Deterministic Min-cut in Poly-logarithmic Max-flows

Jason Li

Joint work with Debmalya Panigrahi (Duke)

Introduction

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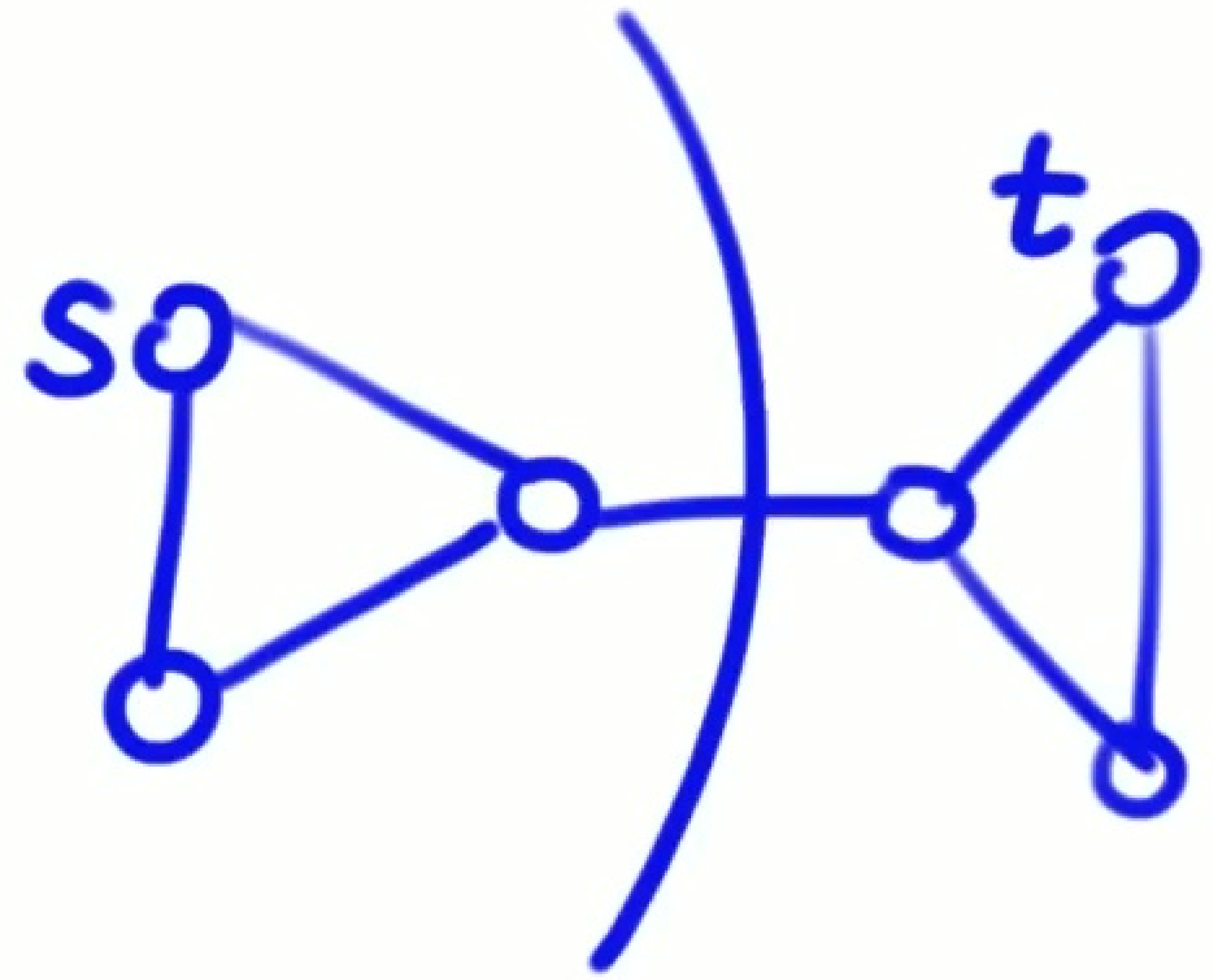

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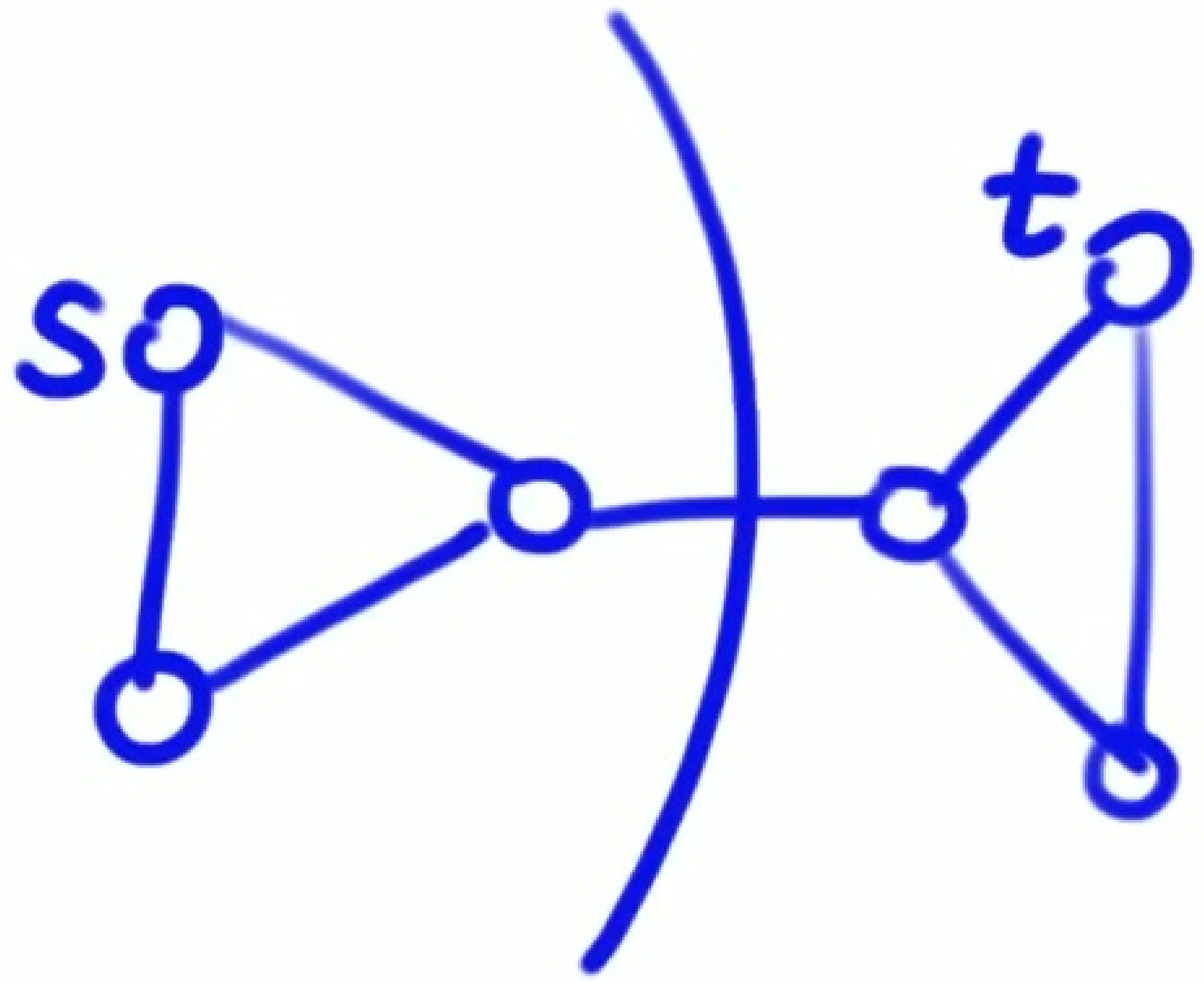

s-t mincut: ...cut whose removal disconnects vertices s and t

Max-flow min-cut theorem: s-t min-cut = s-t max-flow,
can recover s-t min-cut given s-t max-flow

Global min-cut algorithms

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[Kawarabayashi and Thorup 2015]: deterministic min-cut on simple graphs in $\tilde{O}(m)$ time

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[This work]: deterministic min-cut for weighted graphs in $O(m^{1+\varepsilon})$ time plus polylog(n) calls to s-t max-flow

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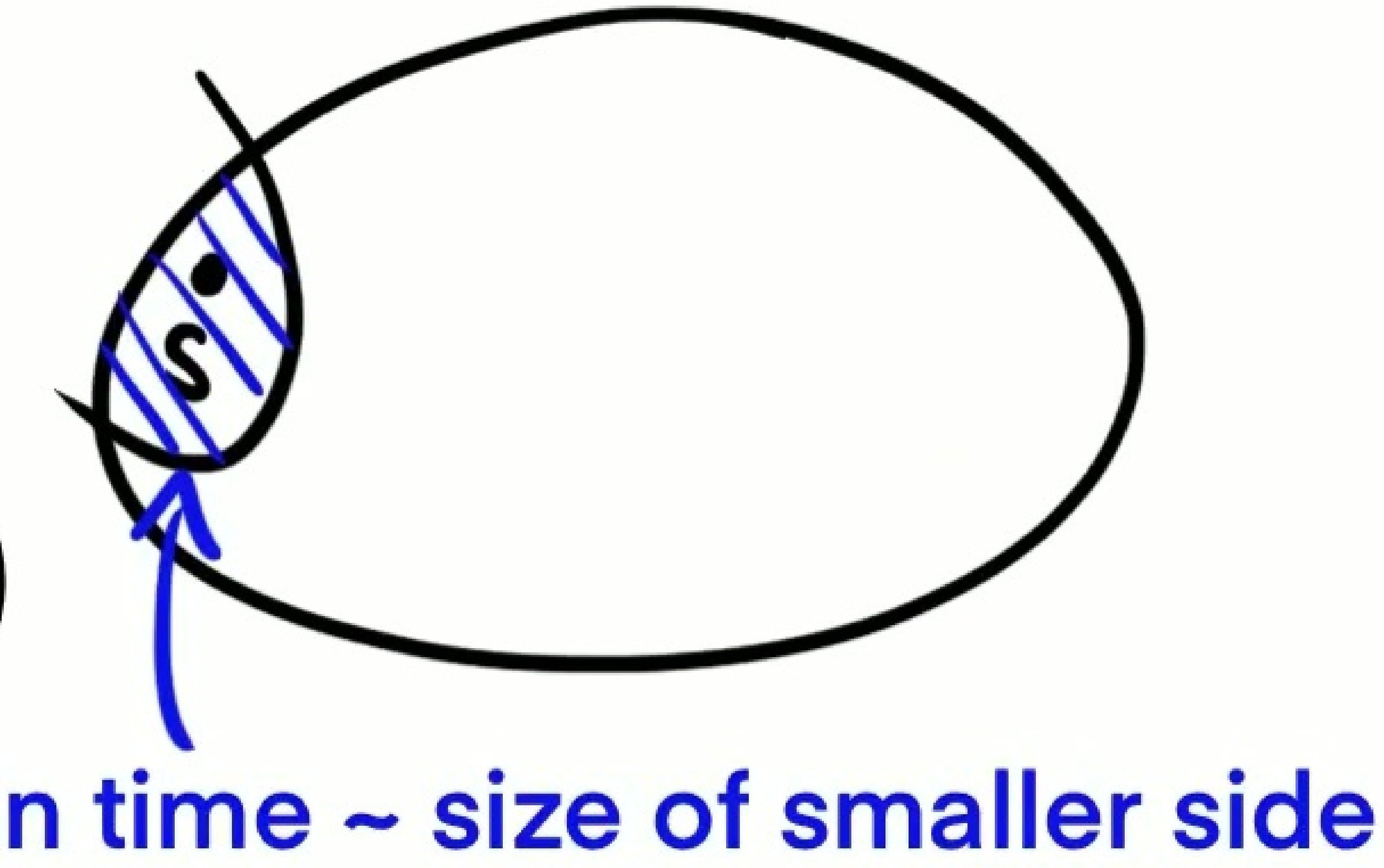
"Modern" approach to algorithm design

Local Graph Cut Algorithms

Fix a seed vertex s. If there exists a good cut “local” to s, then output in sublinear time (without looking at the whole graph)



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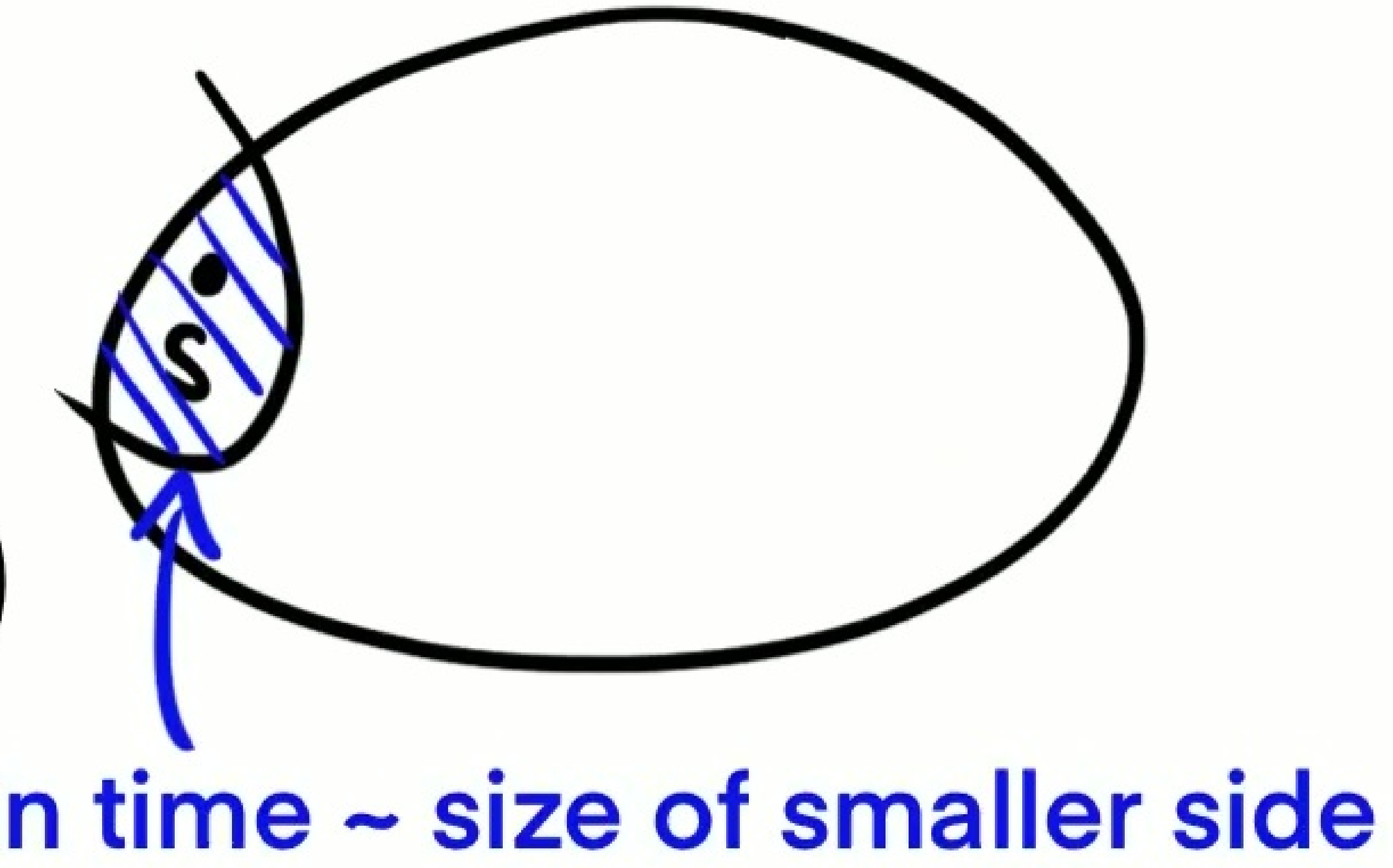

