Faster Exact and Approximate k-cut

Jason Li

Joint with Anupam Gupta (CMU), Euiwoong Lee (NYU)

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Min k-cut Problem

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- [GLL'18] 1.9997-apx in FPT time $f(k)\,\mathrm{poly}(n)$
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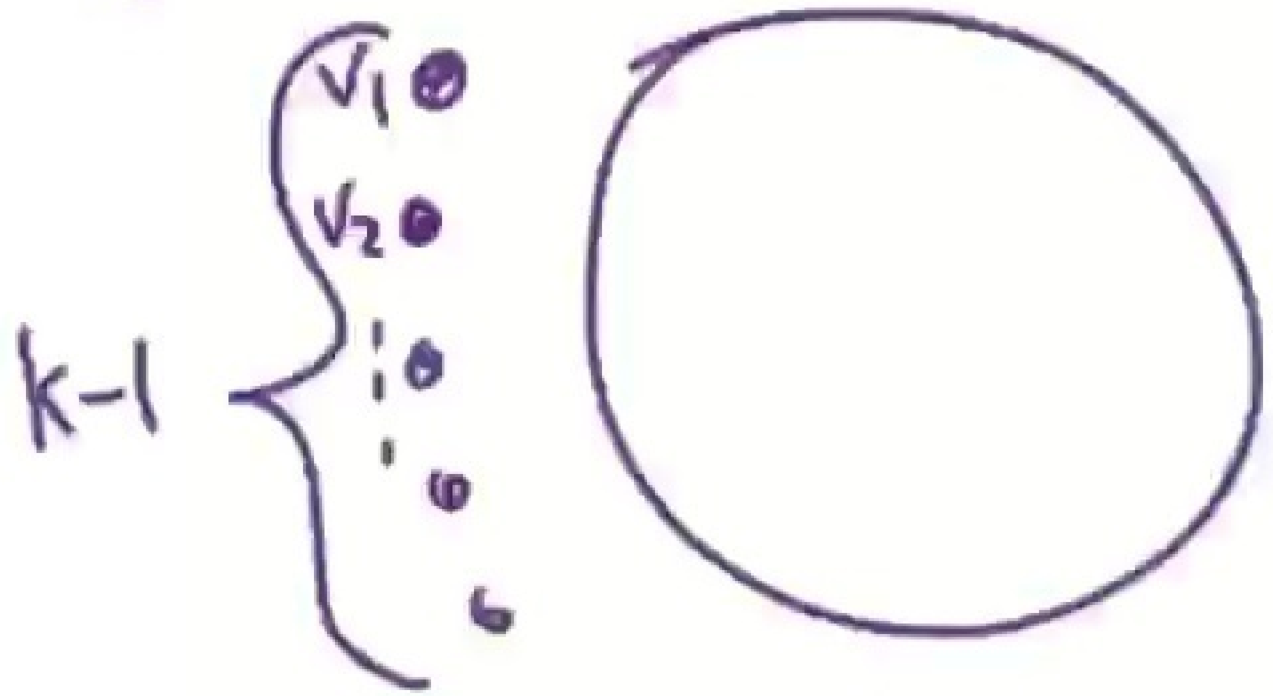

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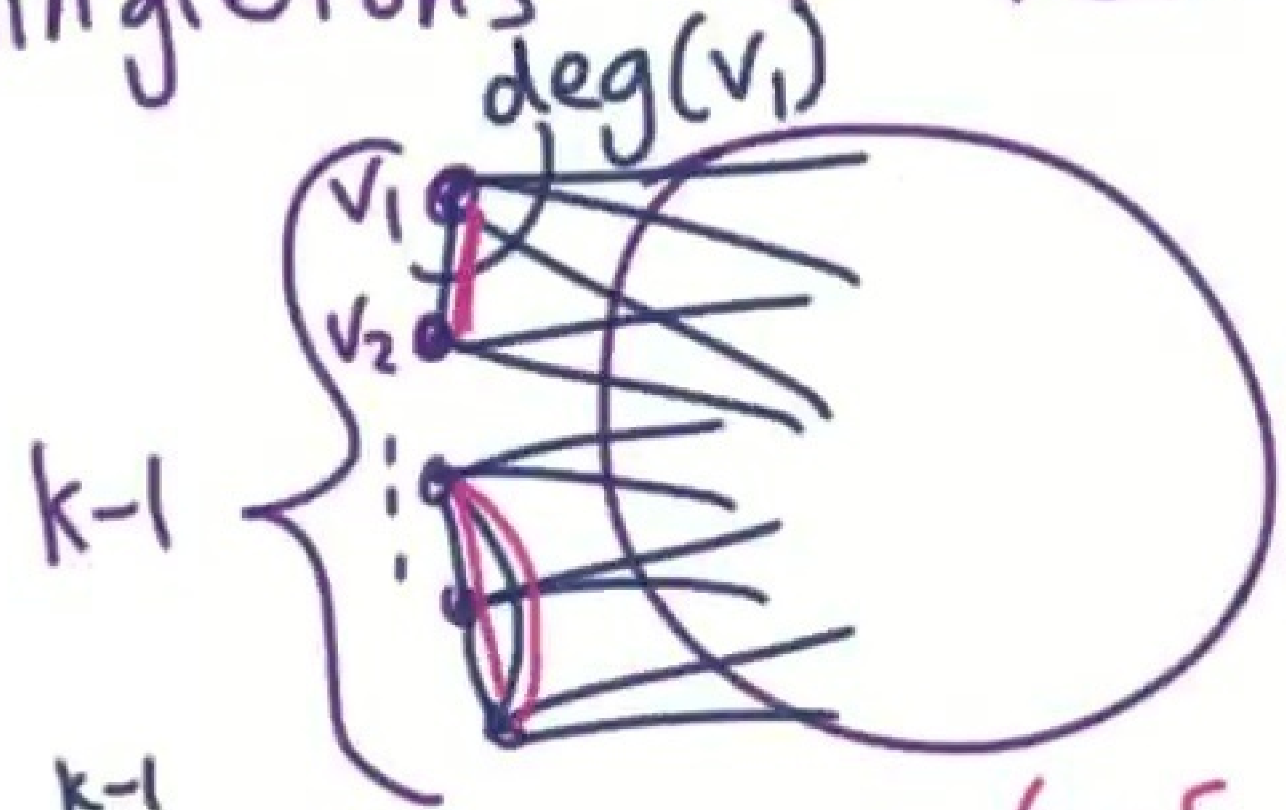

- $k\text{-cut} = \sum_{i=1}^{k-1} \deg(v_i) - w(E[v_1, v_2, \ldots, v_{k-1}])$

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• If G is regular, minimized when

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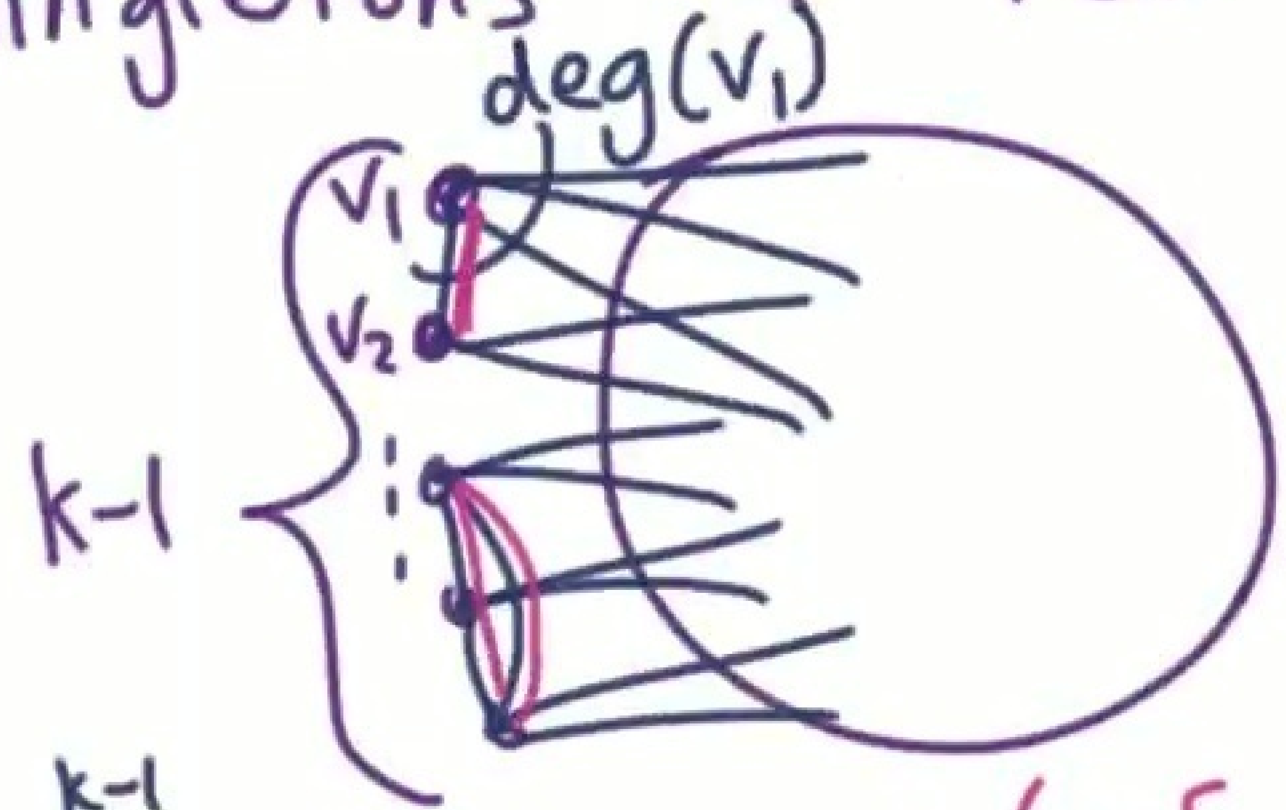

