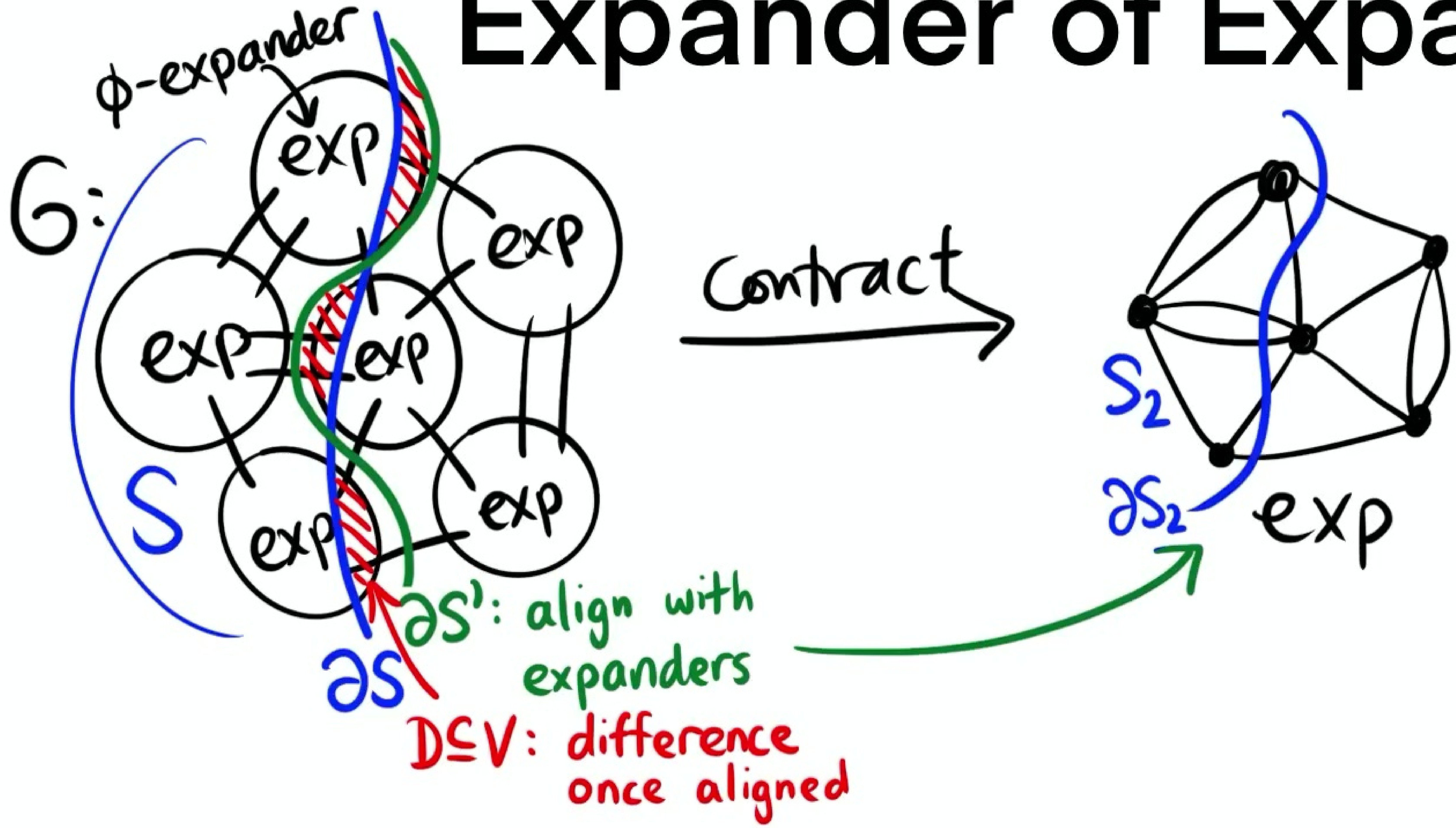

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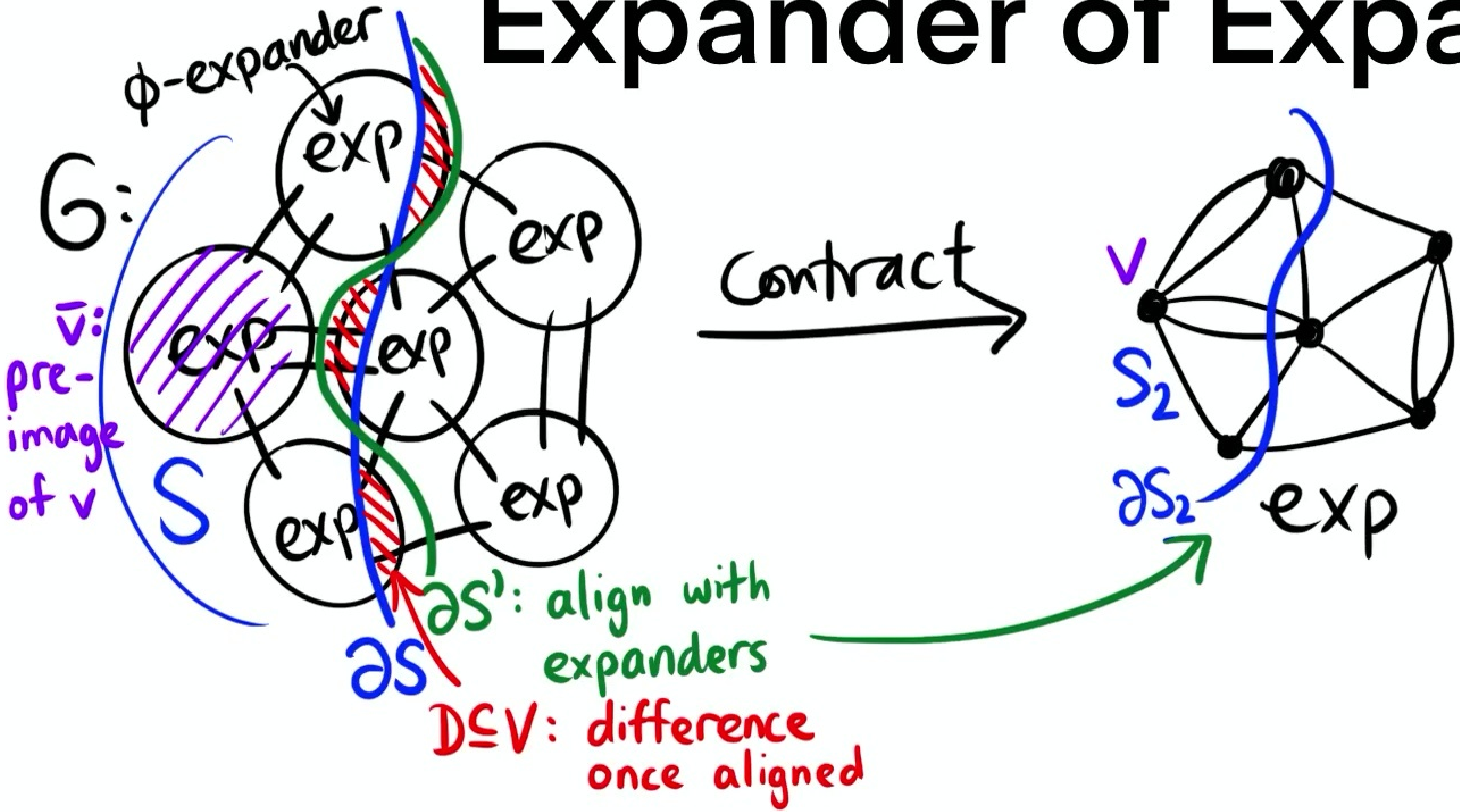