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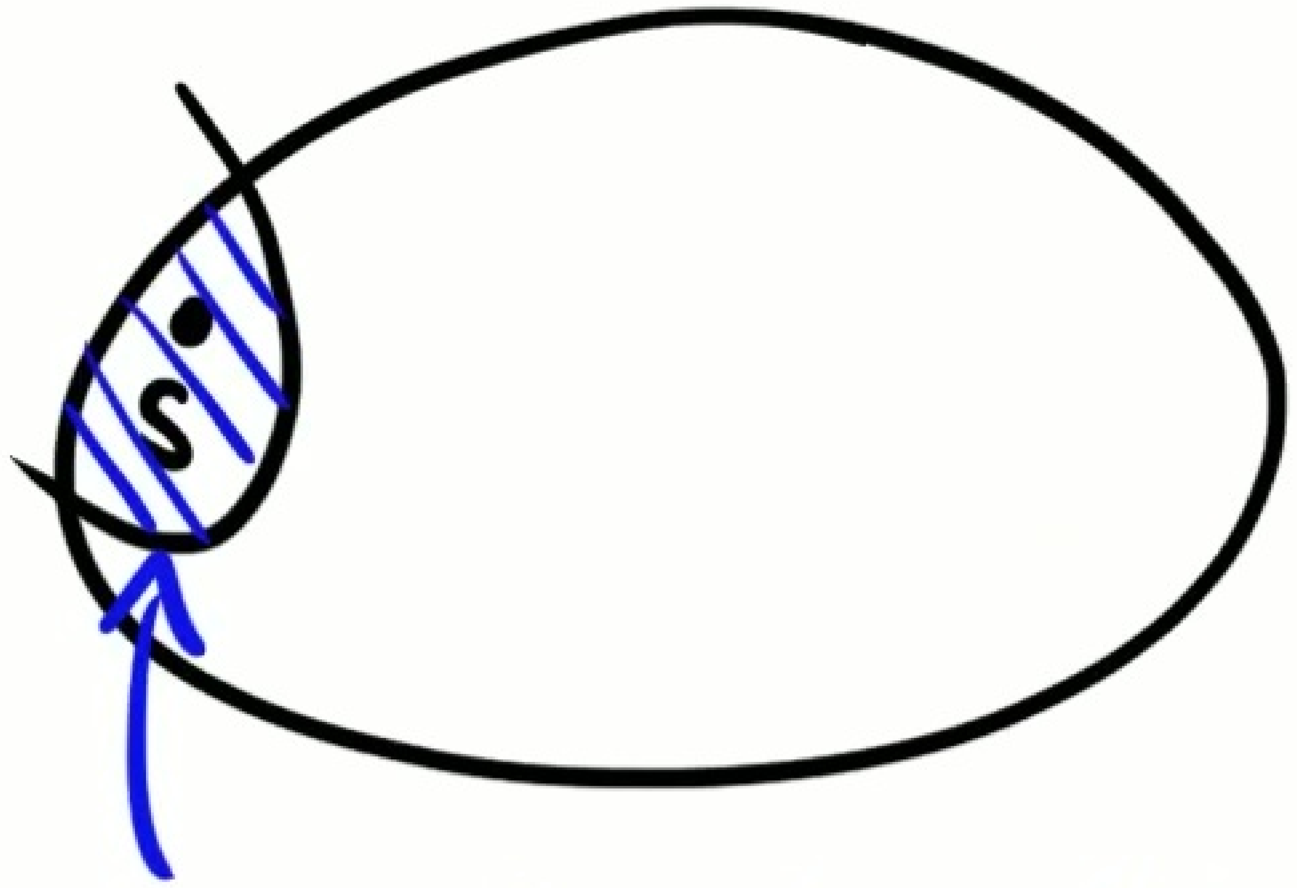

Run in time ~ size of smaller side

"PageRank Nibble" [Andersen, Chung, Lang '06]:
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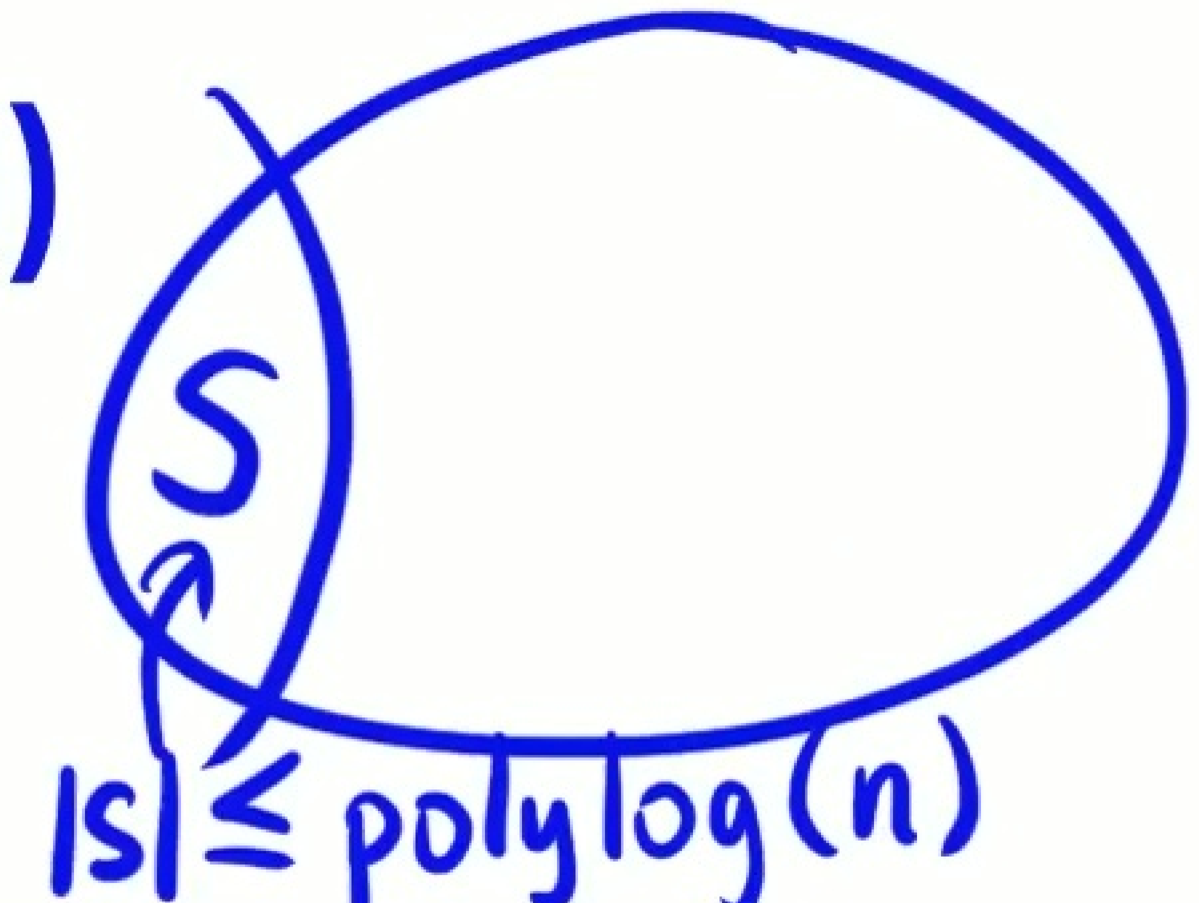
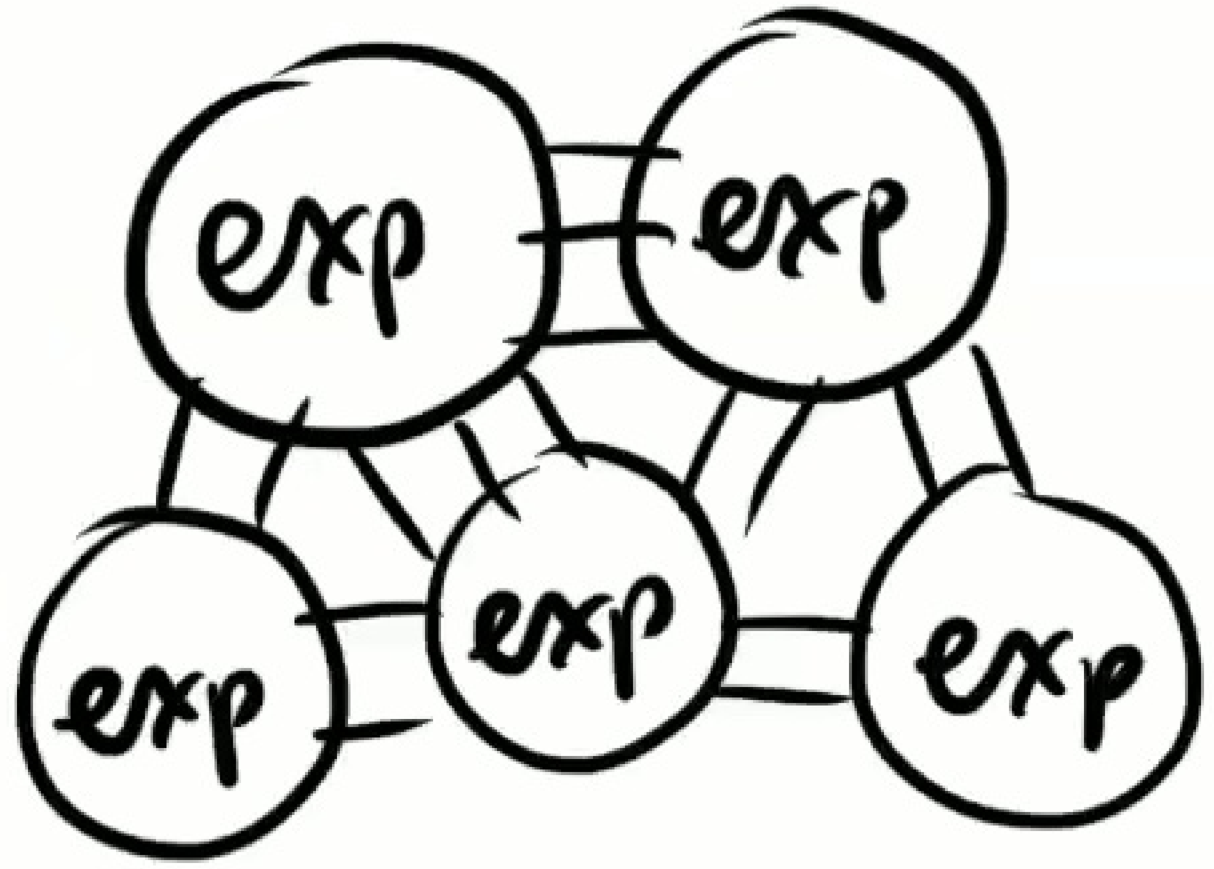
This work: if global min-cut has polylog(n) vertices on smaller side ("unbalanced"), then can find in polylog(n) s-t max-flows



Expander Decomposition

Solve when graph is an expander (easy case)

For general graphs, decompose into expanders, solve on each expander, and recurse



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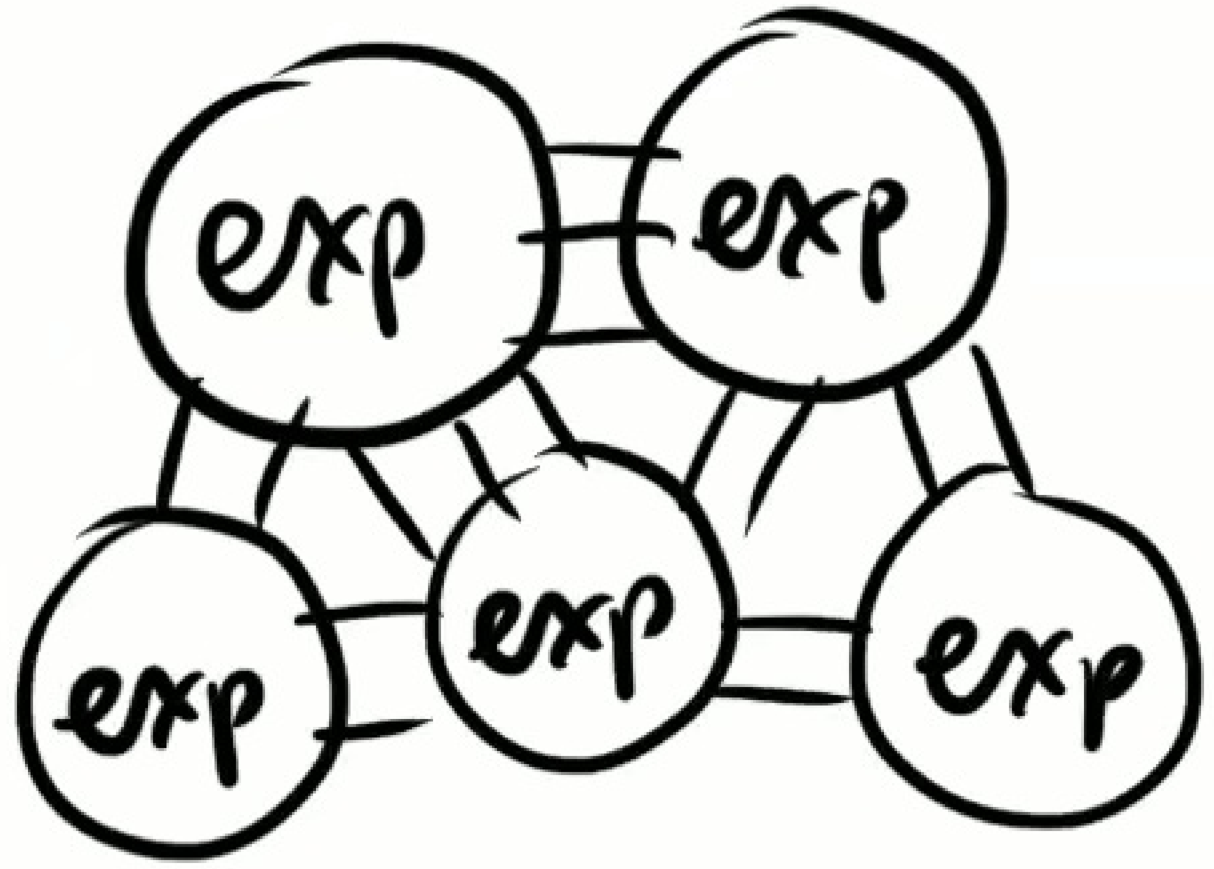