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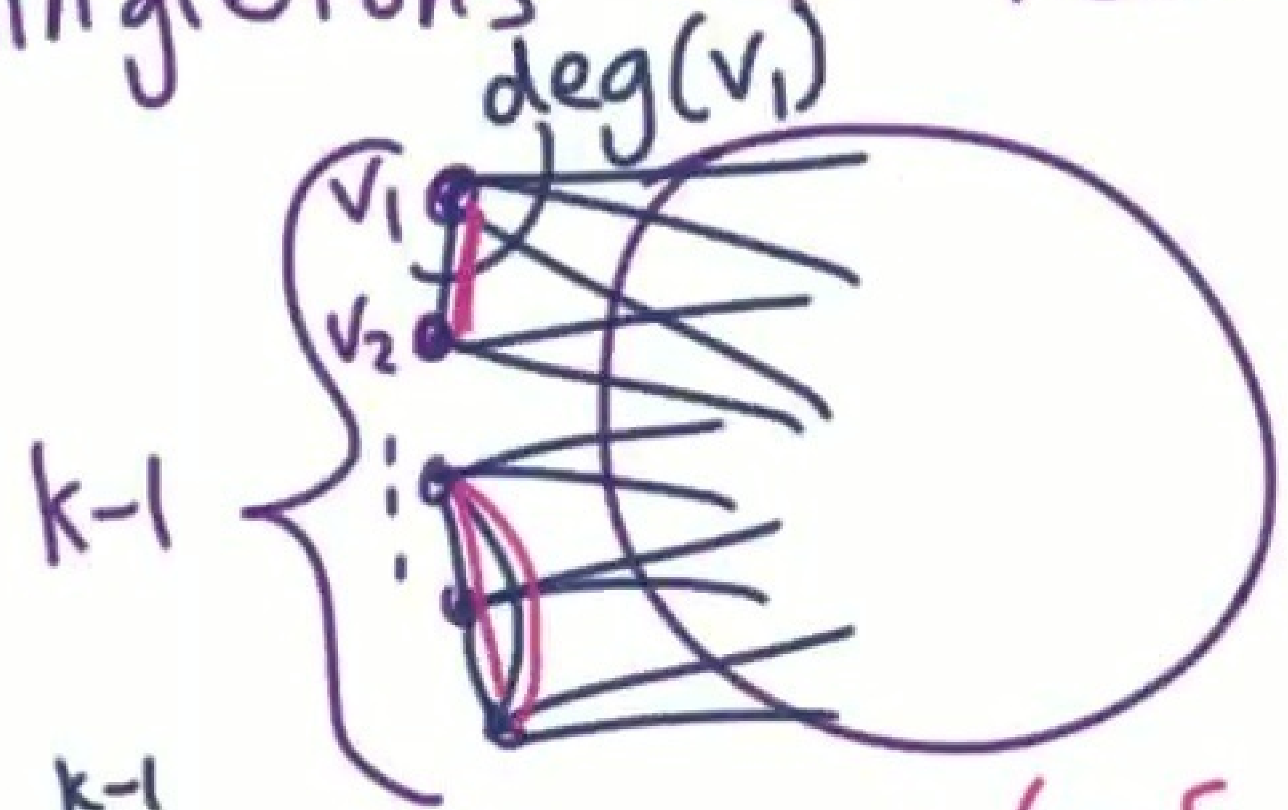

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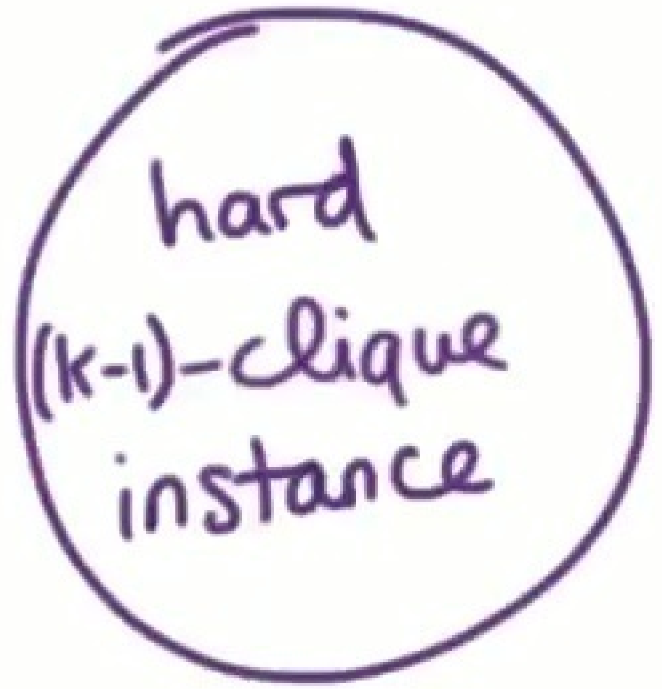

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Thorup's tree packing

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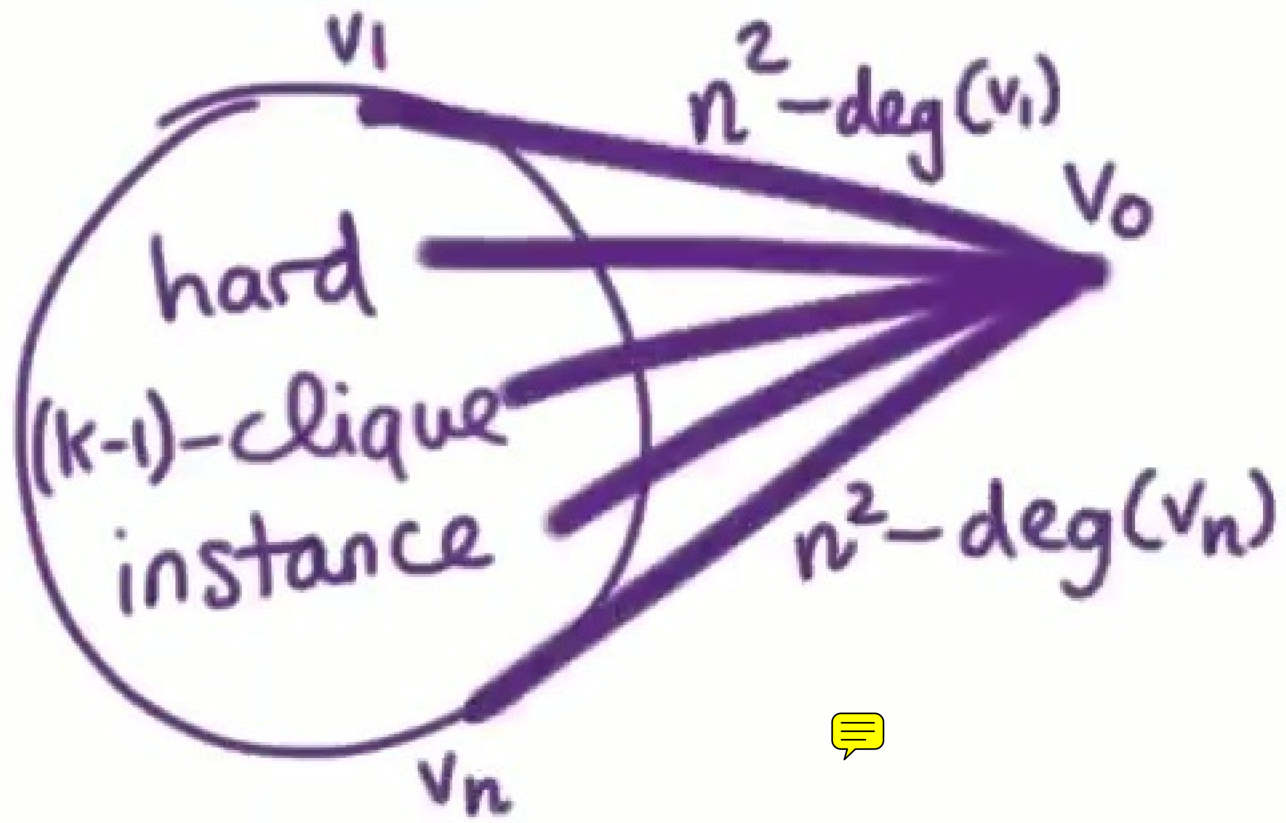