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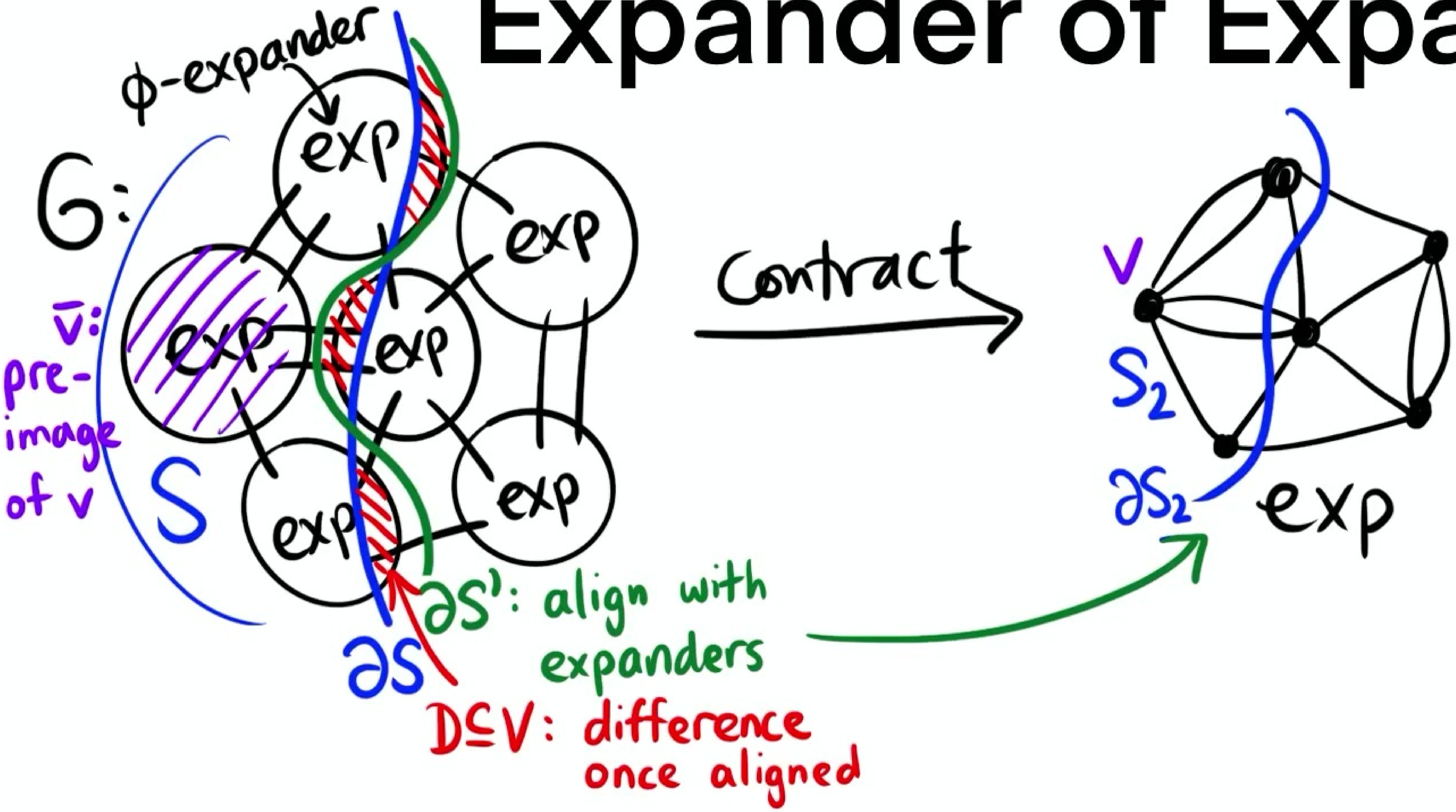