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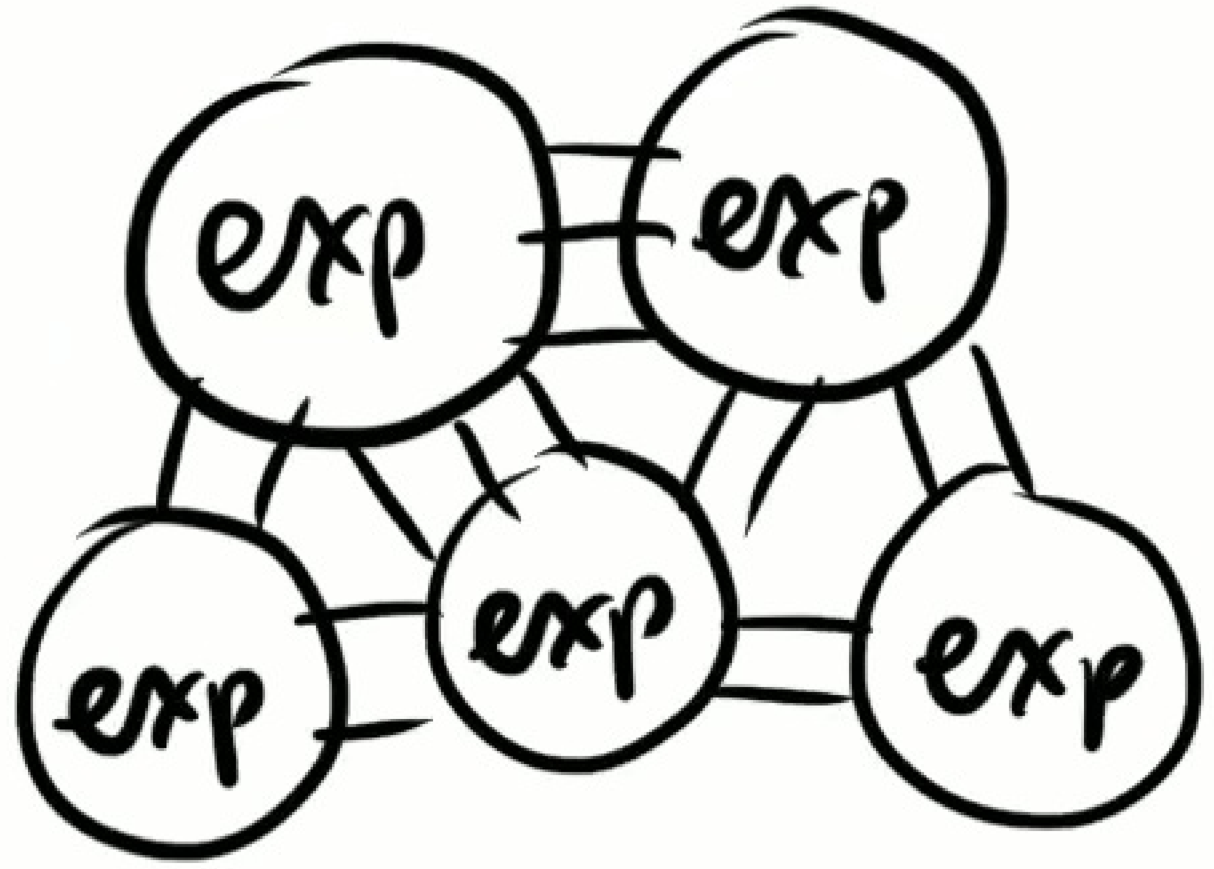

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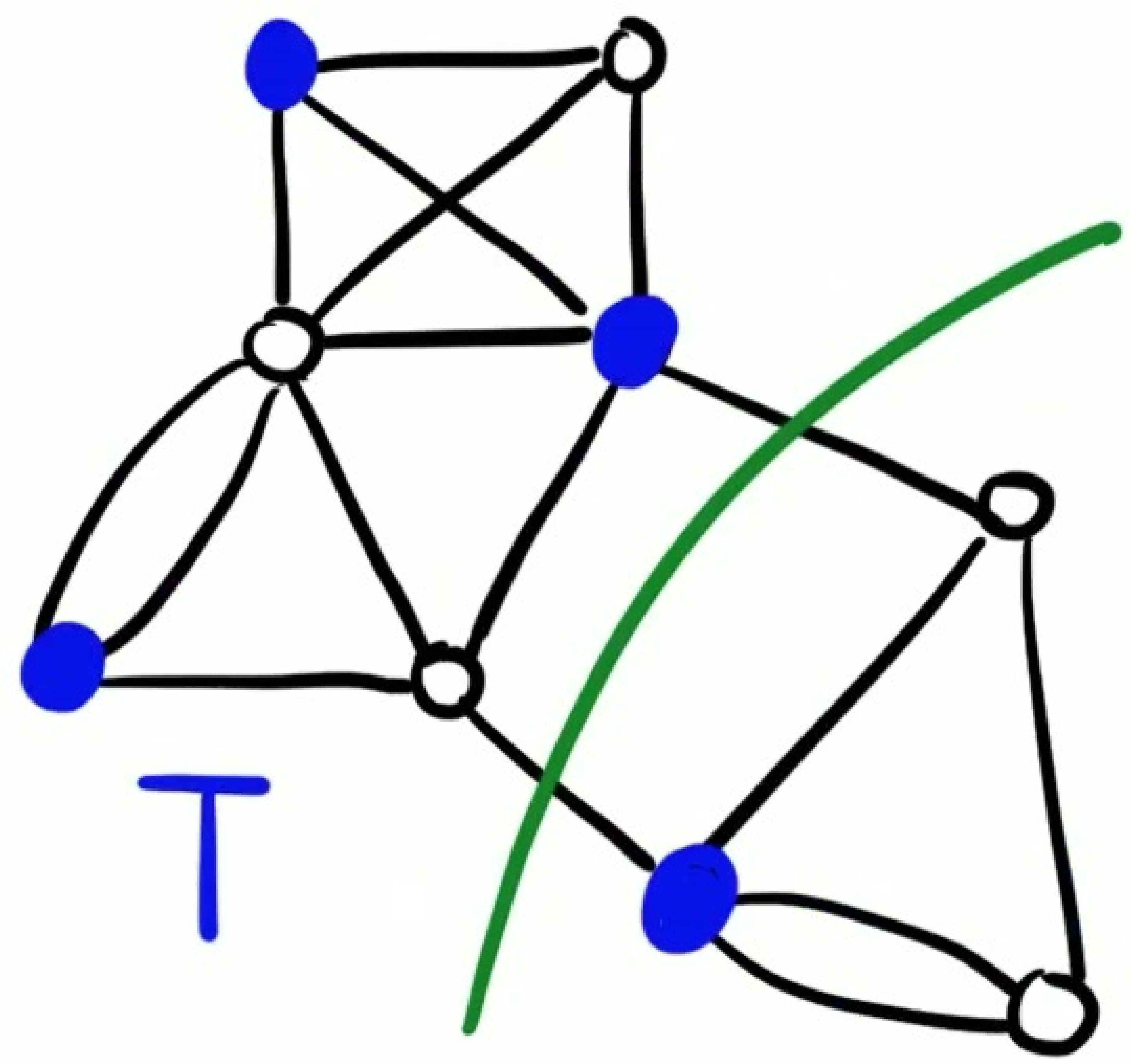
For min-cut: when graph is an expander, the min-cut must be unbalanced! (polylog(n) vertices on smaller side.) So local algorithm works.

Local algorithm: Isolators

Suppose (S, V\S) is a min-cut.

$T \subseteq V$ is an isolator if $|S \cap T| = 1$



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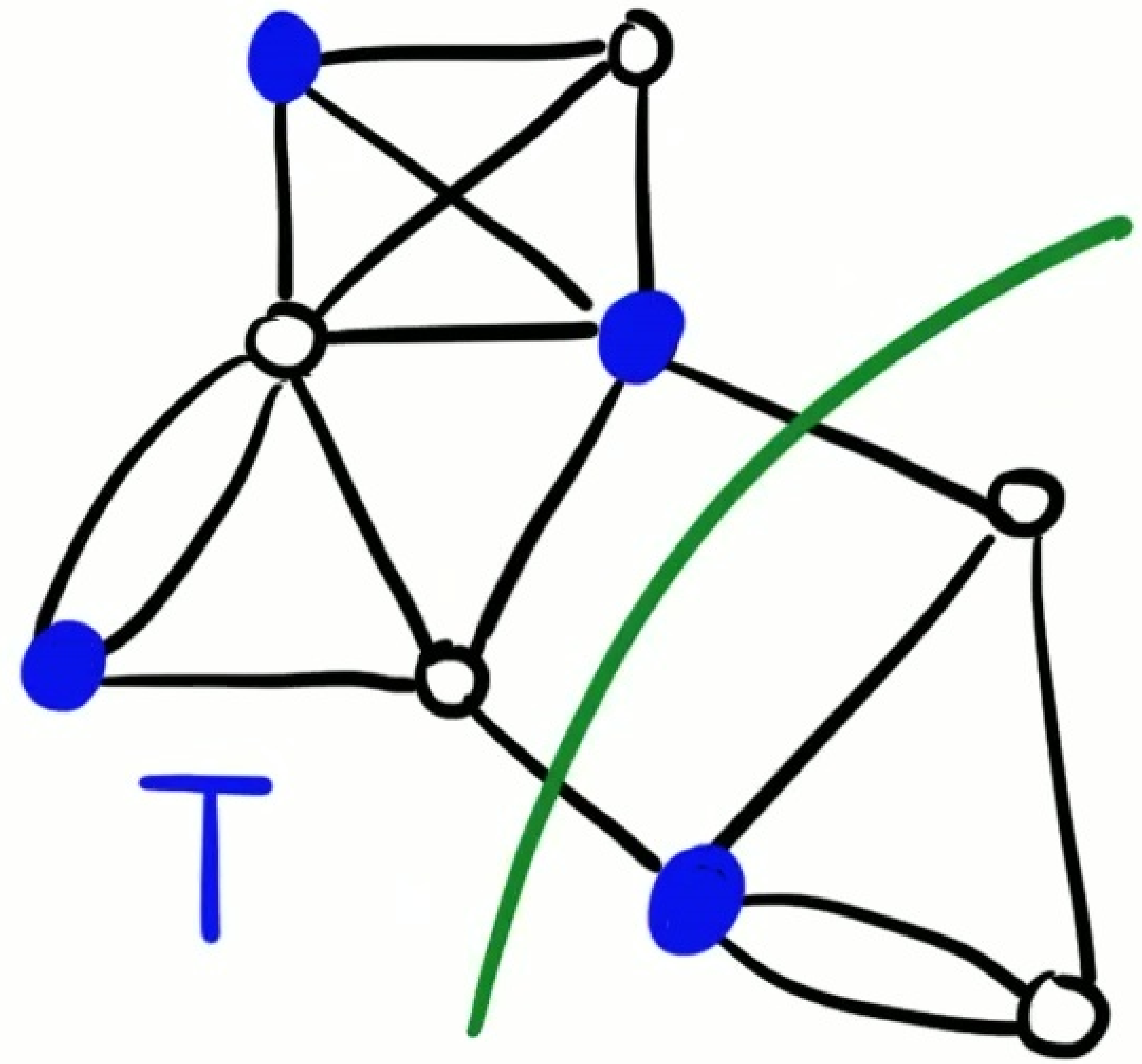

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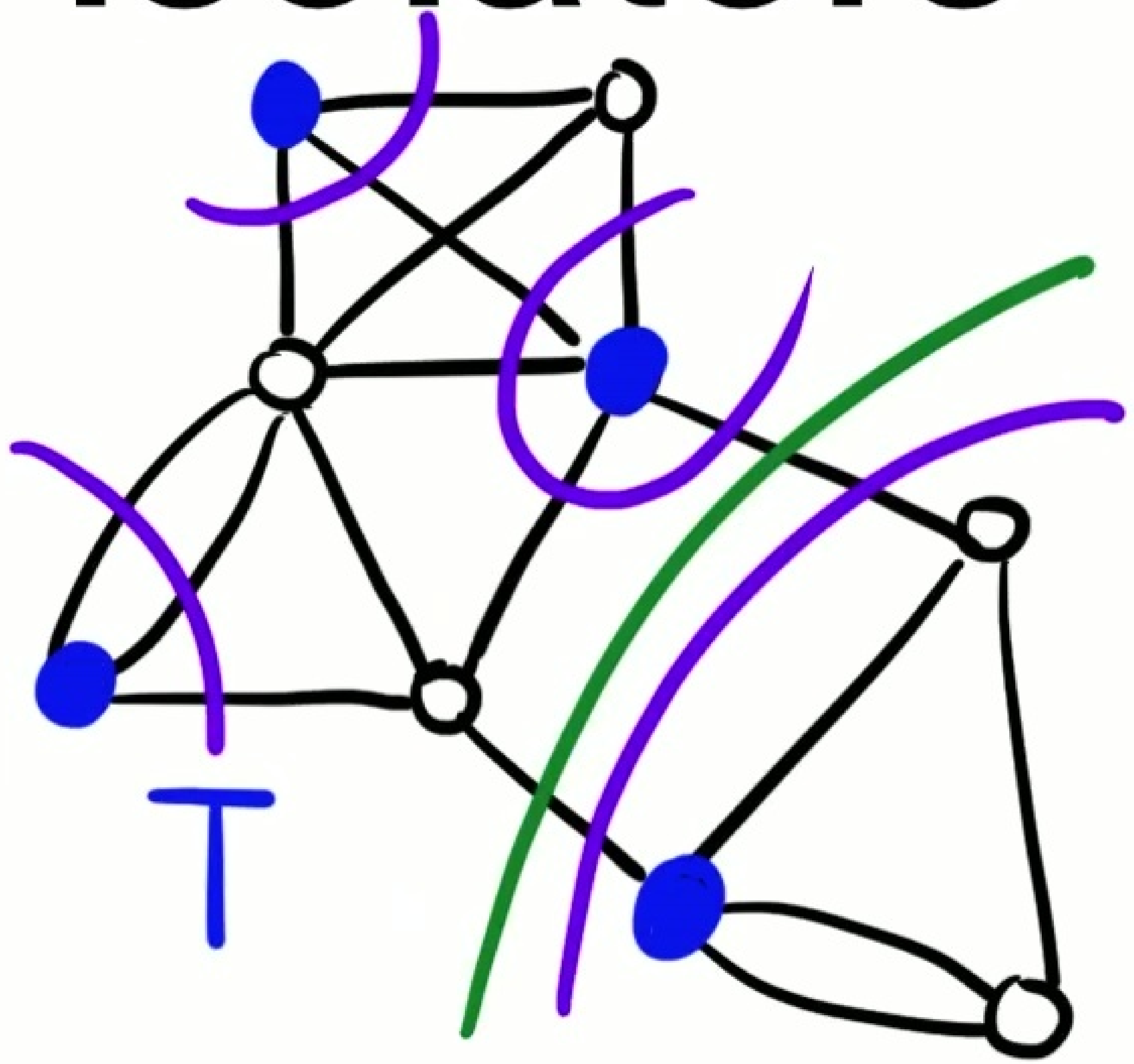