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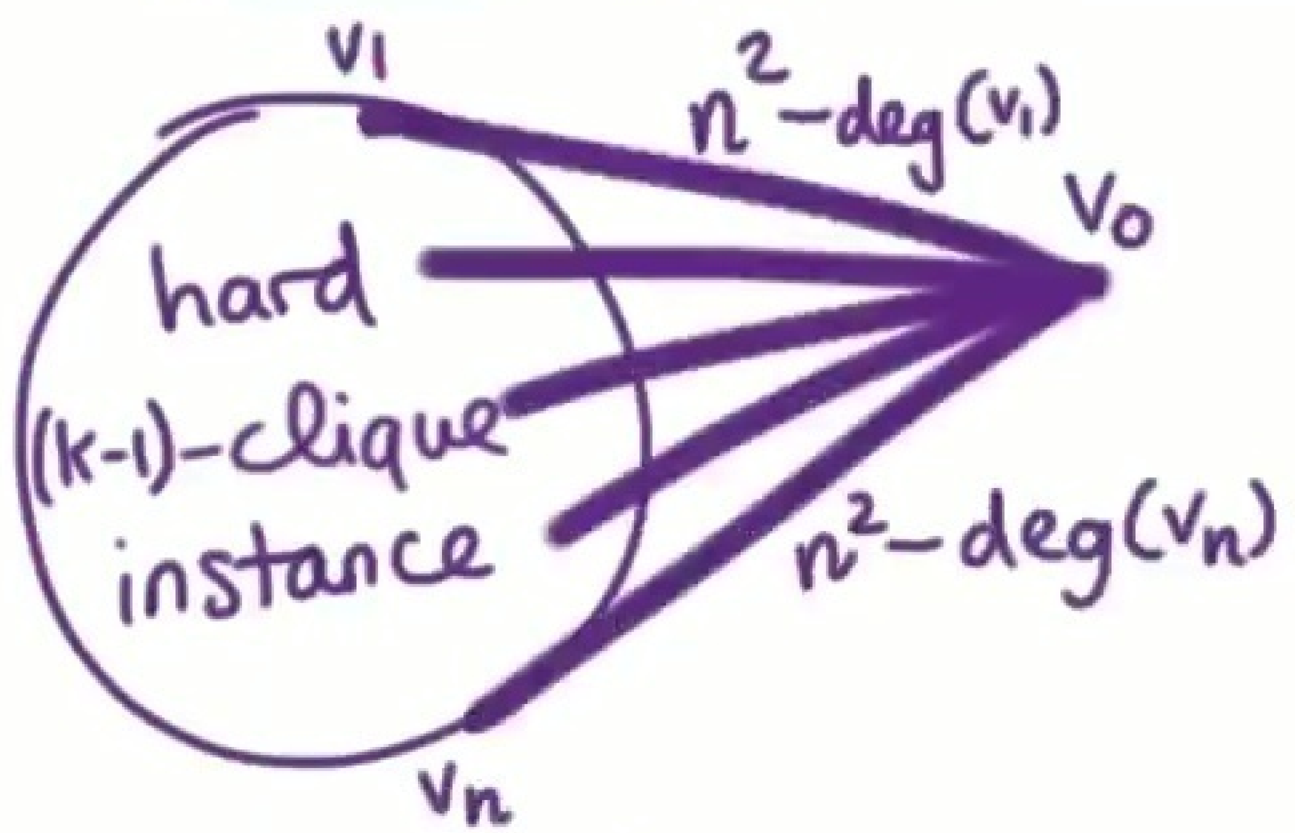

- $\deg(v) = n^2 \quad \forall v$
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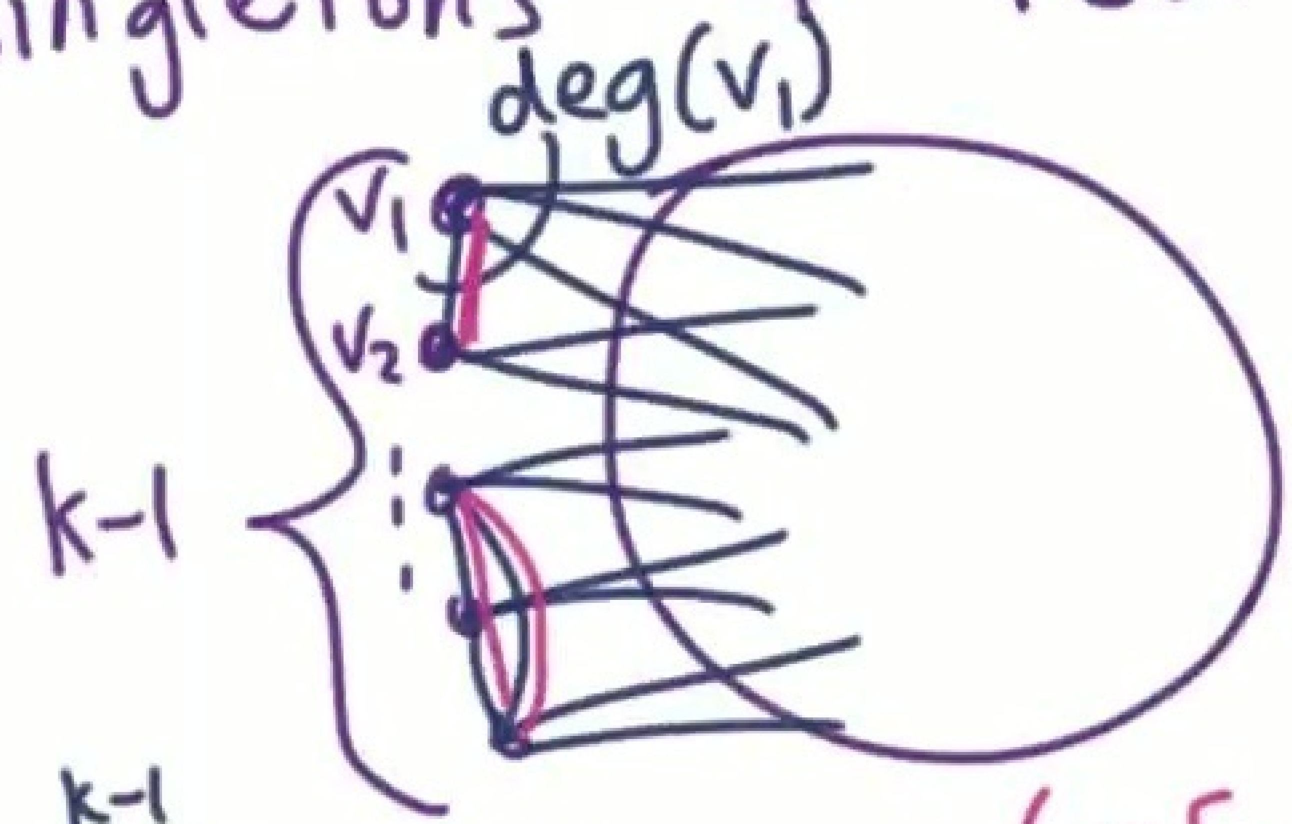