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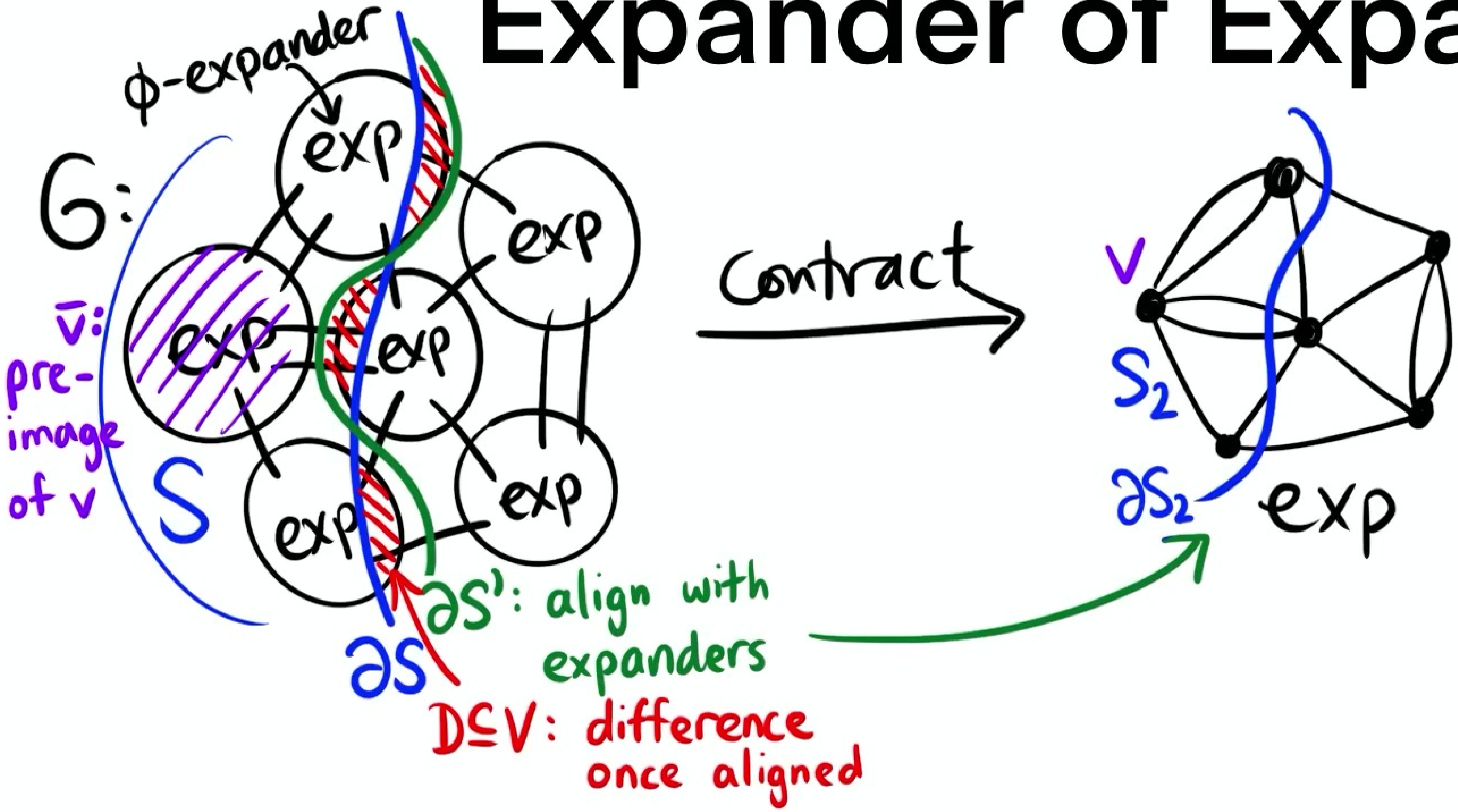