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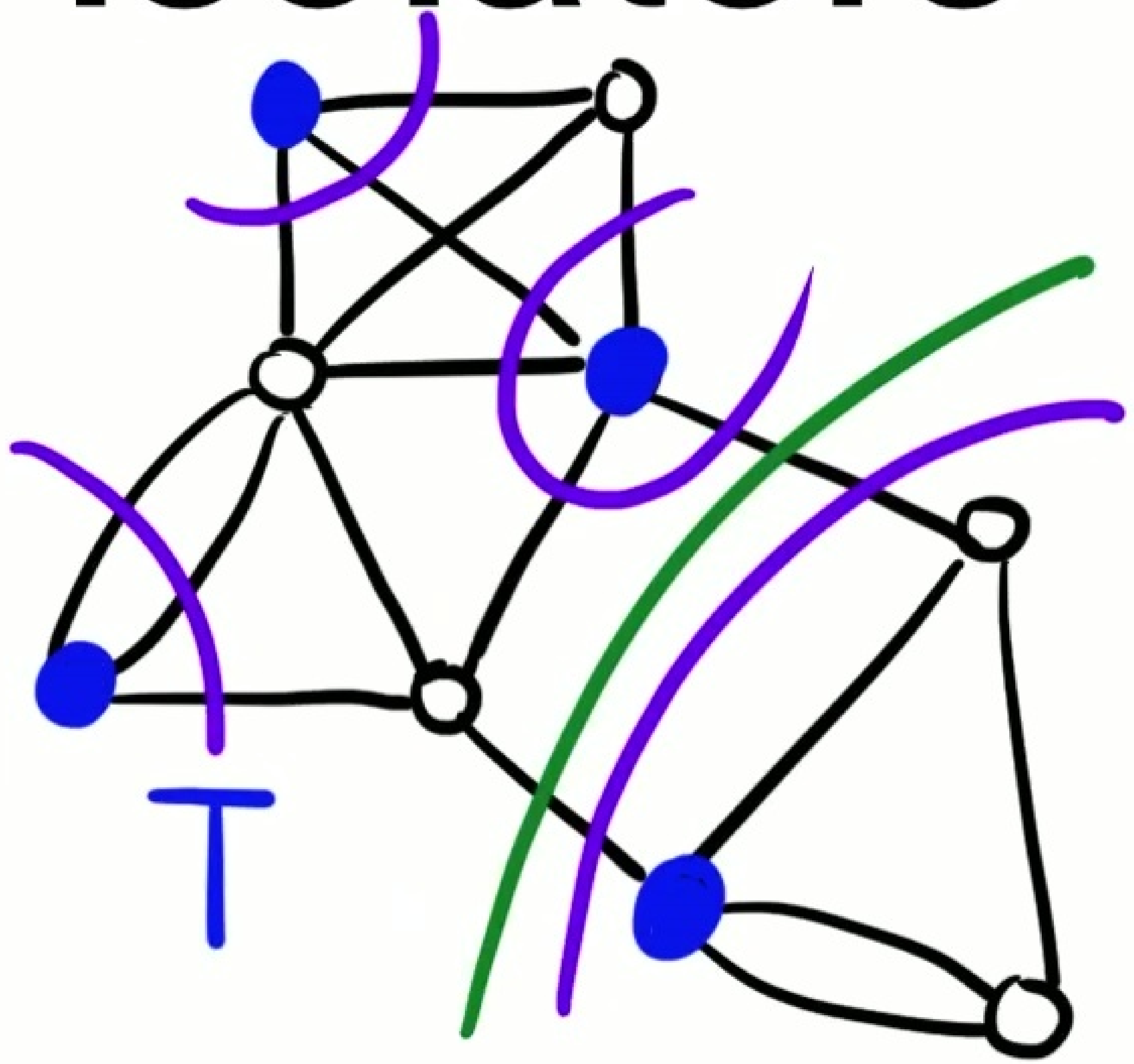

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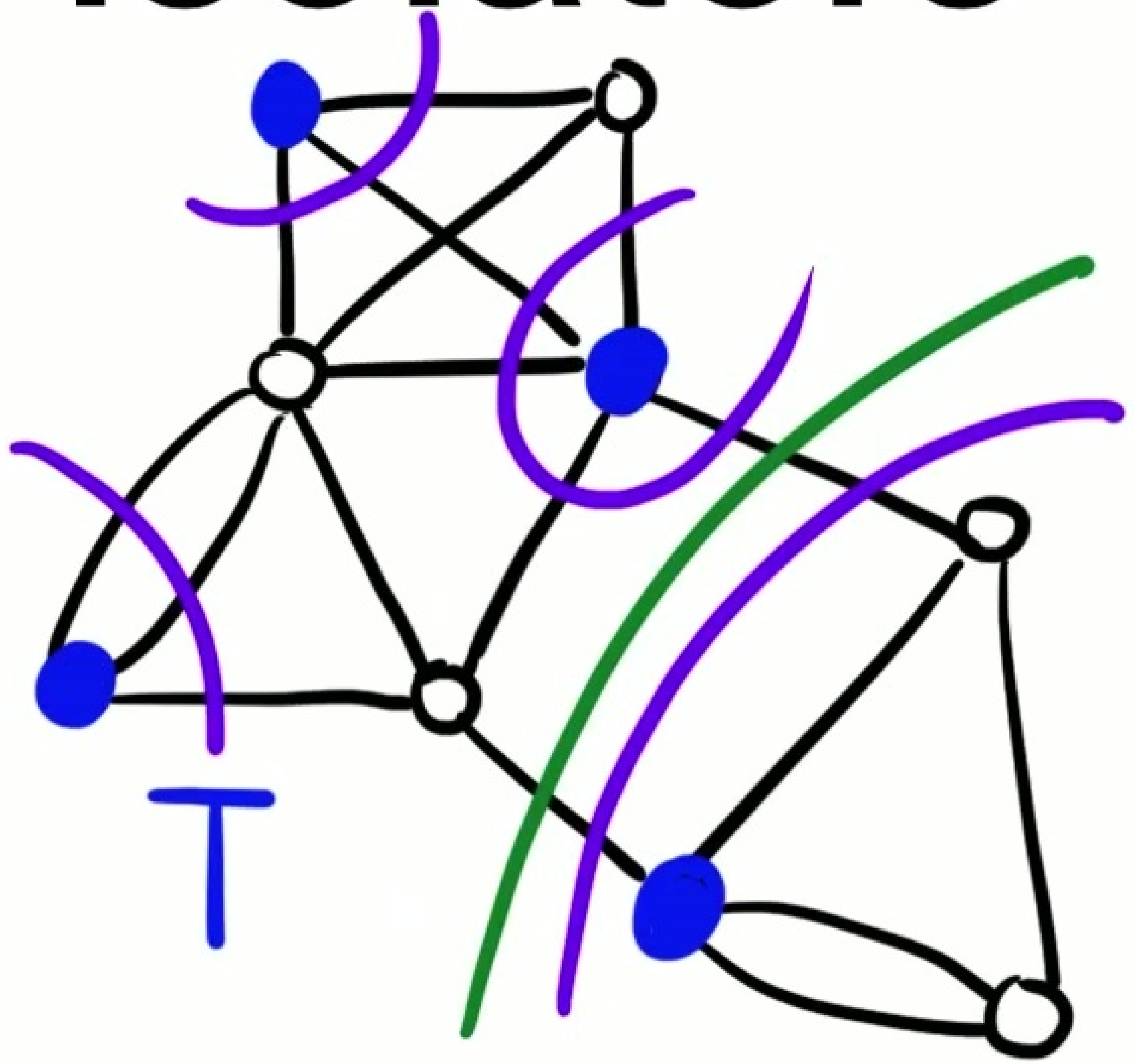

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If T is an isolator, then one of the cuts returned is the min-cut!

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  - Want: each pair s,t in T is separated in at least one of them
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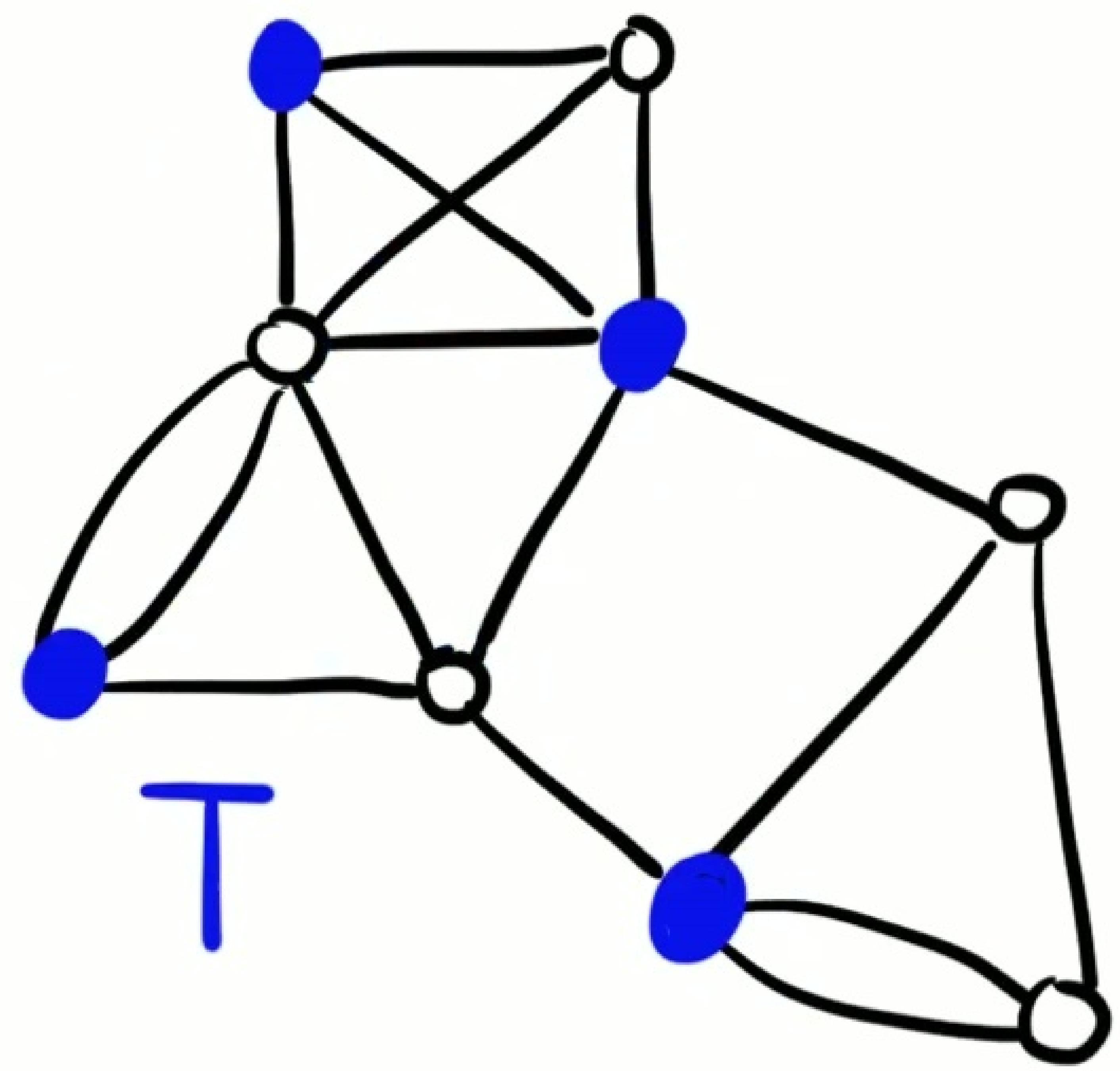

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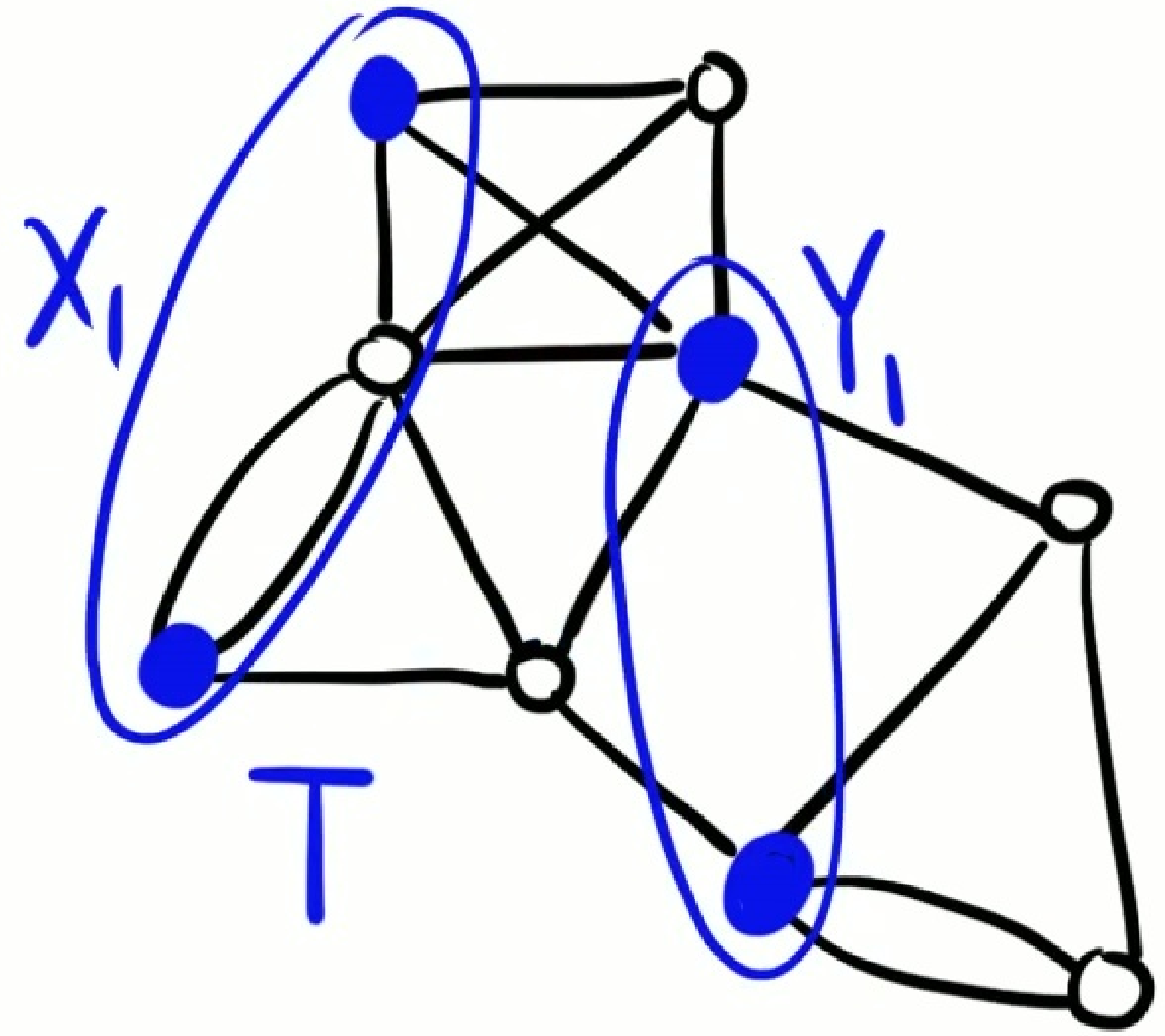