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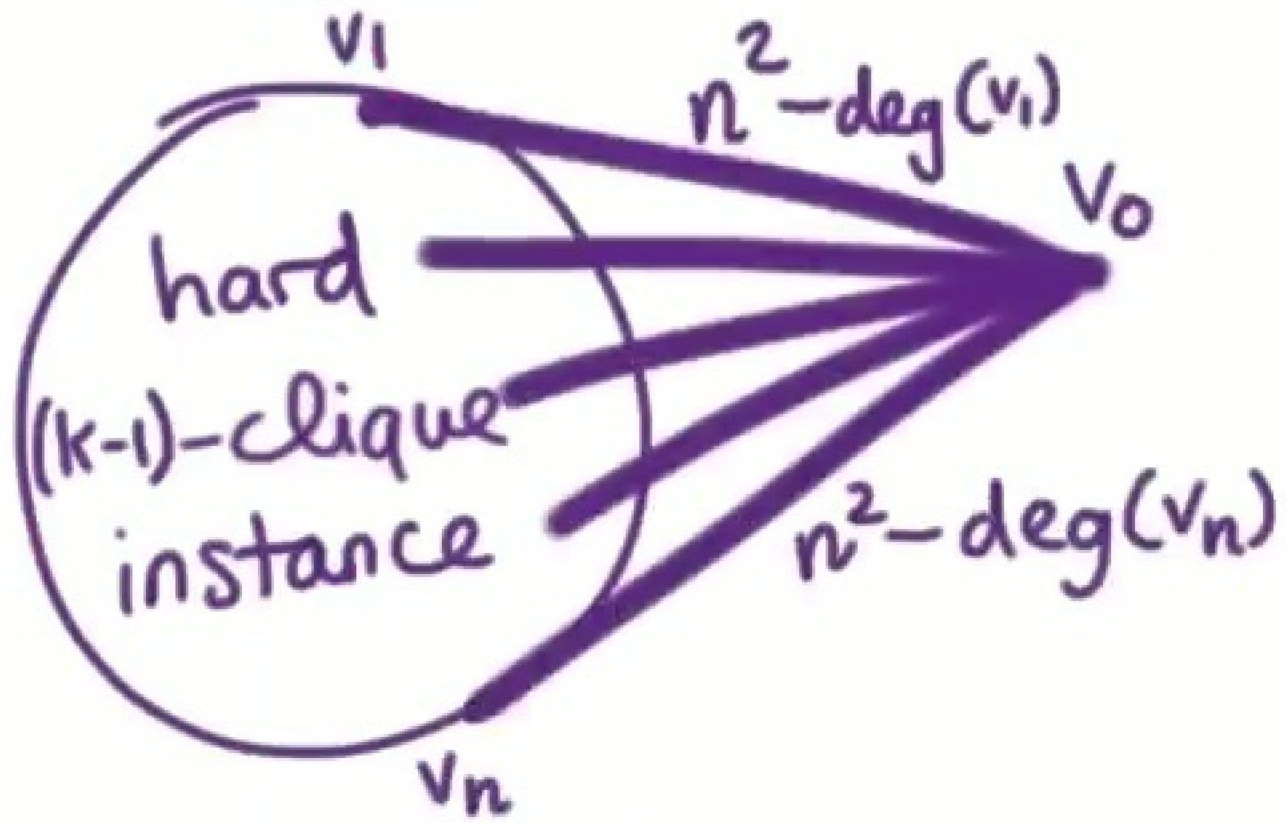

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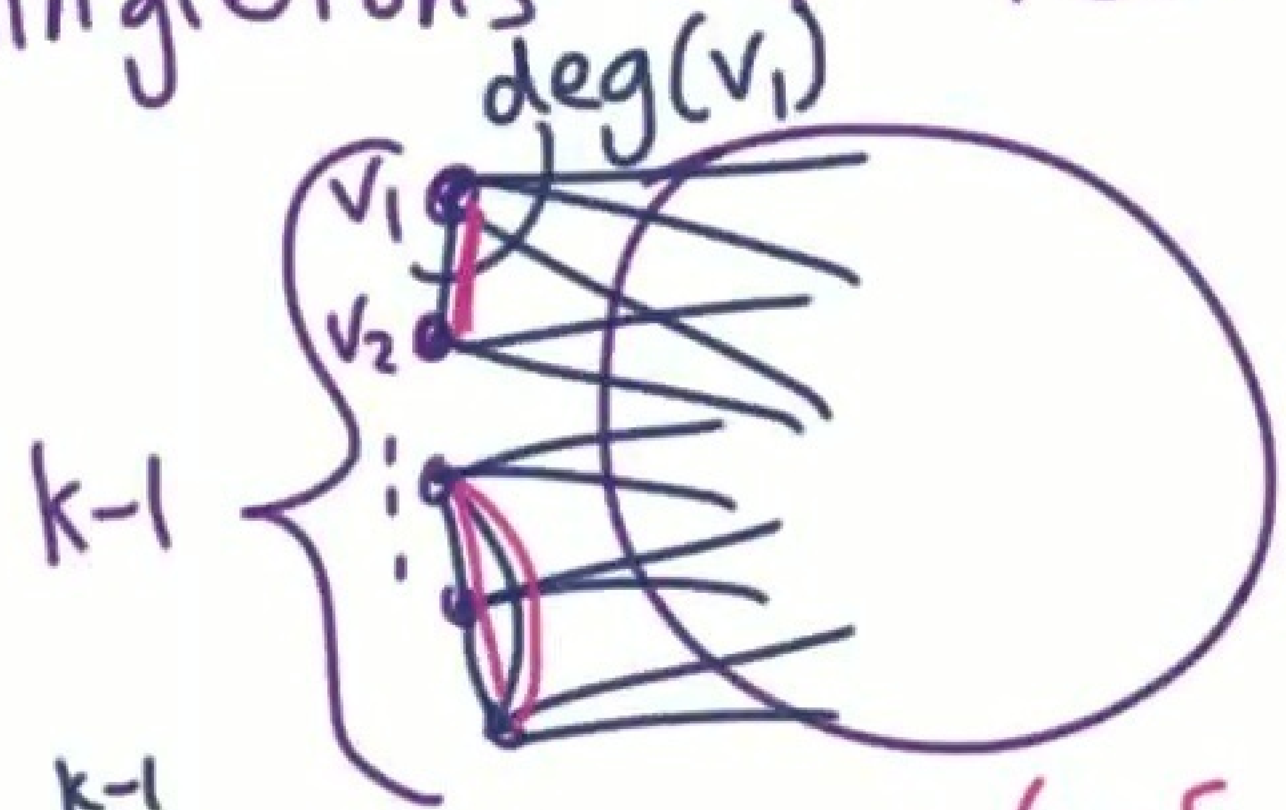

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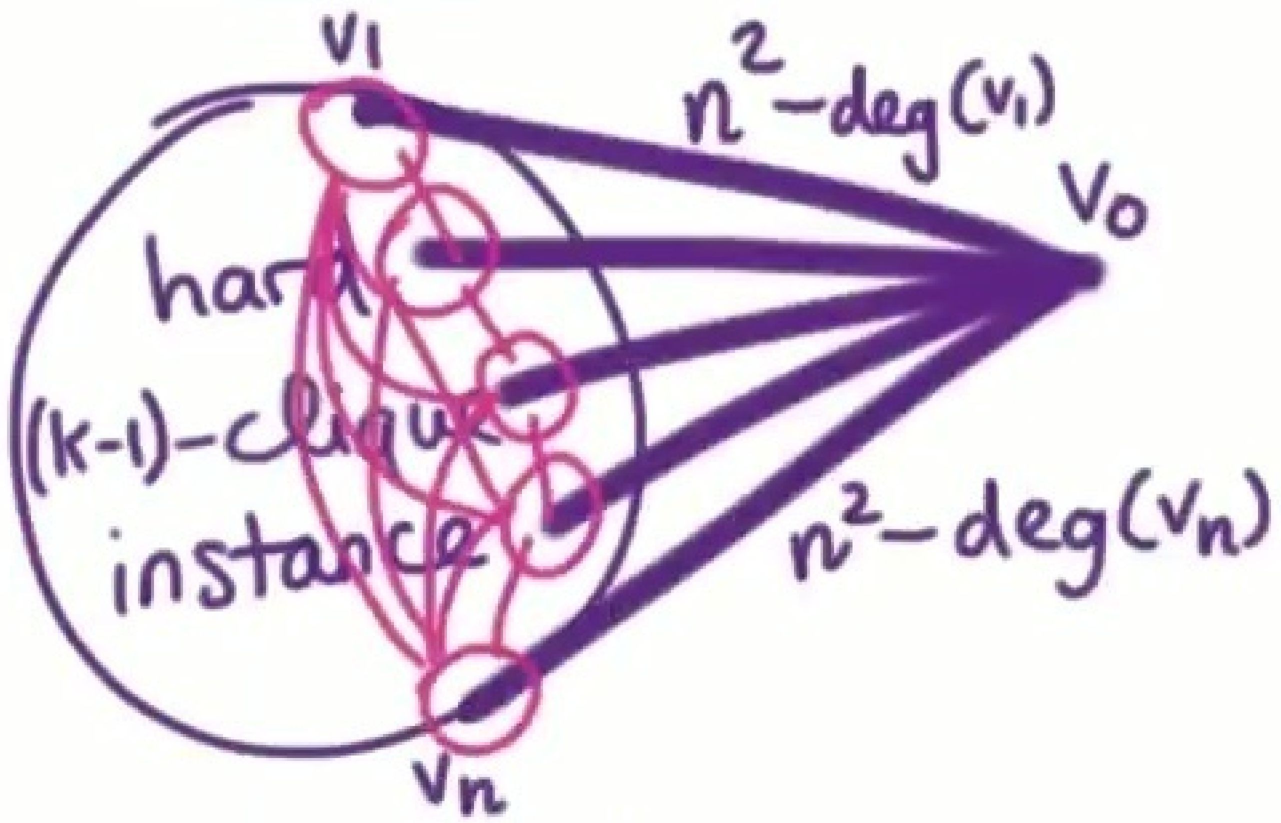

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  $\Longrightarrow (k-1)$ singletons
- $(k-1)$-clique $\Longleftrightarrow$ min k-cut is $(k-1)n^2 - \binom{k}{2}$

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Reduction to (k-1)-resp.