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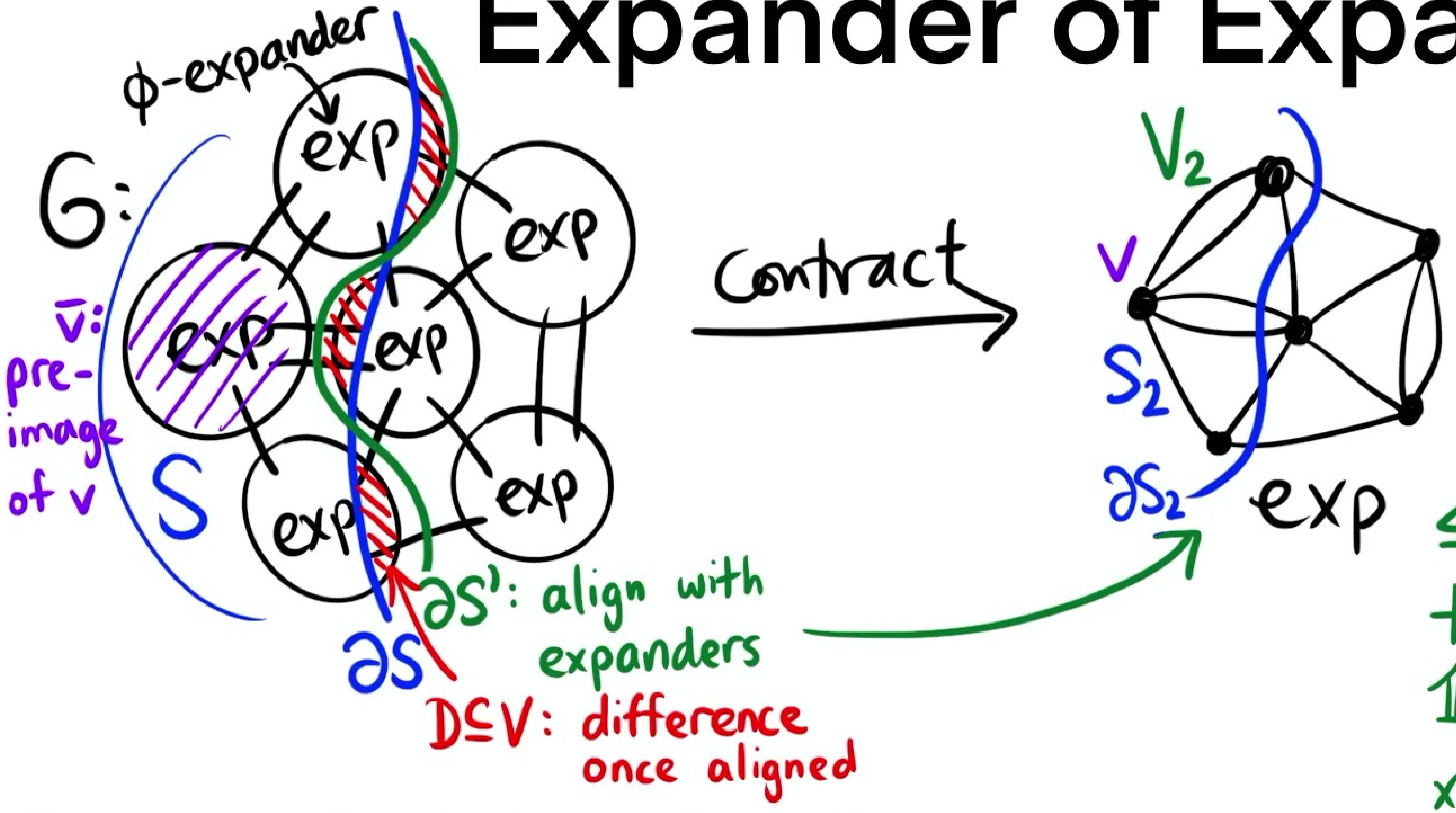

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$\le (2\alpha/\phi)^2$ terms $\mathbb{1}_x^T L_G \mathbb{1}_y$ for

$$x, y = \begin{cases} \bar{v} \text{ for some } v \in V_2 \\ u \text{ for some } u \in V \end{cases}$$