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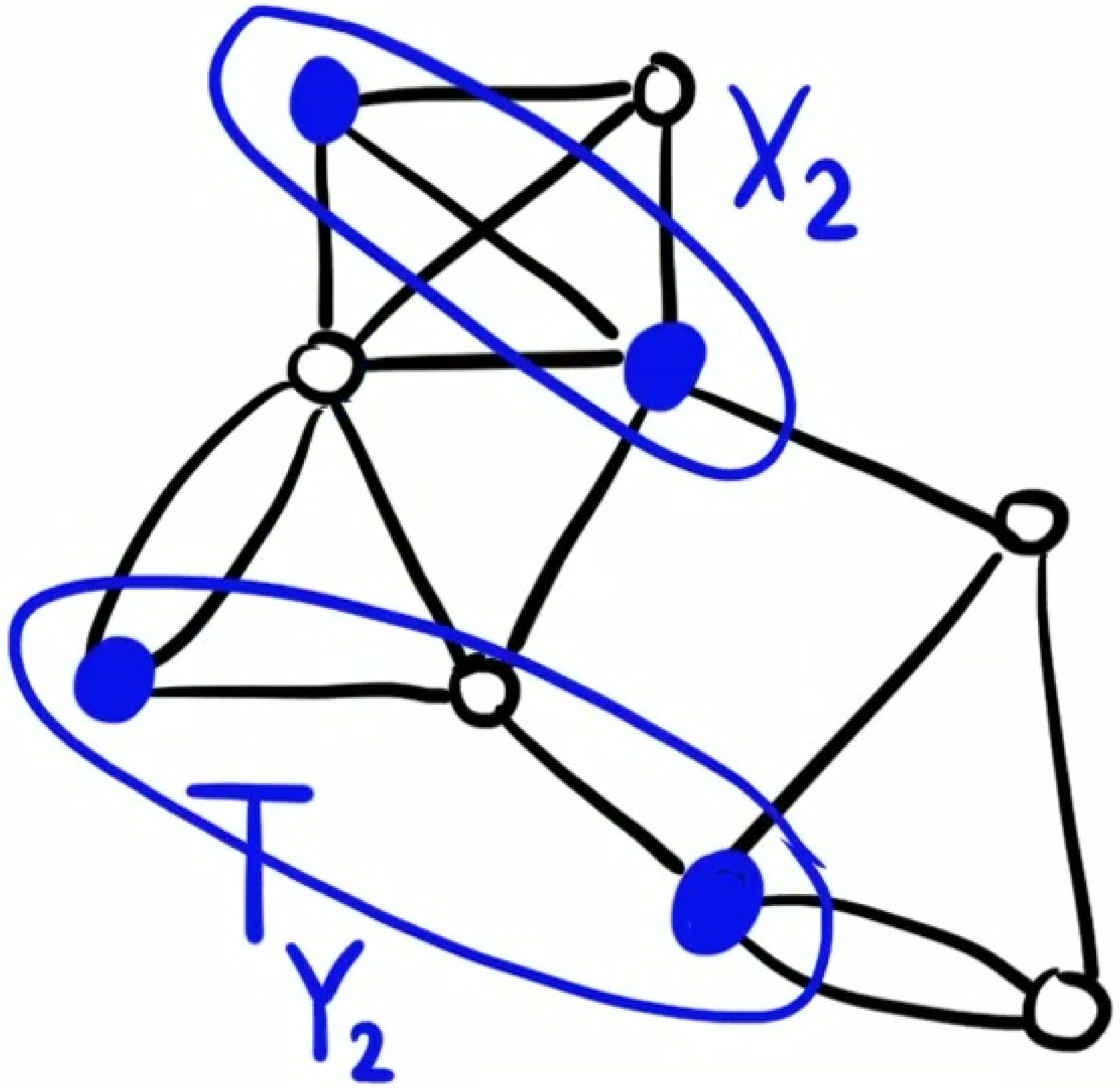

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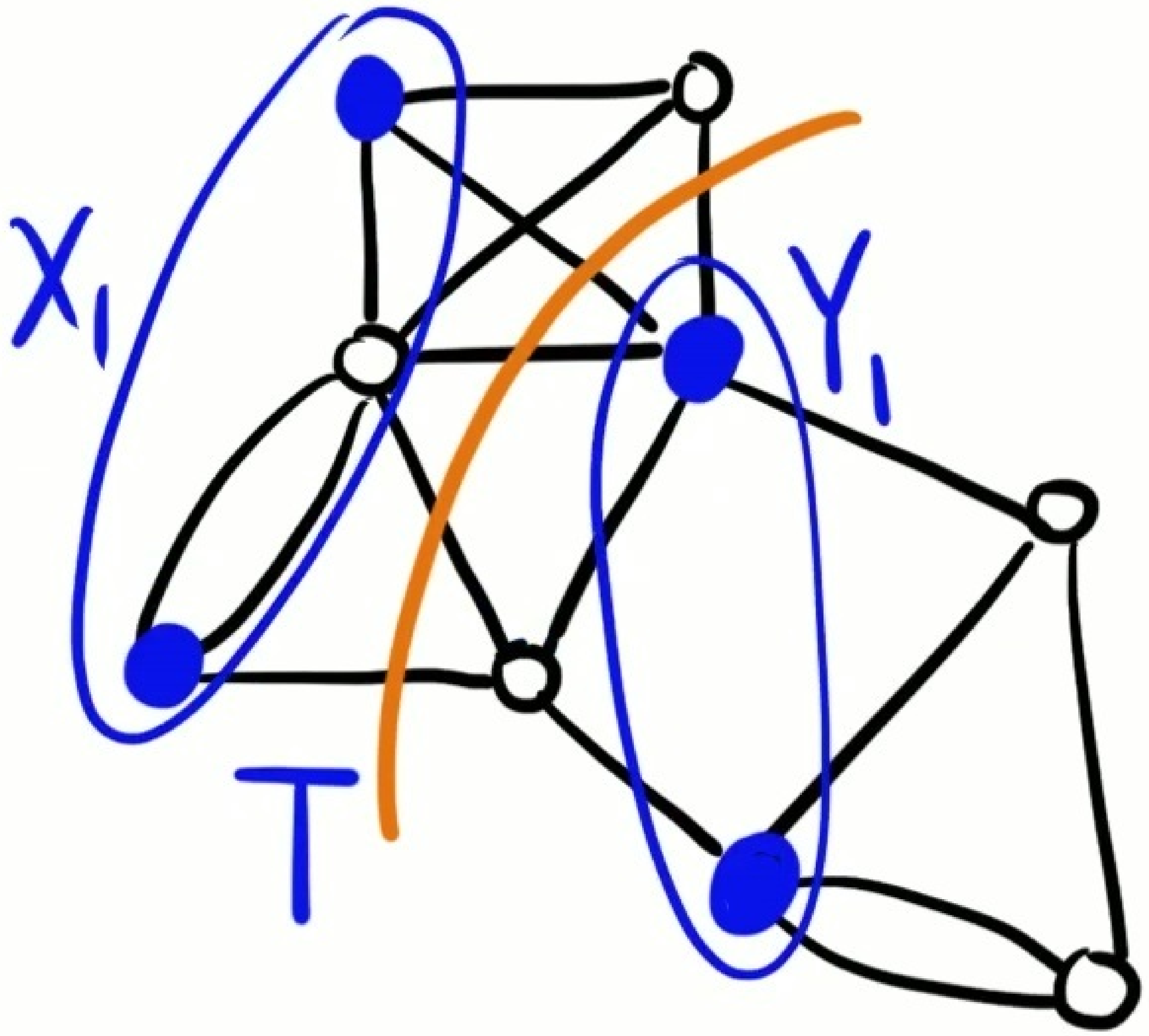

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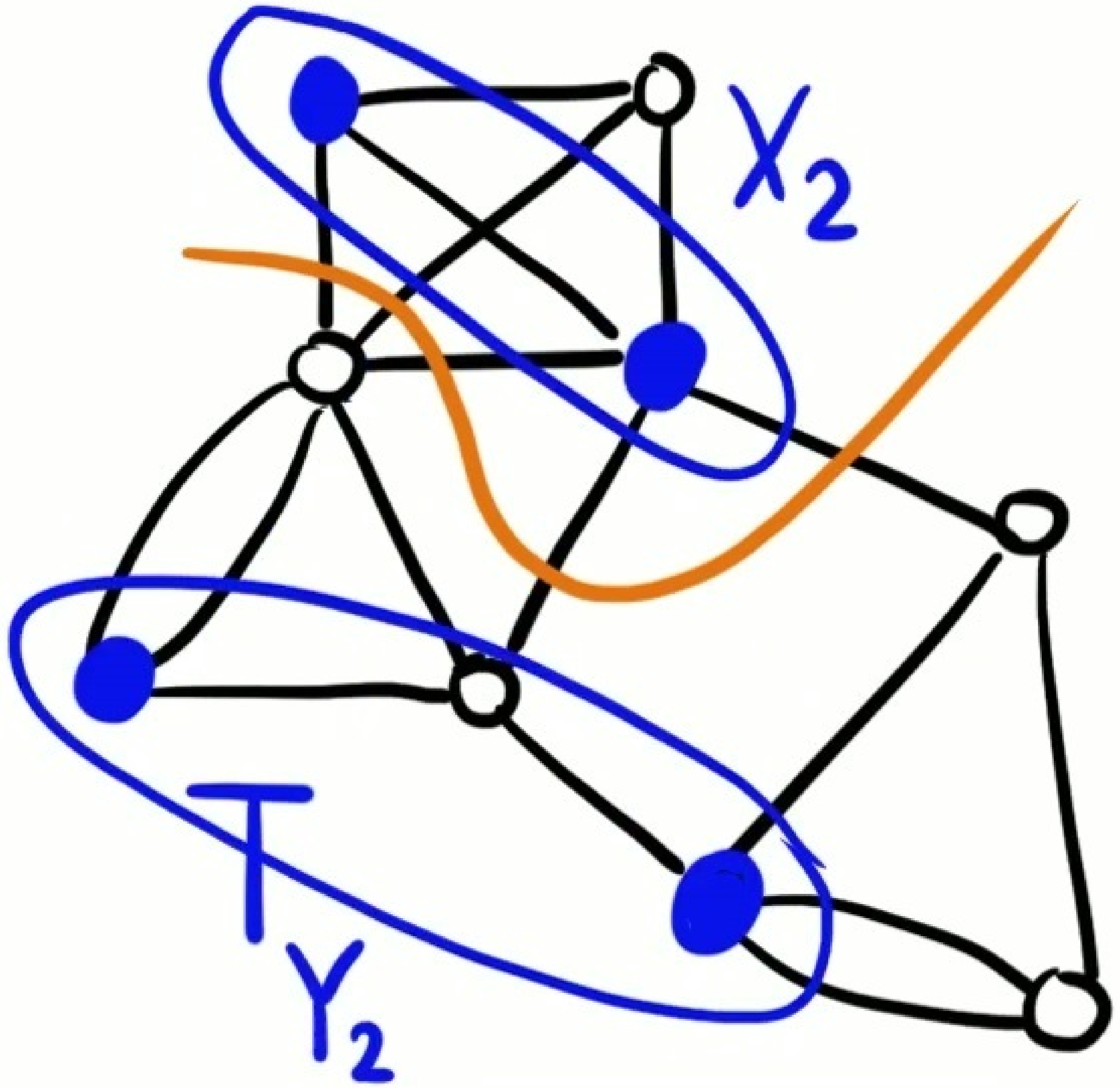

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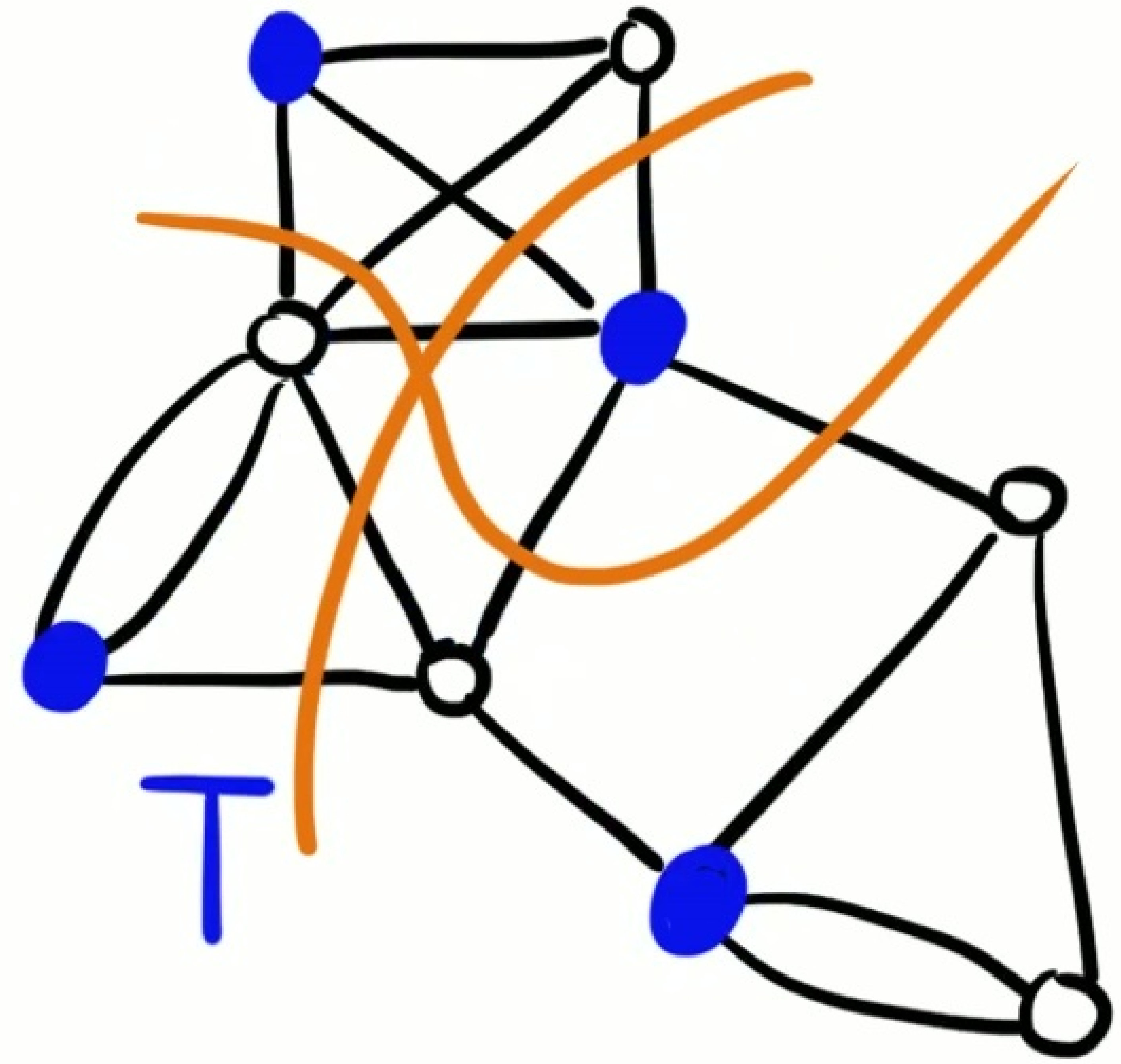