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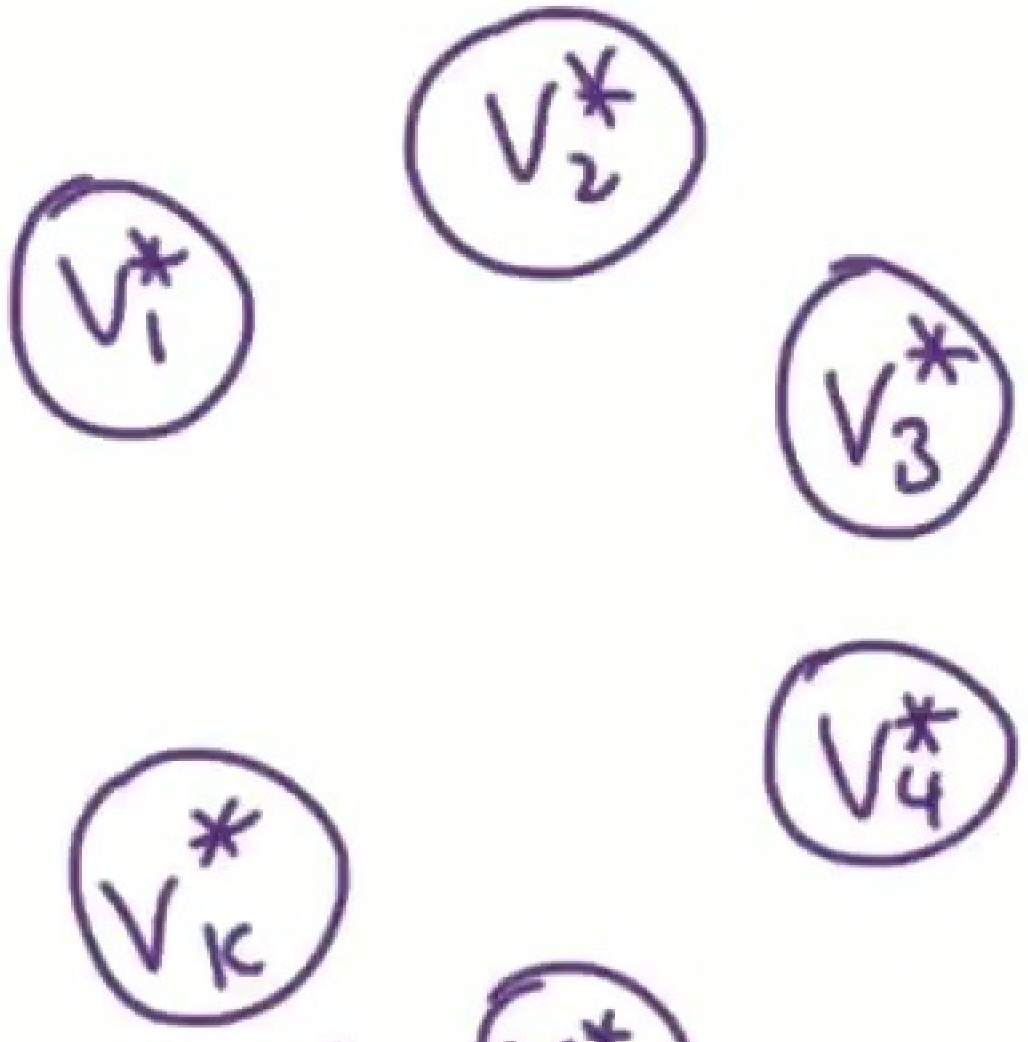

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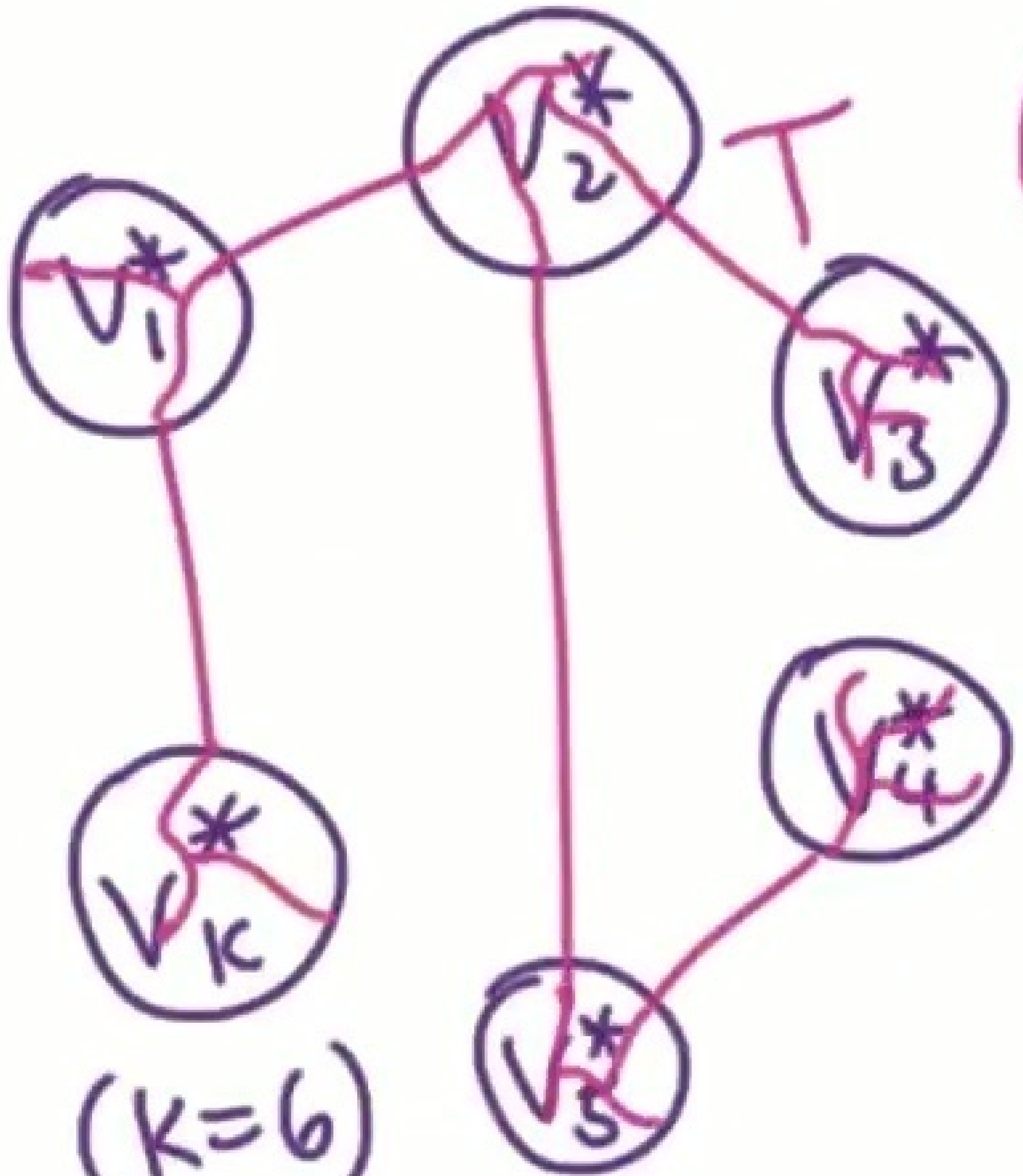