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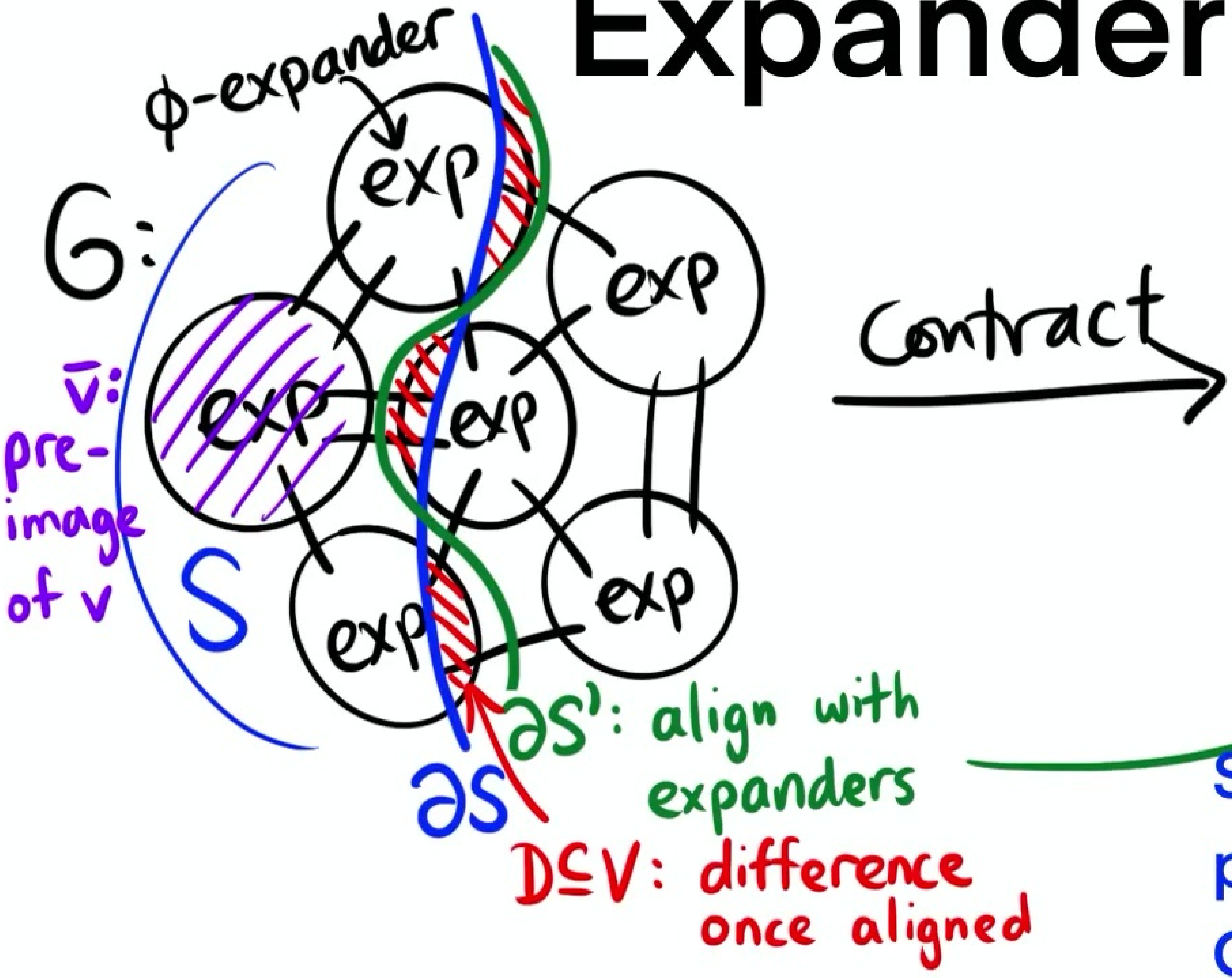