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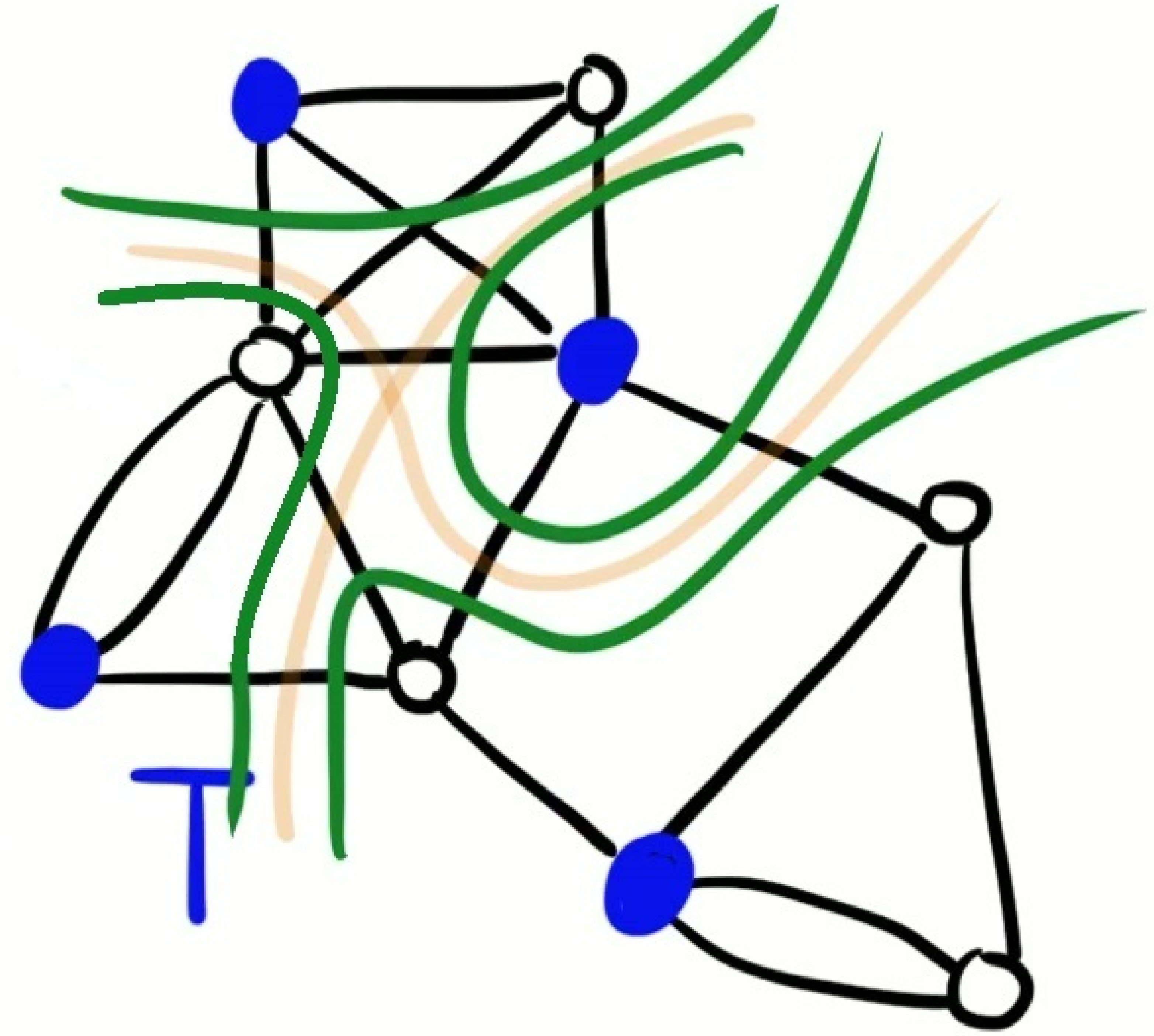

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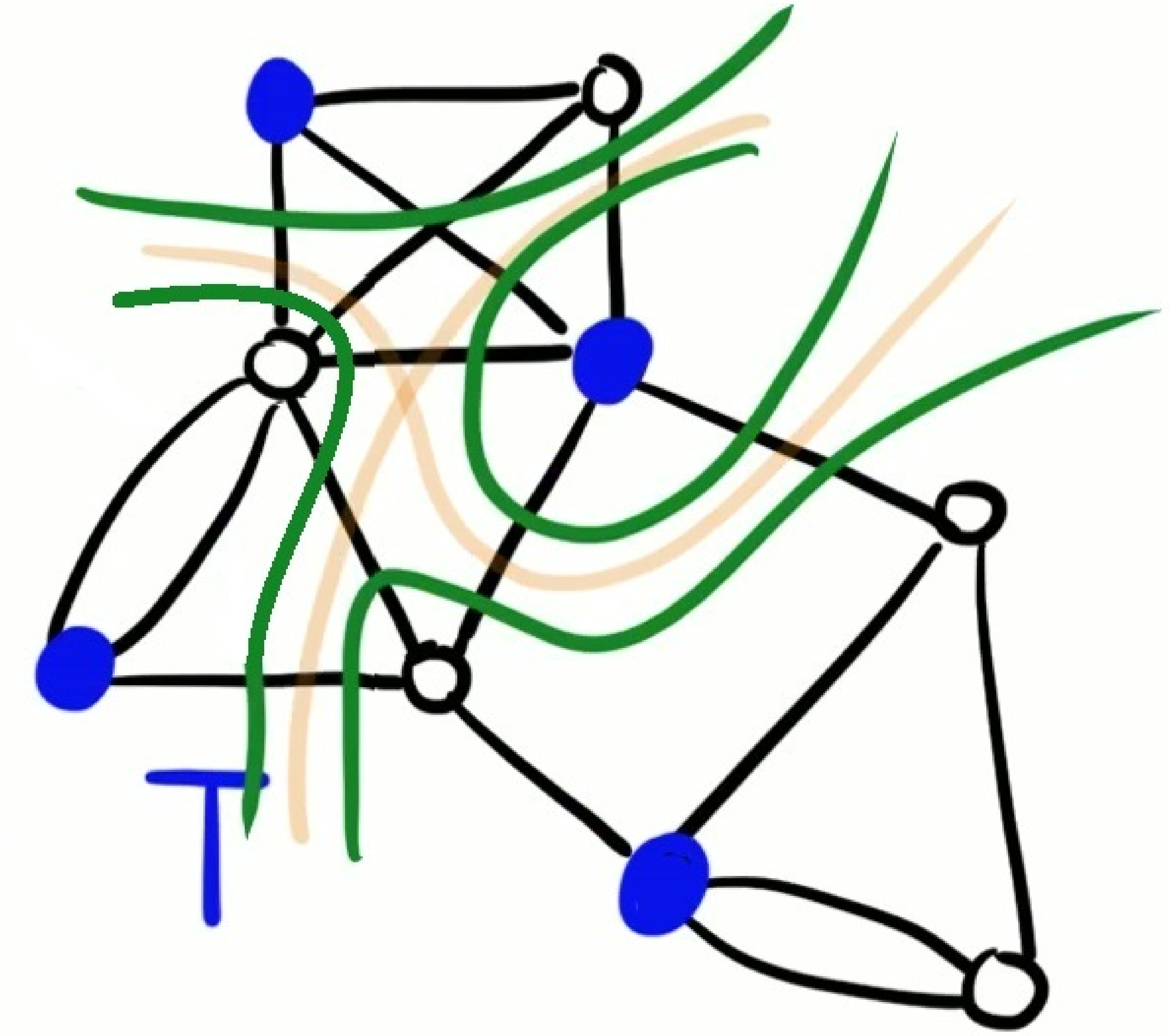

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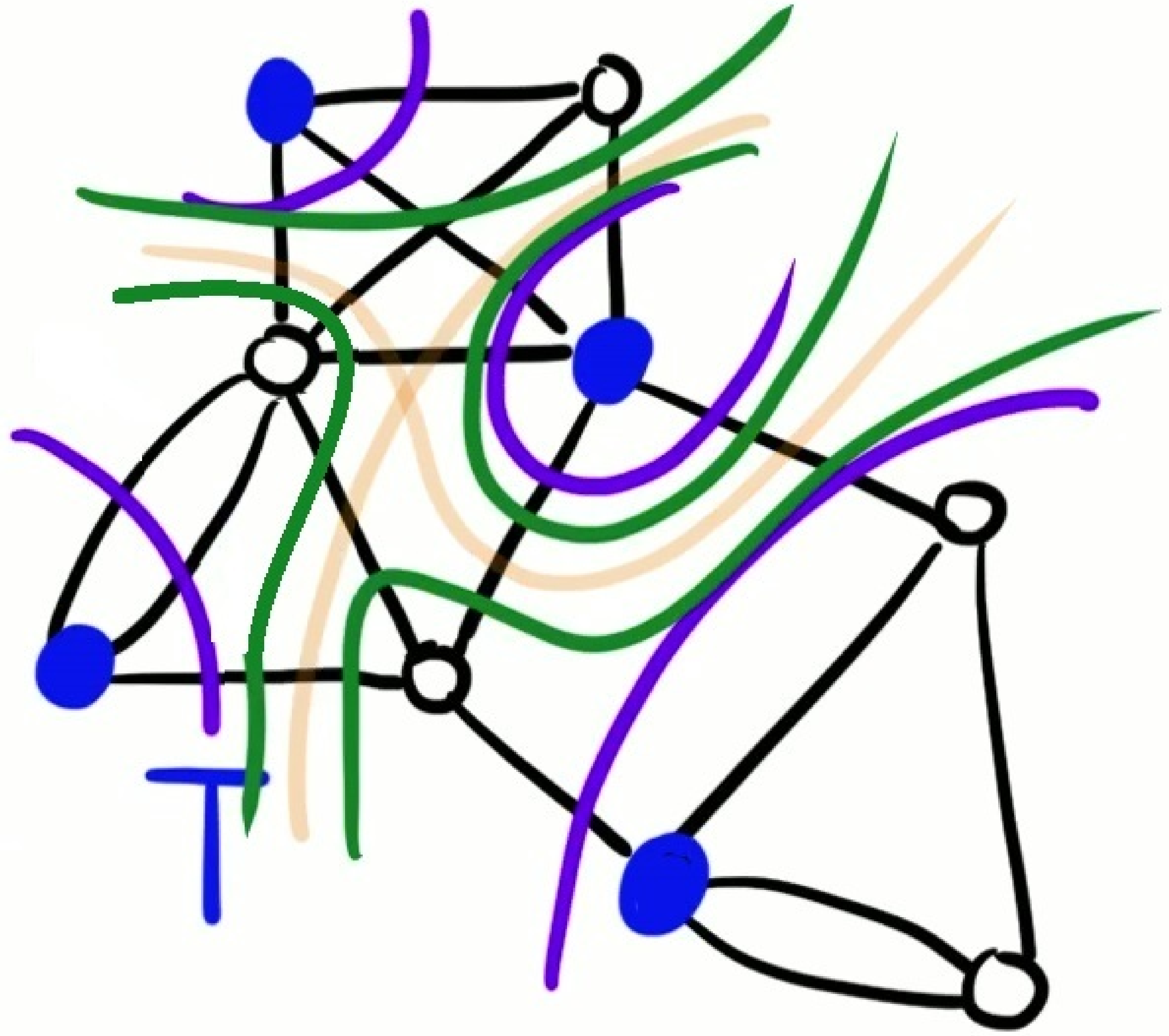

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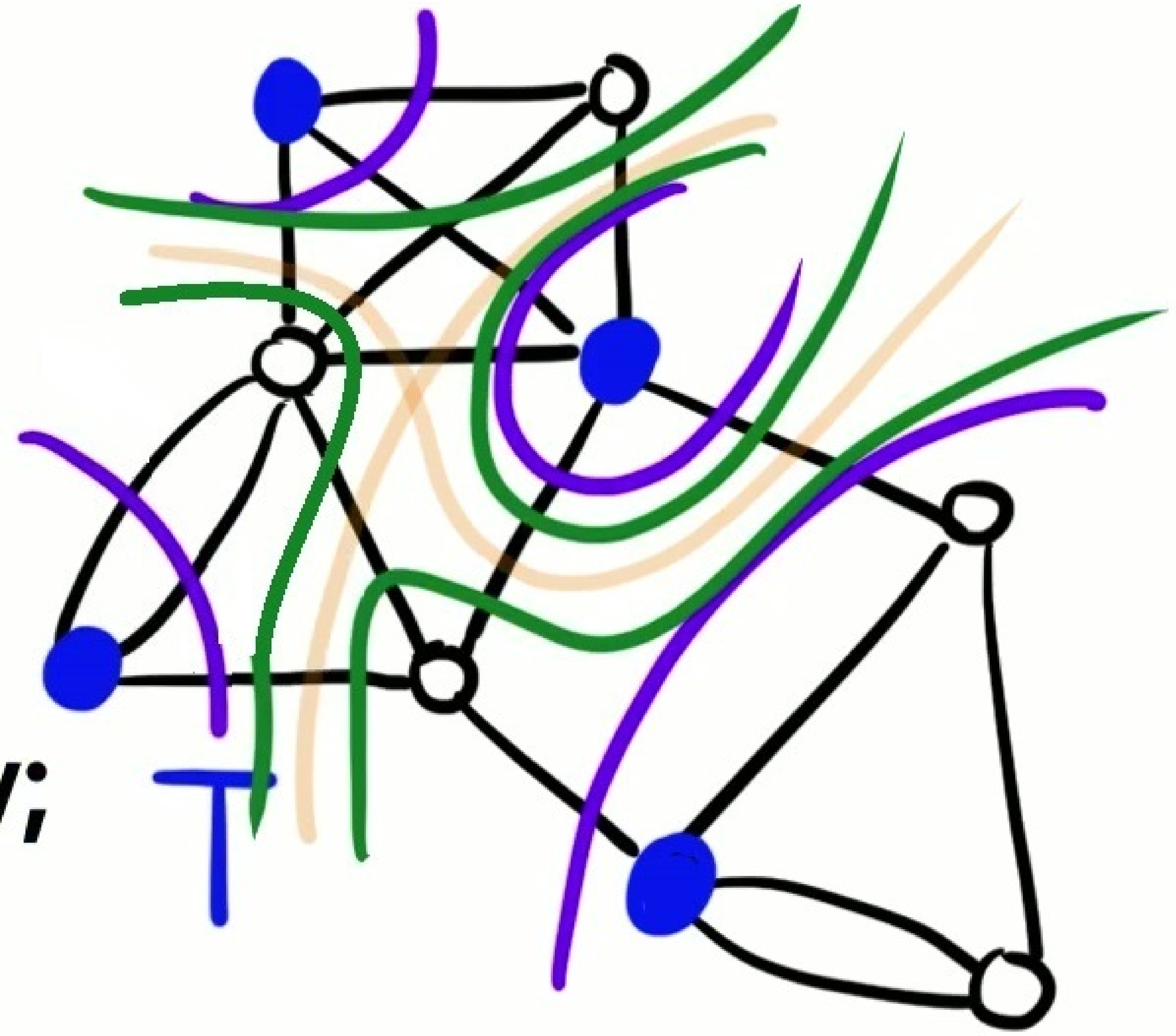