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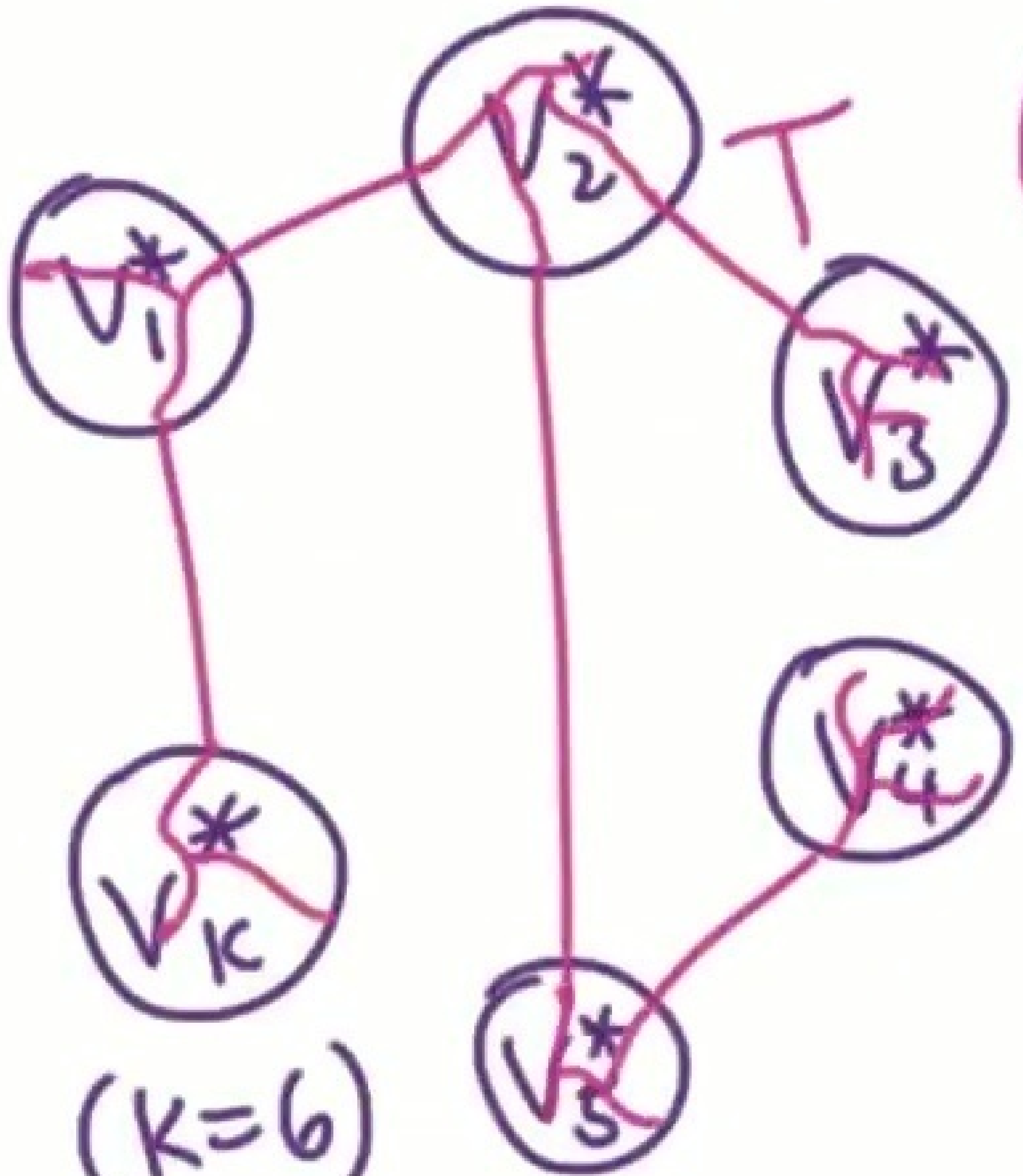

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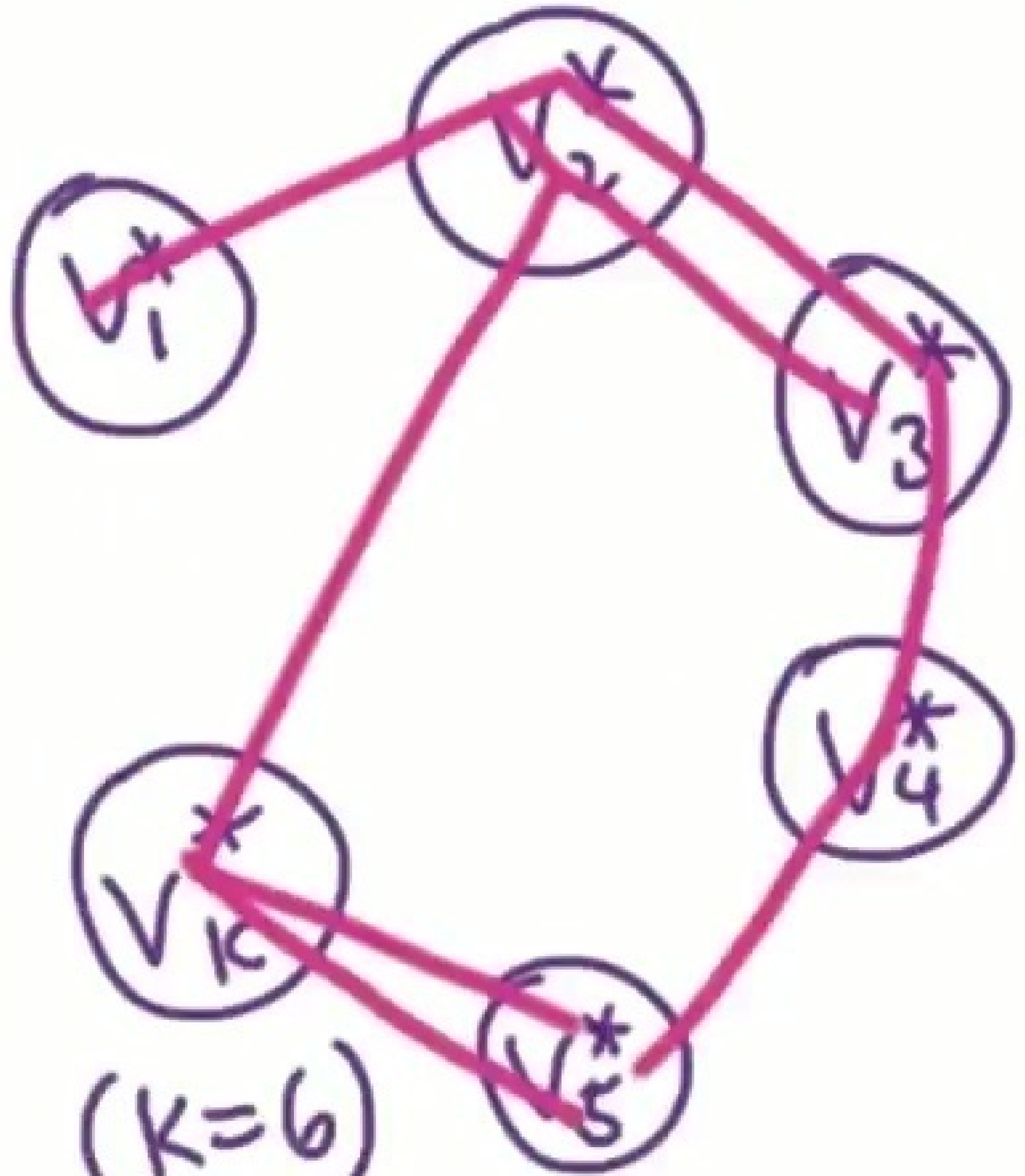