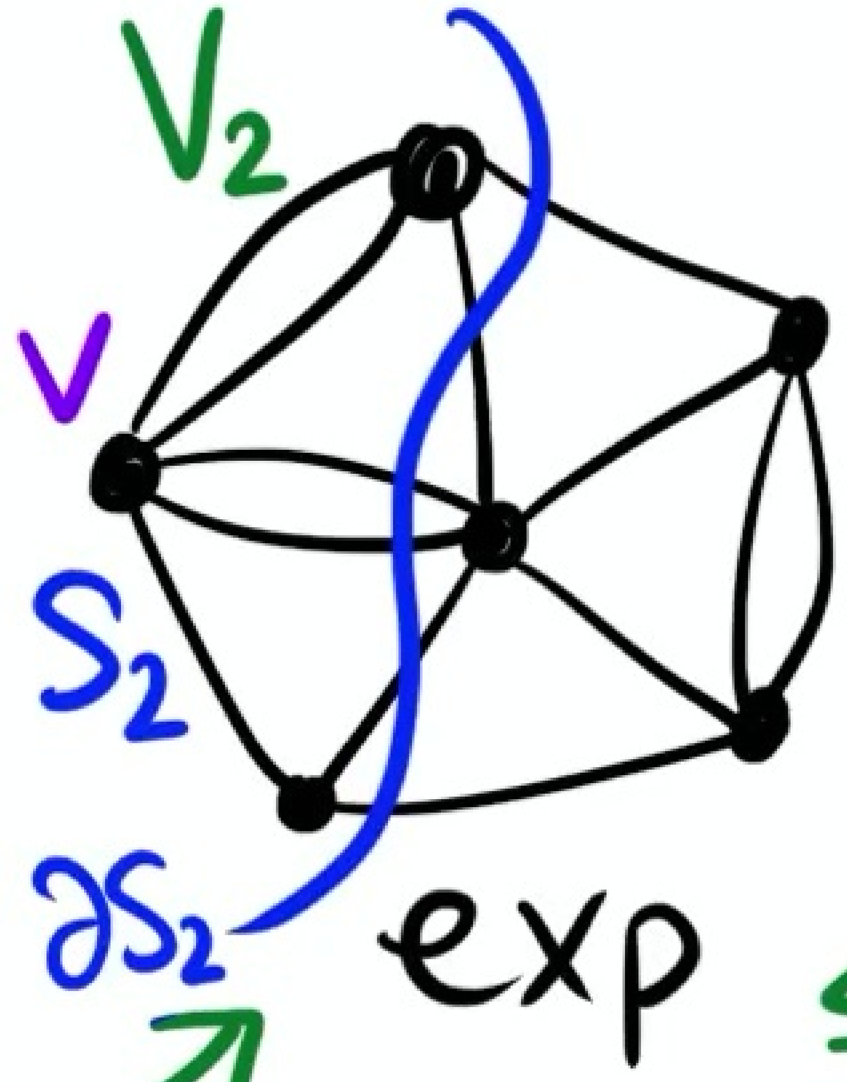

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$\le (2\alpha/\phi)^2$ terms

Suffices to preserve these!

$\mathbb{1}_x^T L_G \mathbb{1}_y$ for $\left[x,y = \begin{cases} \bar{v} & \text{for some } v\in V_2 \\ u & \text{for some } u\in V \end{cases} \right.$

O(m+n) total:

each edge e ∈ E belongs to ≤4 of them

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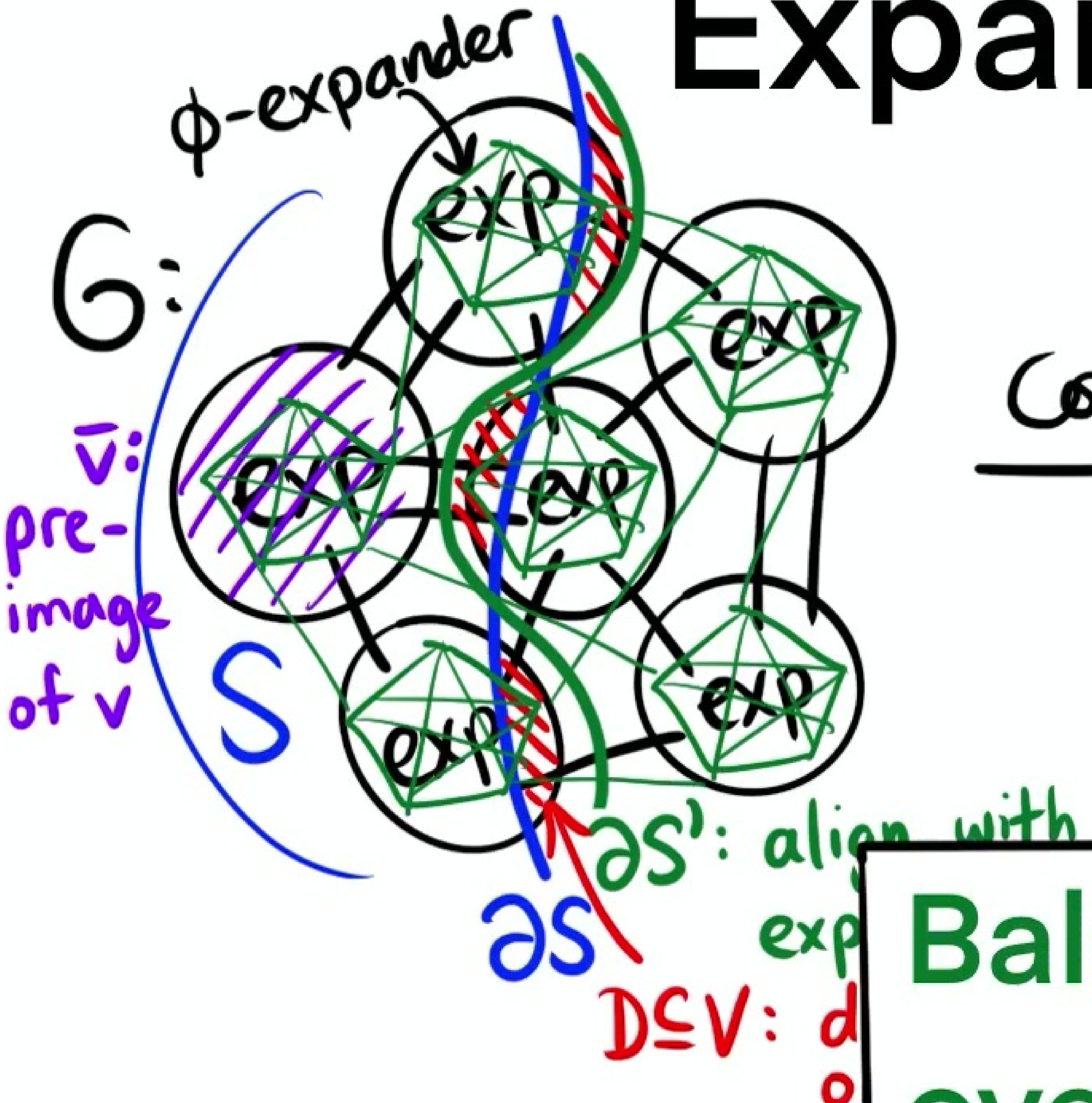
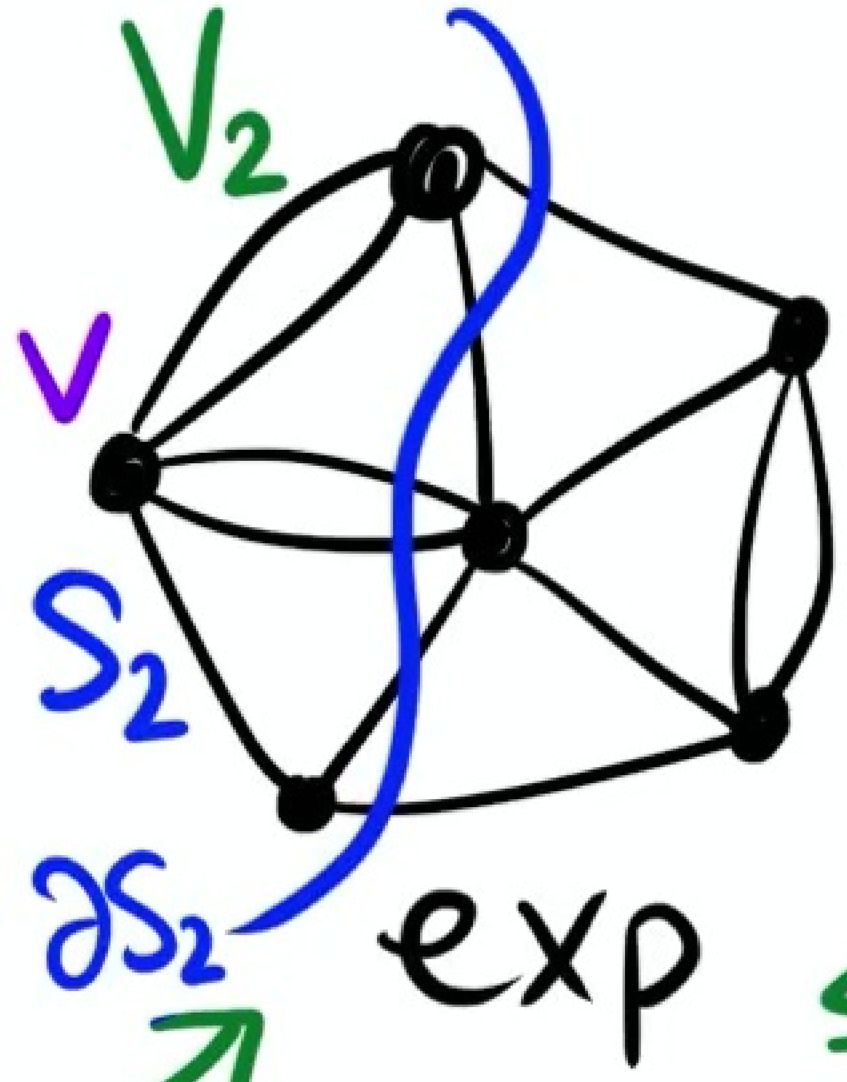


Contract



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Balanced cuts: overlay expander of expanders