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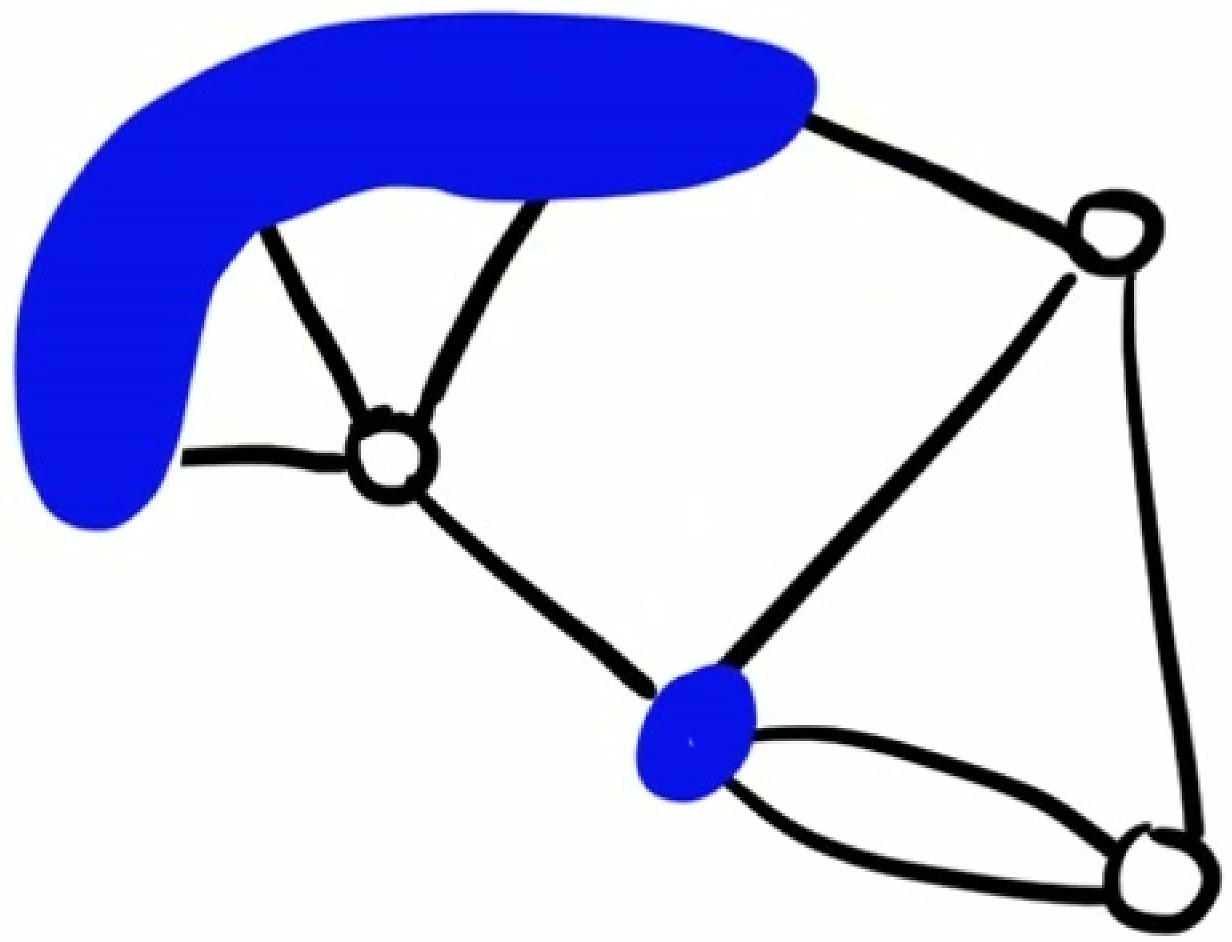

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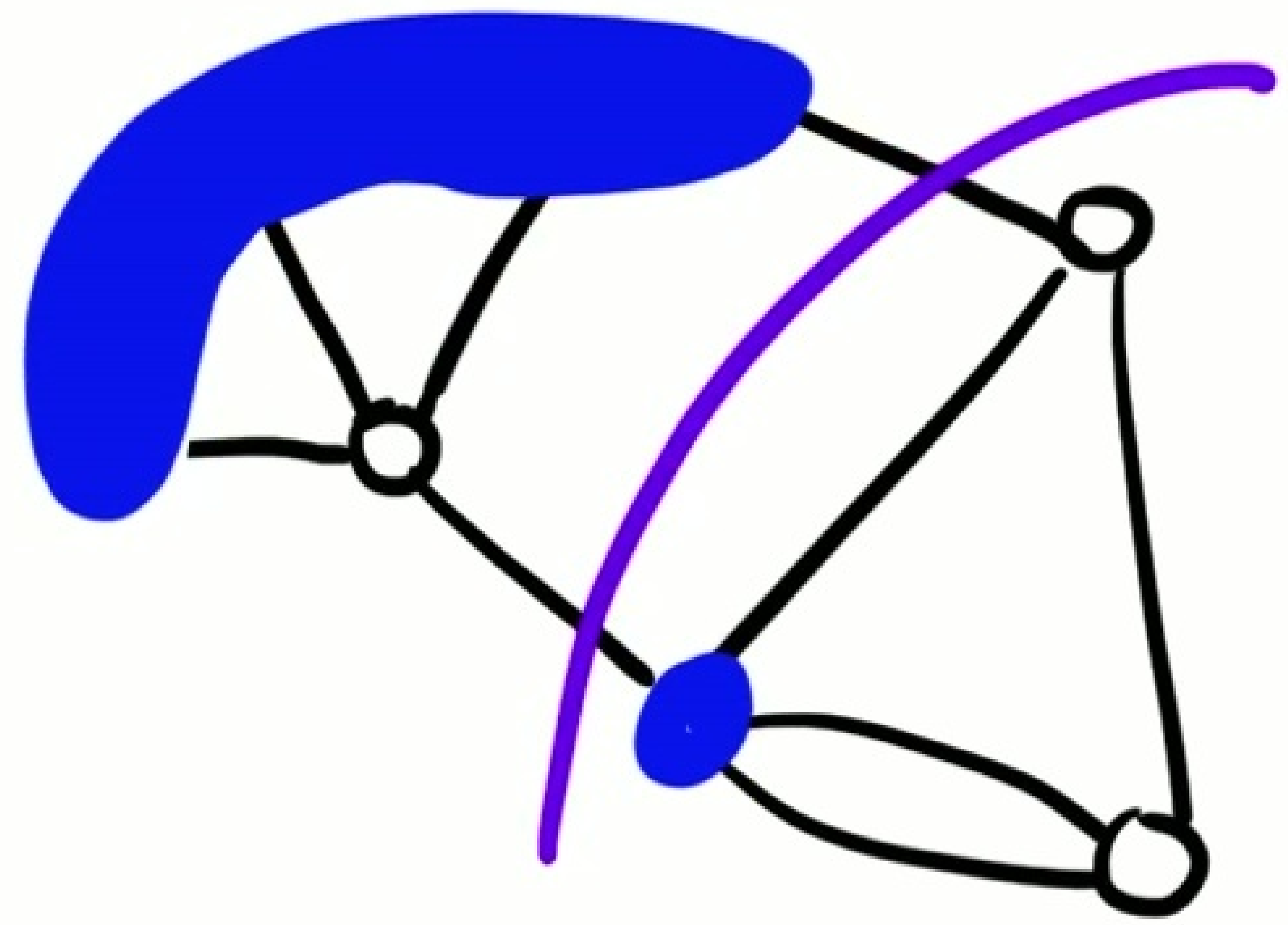

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- Sample each vertex with probability $1/2^i$
  - With constant probability, sample exactly one in S
    => sampled set T is an isolator
- Run the isolator algorithm for all i, output smallest (t, T\t)-min-cut found

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Unbalanced case: T = V is an almost-isolator if $|S| < \text{polylog}(n)$. Algorithm calls polylog(n) max-flows

Balanced case: det. sparsification

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Algorithm: start with T=V

run unbalanced case, then sparsify T,

repeat until $|T| = 1$

return smallest $(t, T \setminus t)$-min-cut found

Conductance and Expanders

Conductance of an (unweighted) graph: $\Phi(G) = \min_{\substack{S \subseteq V \\ vol(S) \leq vol(V \setminus S)}} \frac{|E(S, V \setminus S)|}{vol(S)}$

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Proof: Suppose $vol(S) \leq vol(V \setminus S)$

$$vol(S) = \sum_{v \in S} deg(v) \geq \sum_{v \in S} \lambda = \lambda |S| \quad [\lambda = \text{min-cut}]$$

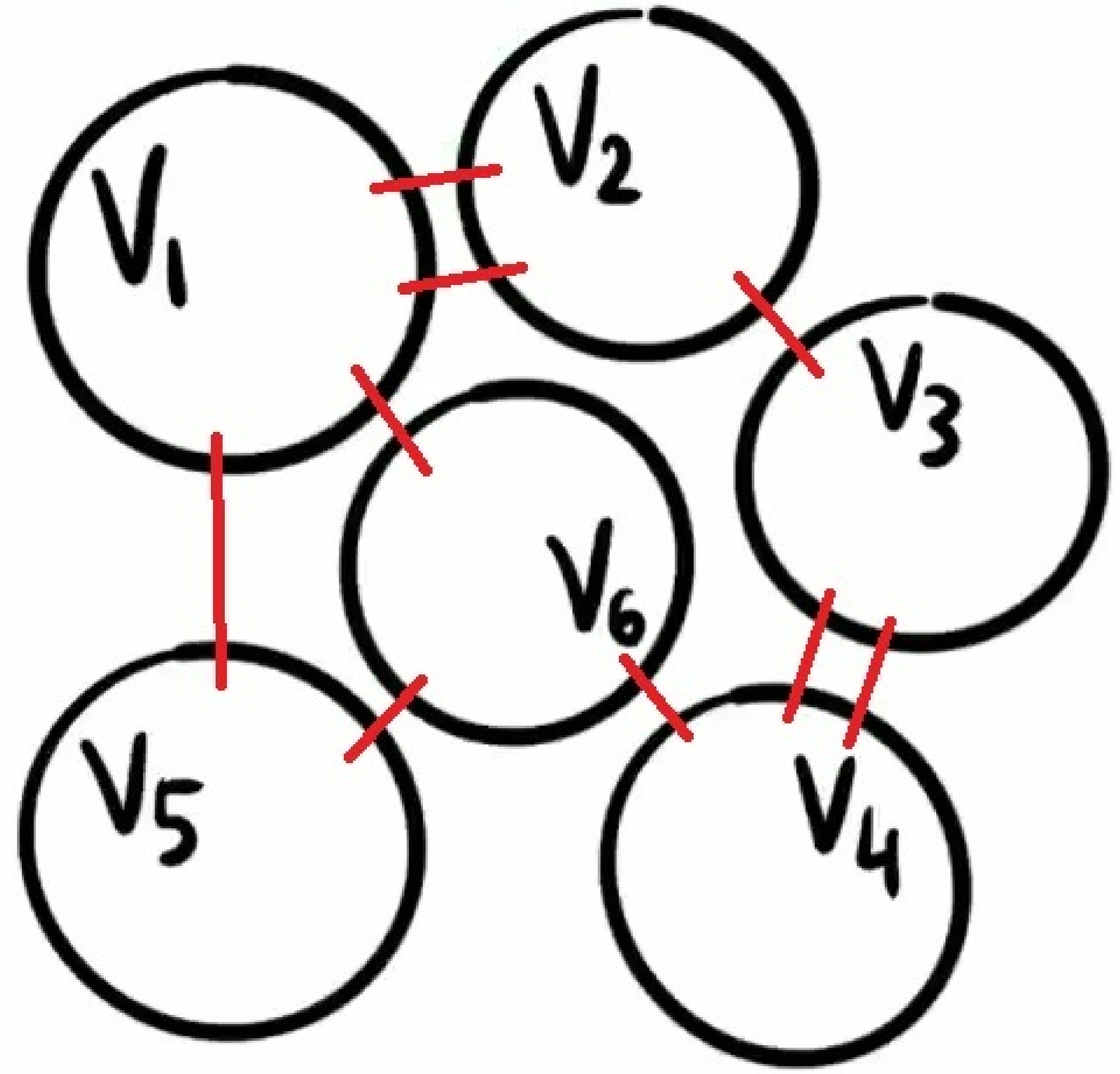
$$\phi \leq \Phi(G) \leq \frac{|E(S, V \setminus S)|}{vol(S)} \leq \frac{\lambda}{\lambda |S|} \iff |S| \leq 1/\phi$$

Deterministic sparsification of T (T=V)

Expander decomposition: partition V into $V_1, \ldots, V_k$ s.t.

(1) Each induced graph $G[V_i]$ is a ϕ-expander

(2) At most half of edges go between expanders



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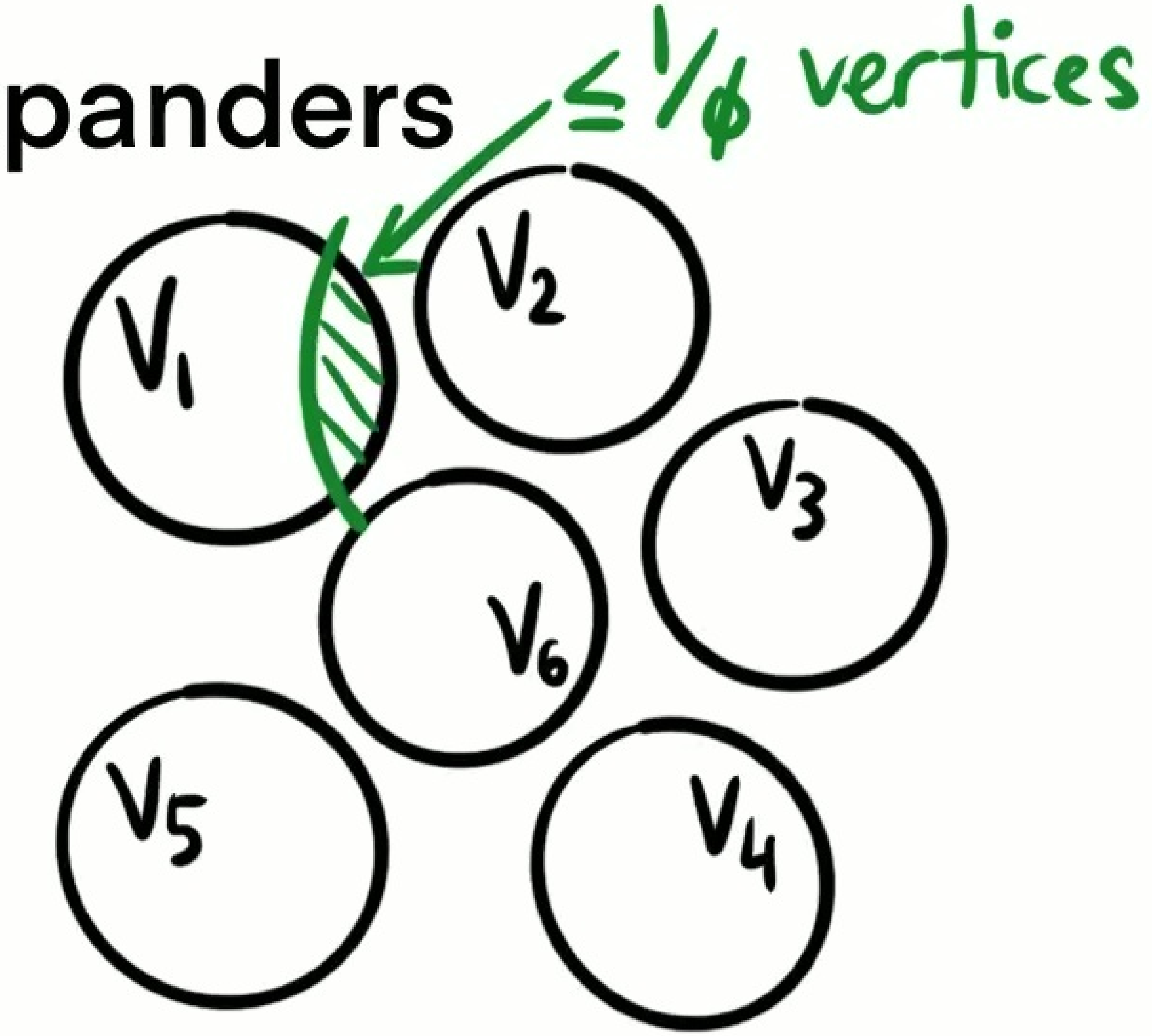