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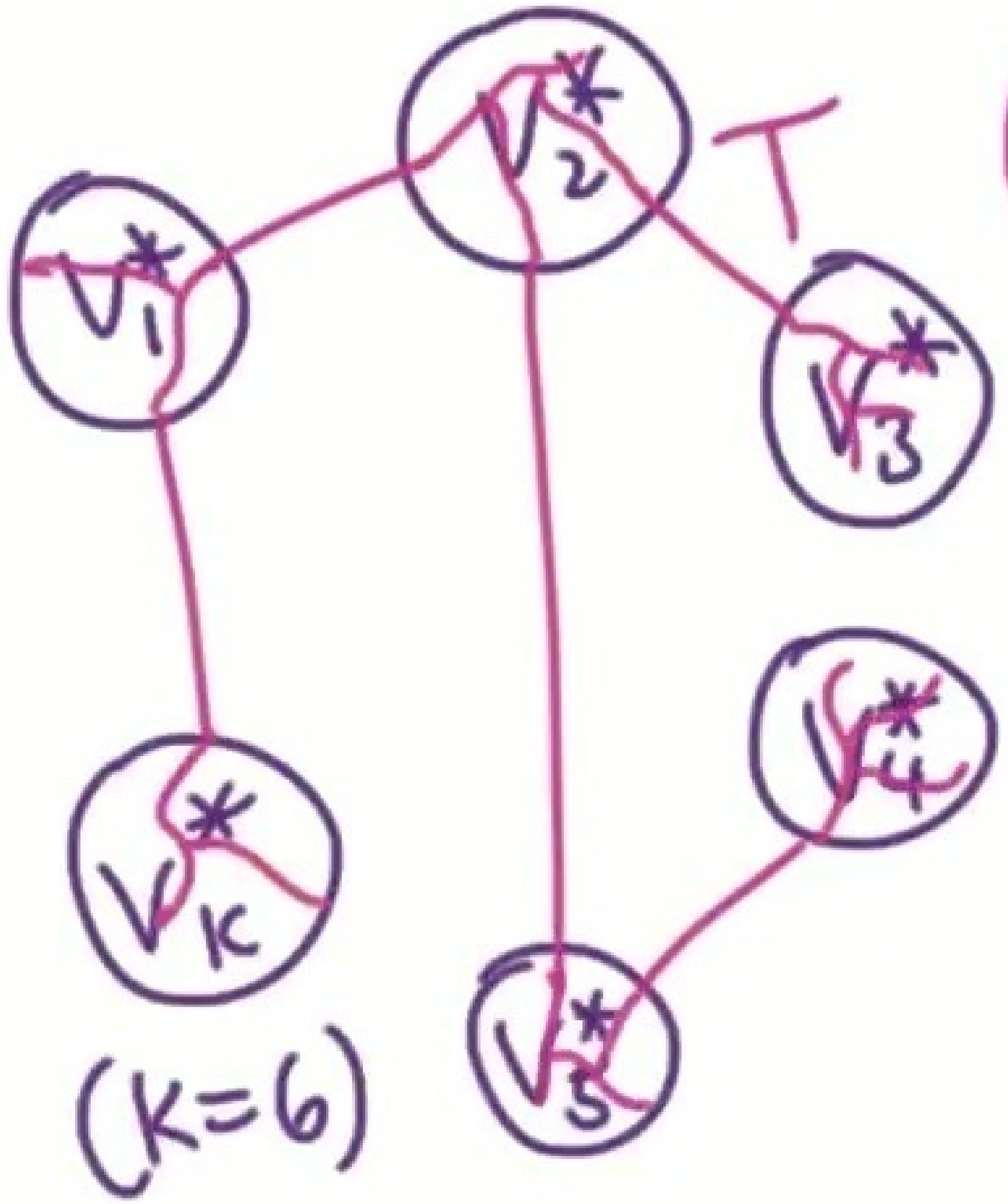

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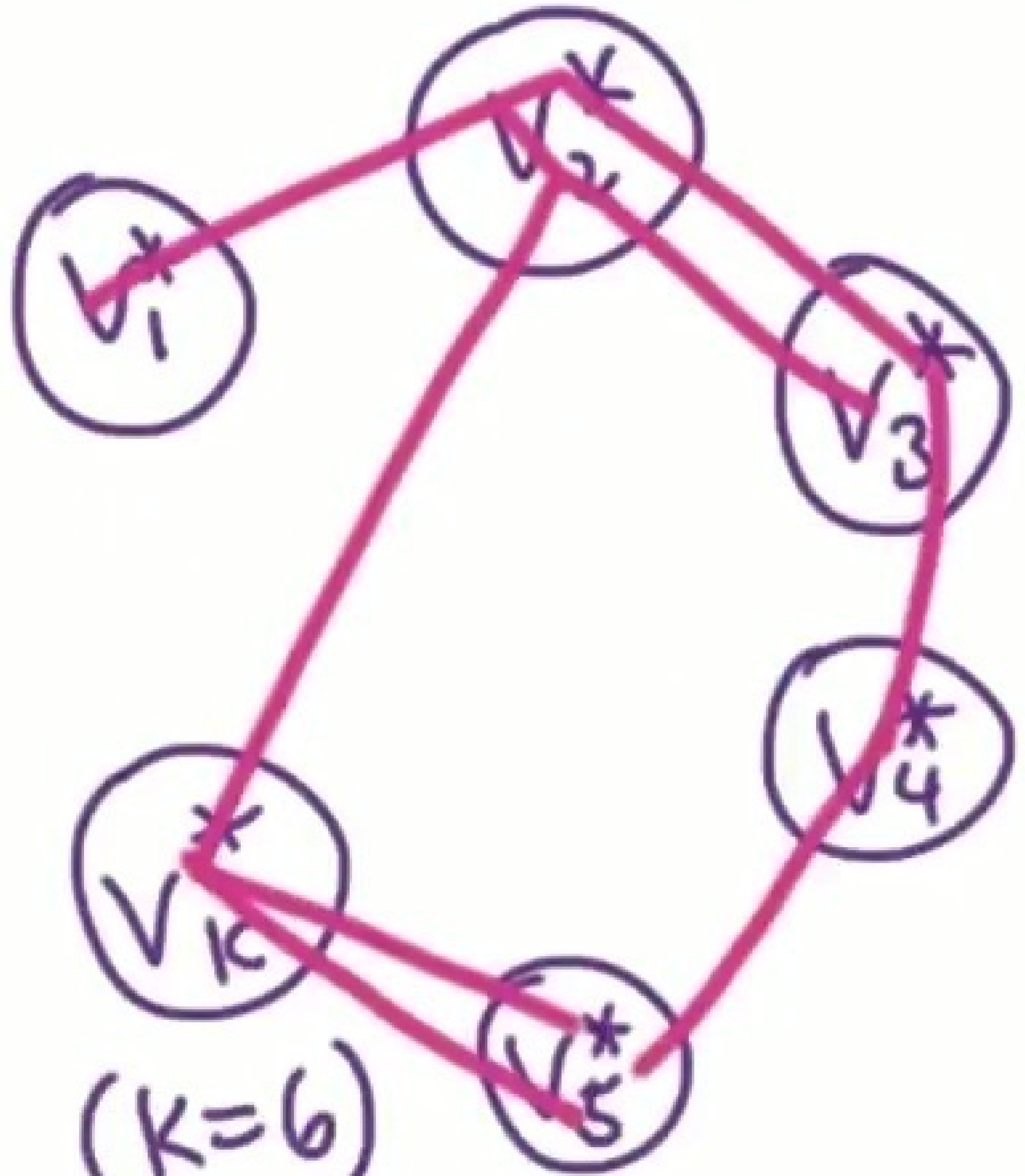

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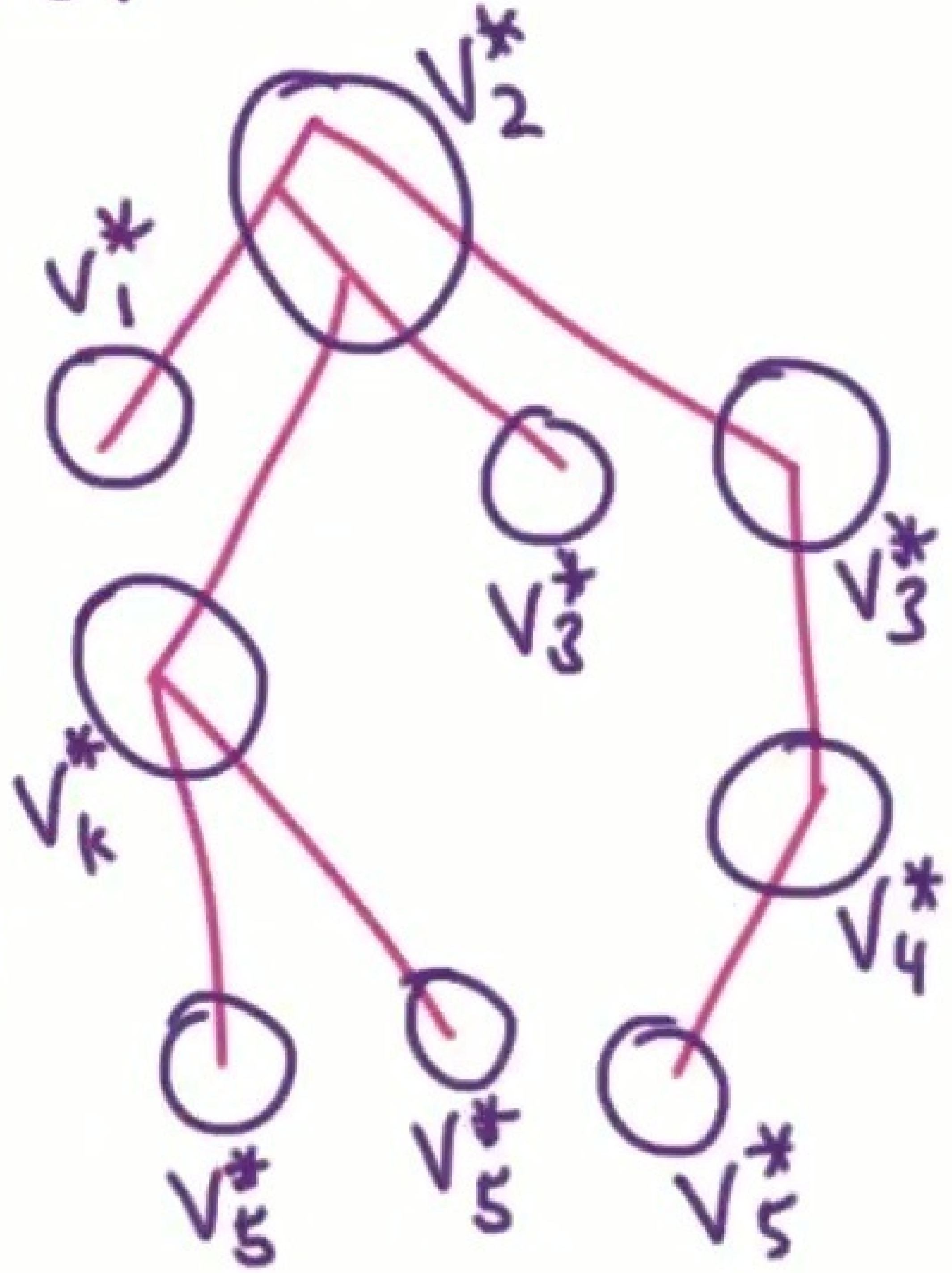