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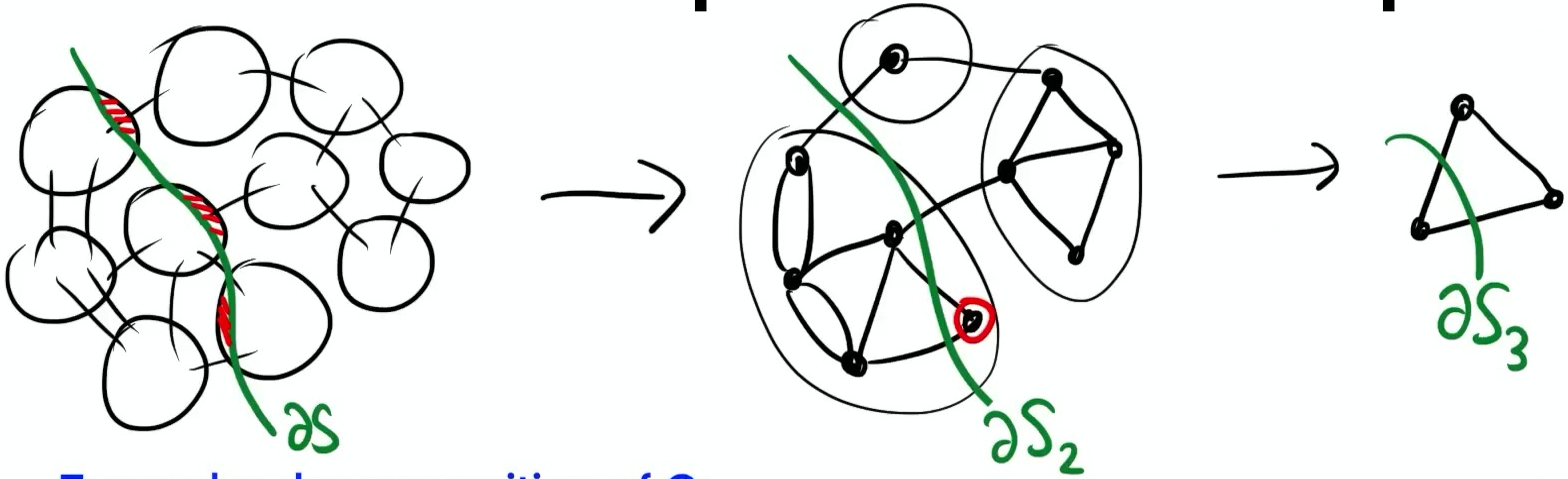
Recursive Expander Decomposition



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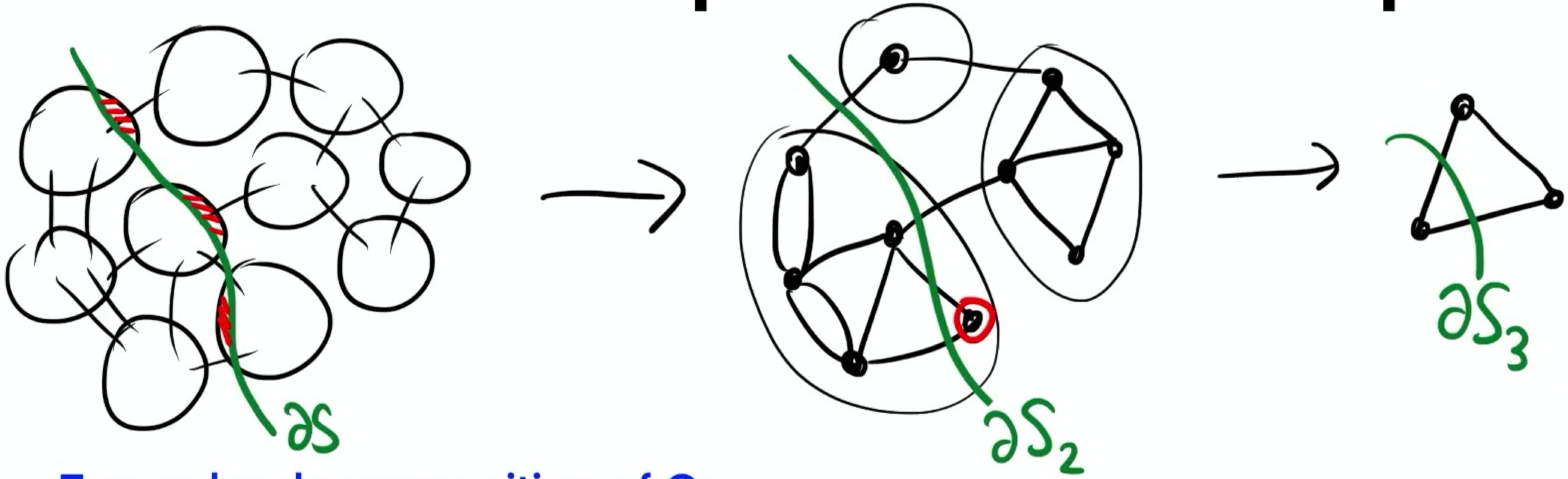
$G[V_i]$ is a ϕ-expander for all i

Recursive Expander Decomposition



Expander decomposition of G:

partition V into $V_1,\dots,V_k$ s.t.

$G[V_i]$ is a ϕ-expander for all i

- \# inter-cluster edges is $\leq \phi$ fraction $\Rightarrow \leq \log_{1/\phi} m$ levels
- "boundary-linked" property to upper bound $|\partial S_2|, |\partial S_3|, \dots$ [GRST SODA'21]

Conclusion

Deterministic mincut in $m^{1+o(1)}$ time by derandomizing skeleton construction in [Karger '96]

Open questions:

- deterministic $(1+\varepsilon)$-approx cut sparsifier?
  - requires understanding structure of balanced cuts
  - spectral approach? Derandomize $\tilde{O}(m)$ time [LS'17] ?
- deterministic mincut in m polylog(n) time?
  - no deterministic expander decomp. known with polylog(n) factors