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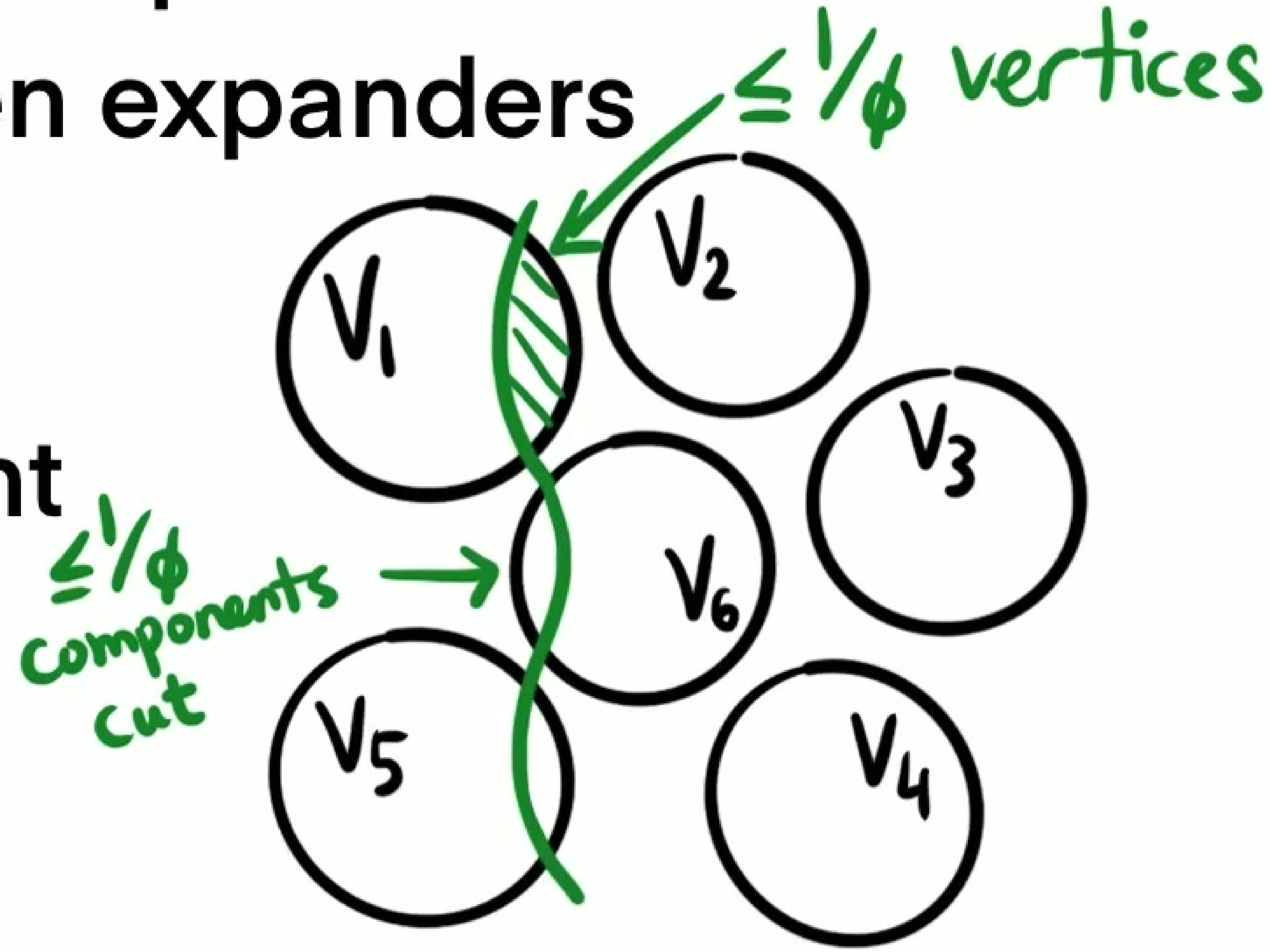

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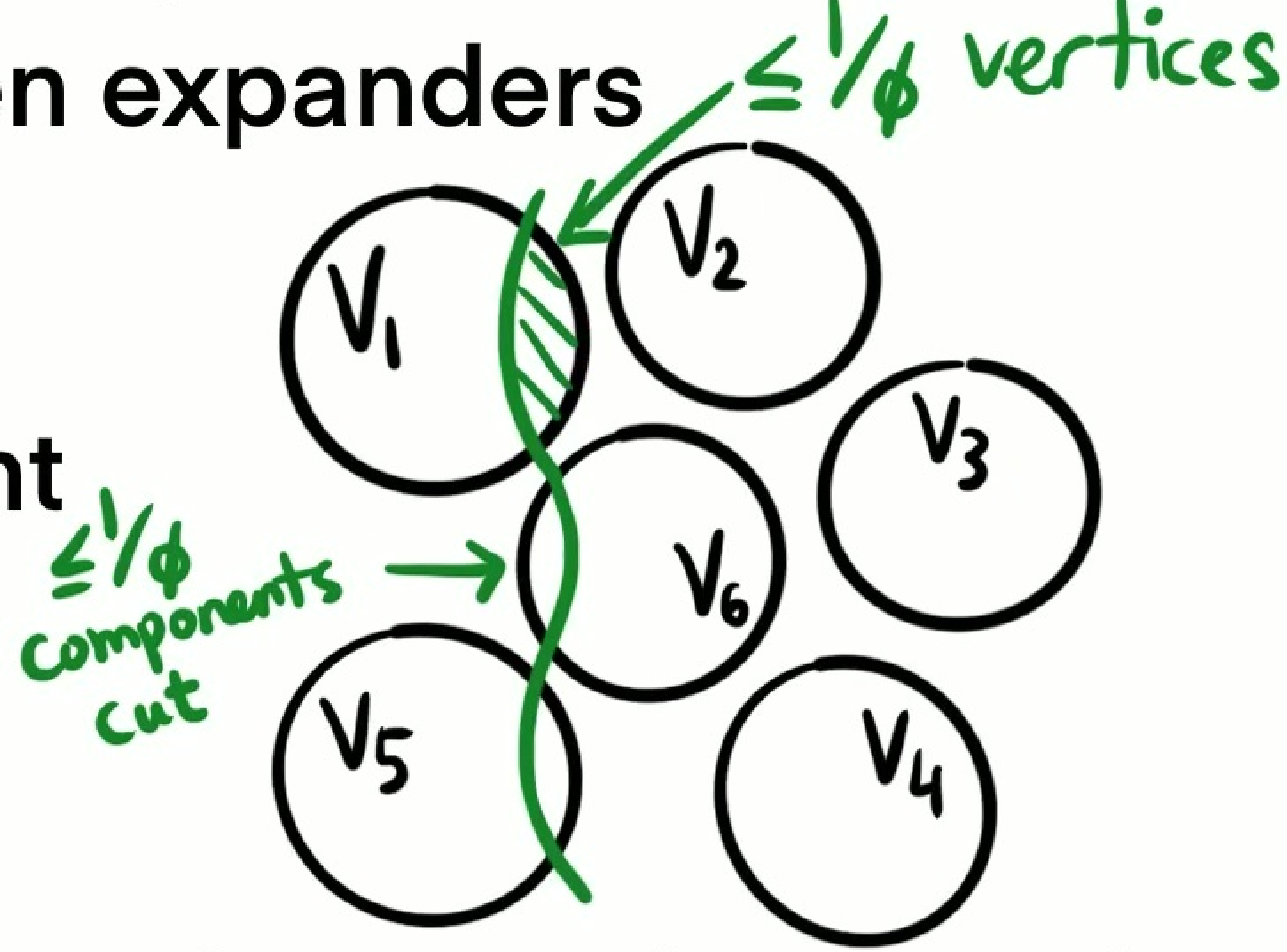

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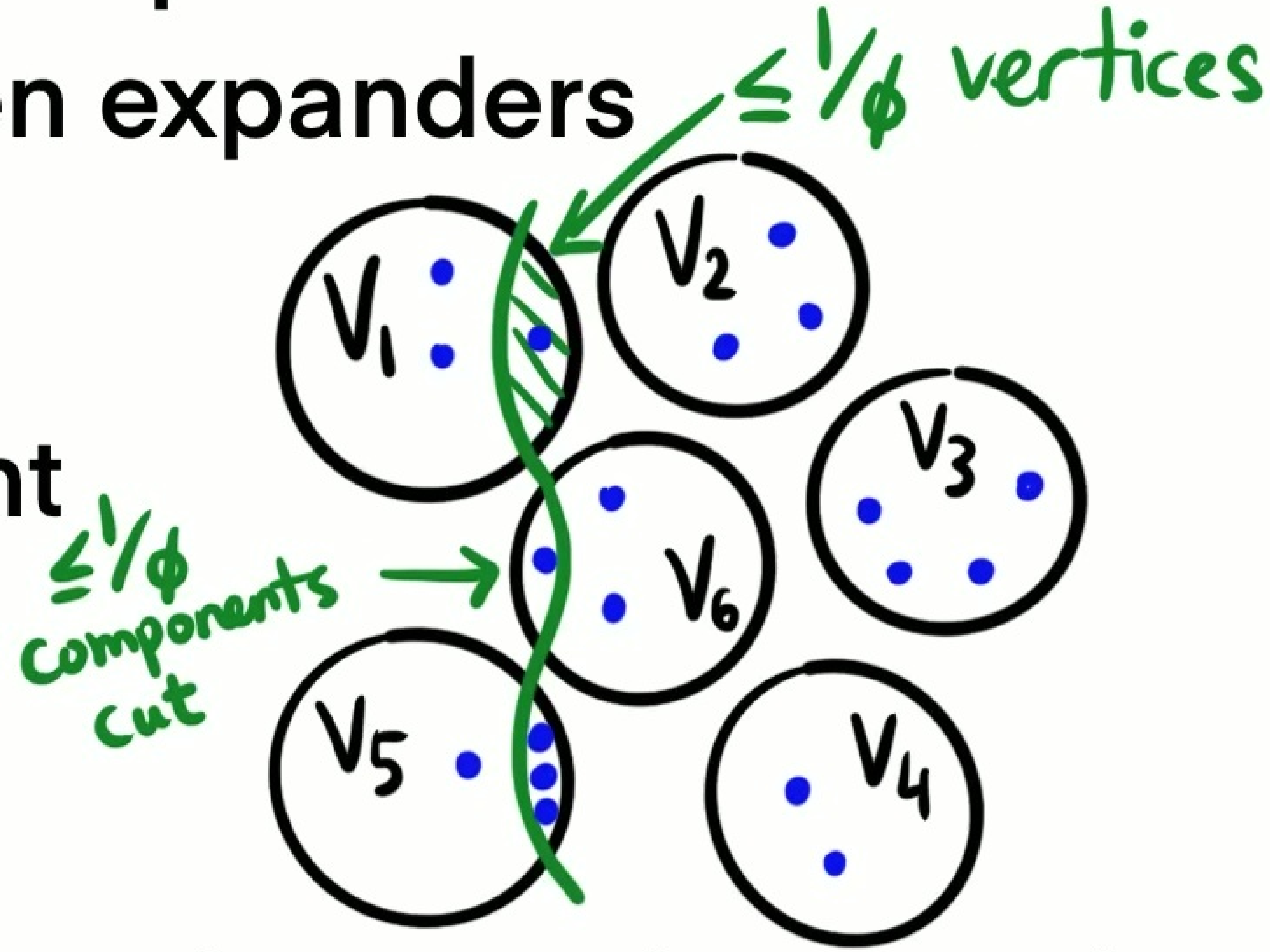

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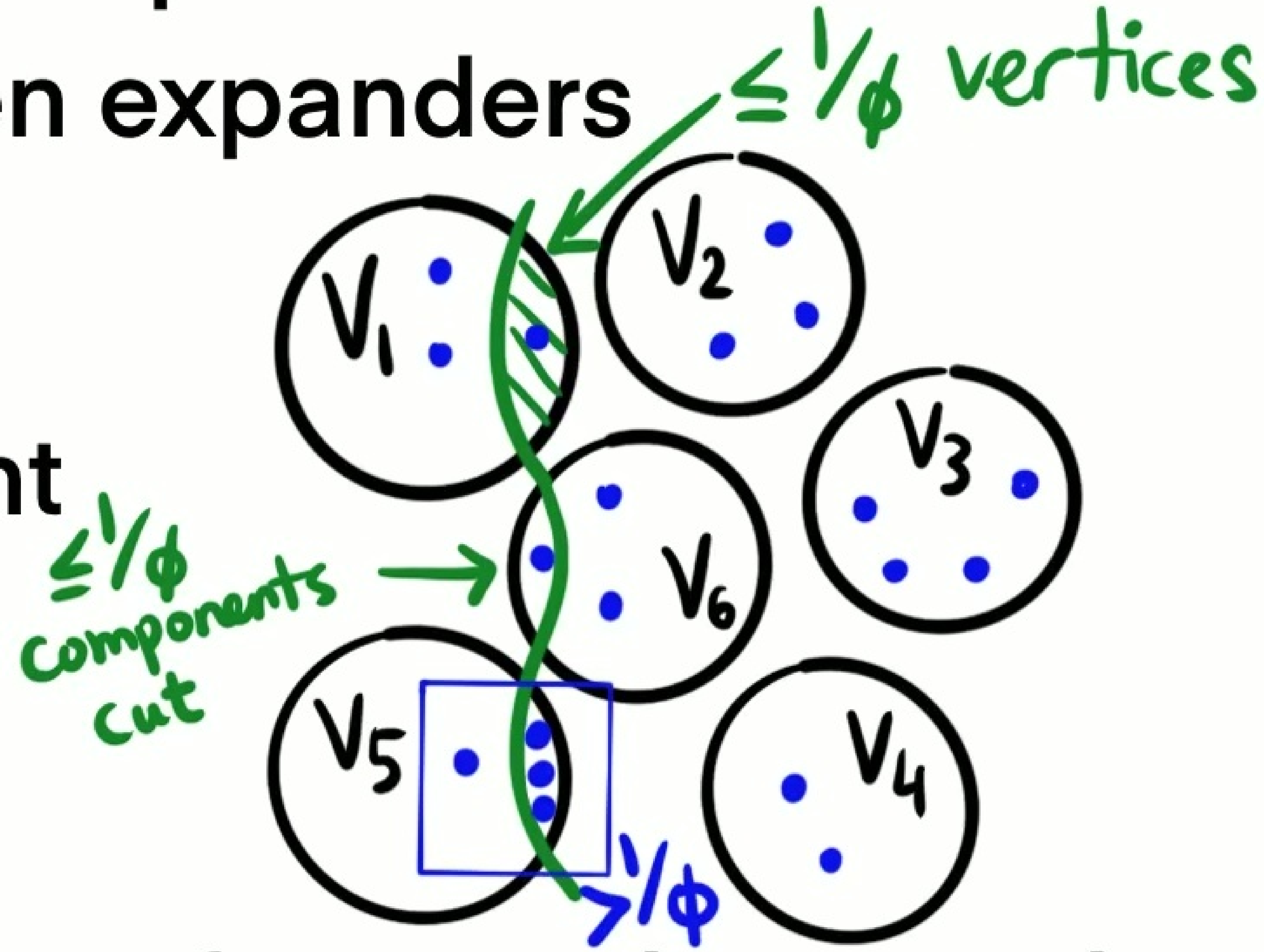

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Other applications of isolators

- Steiner min-cut in polylog(n) max-flows
- More applications?