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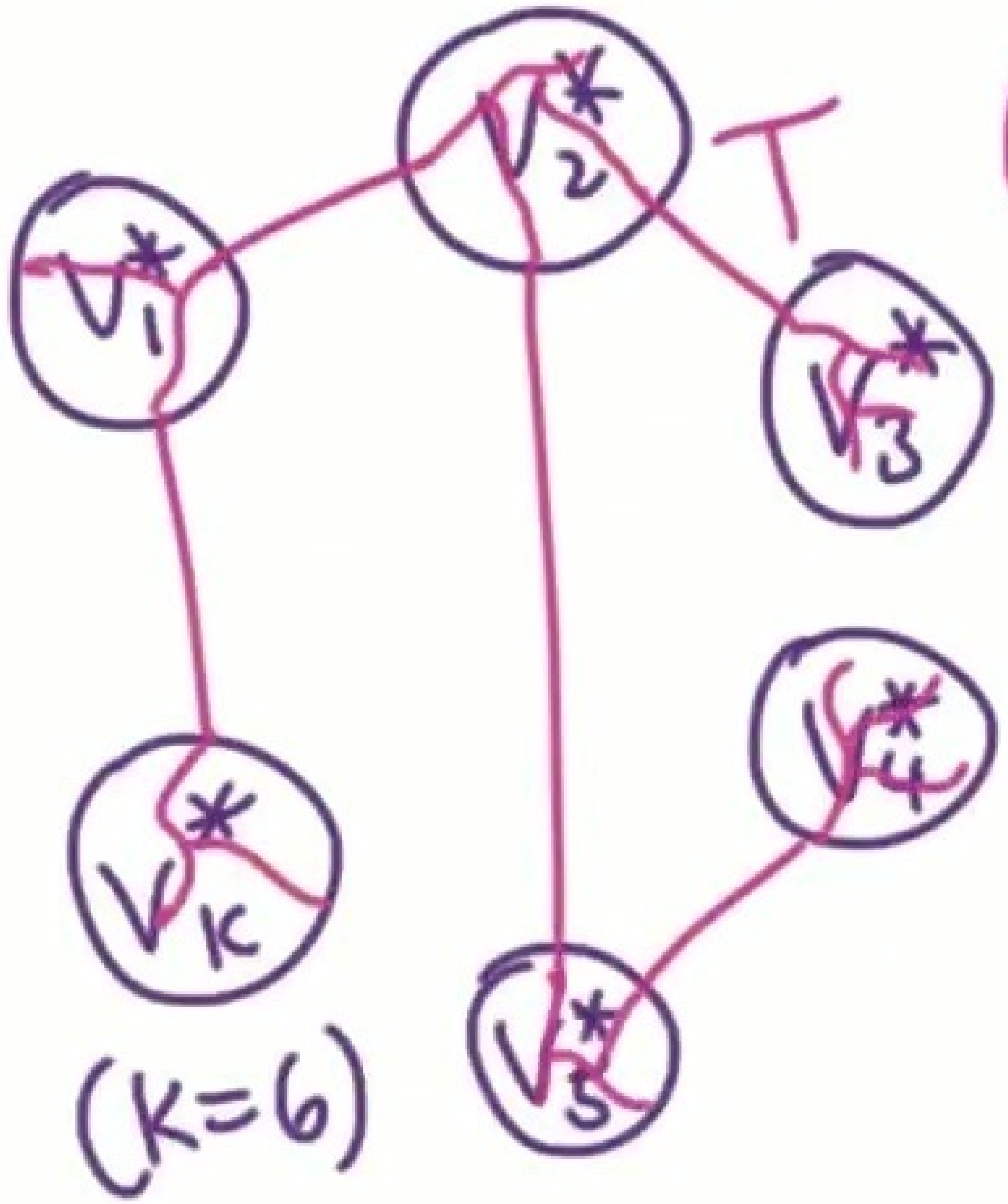

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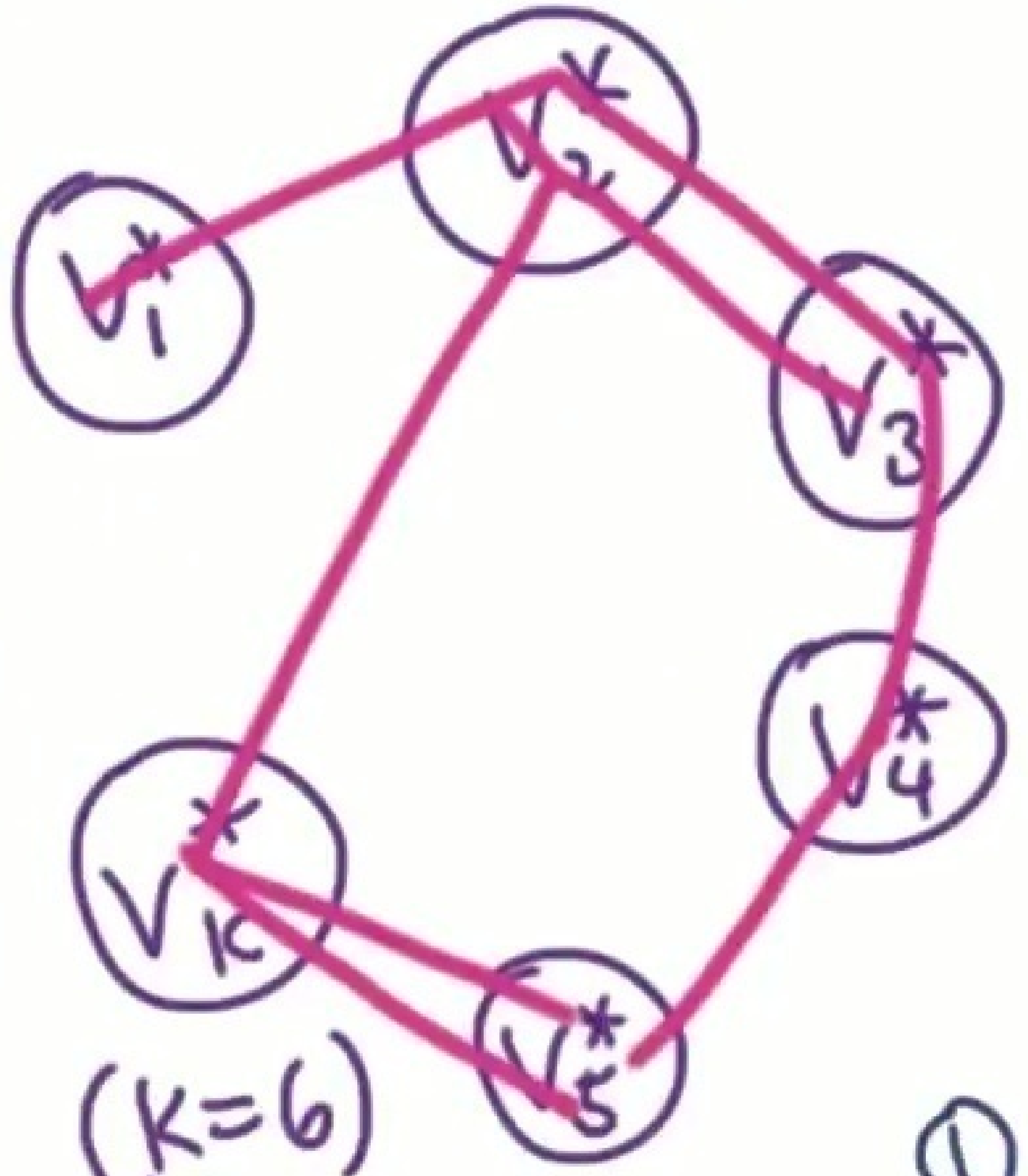

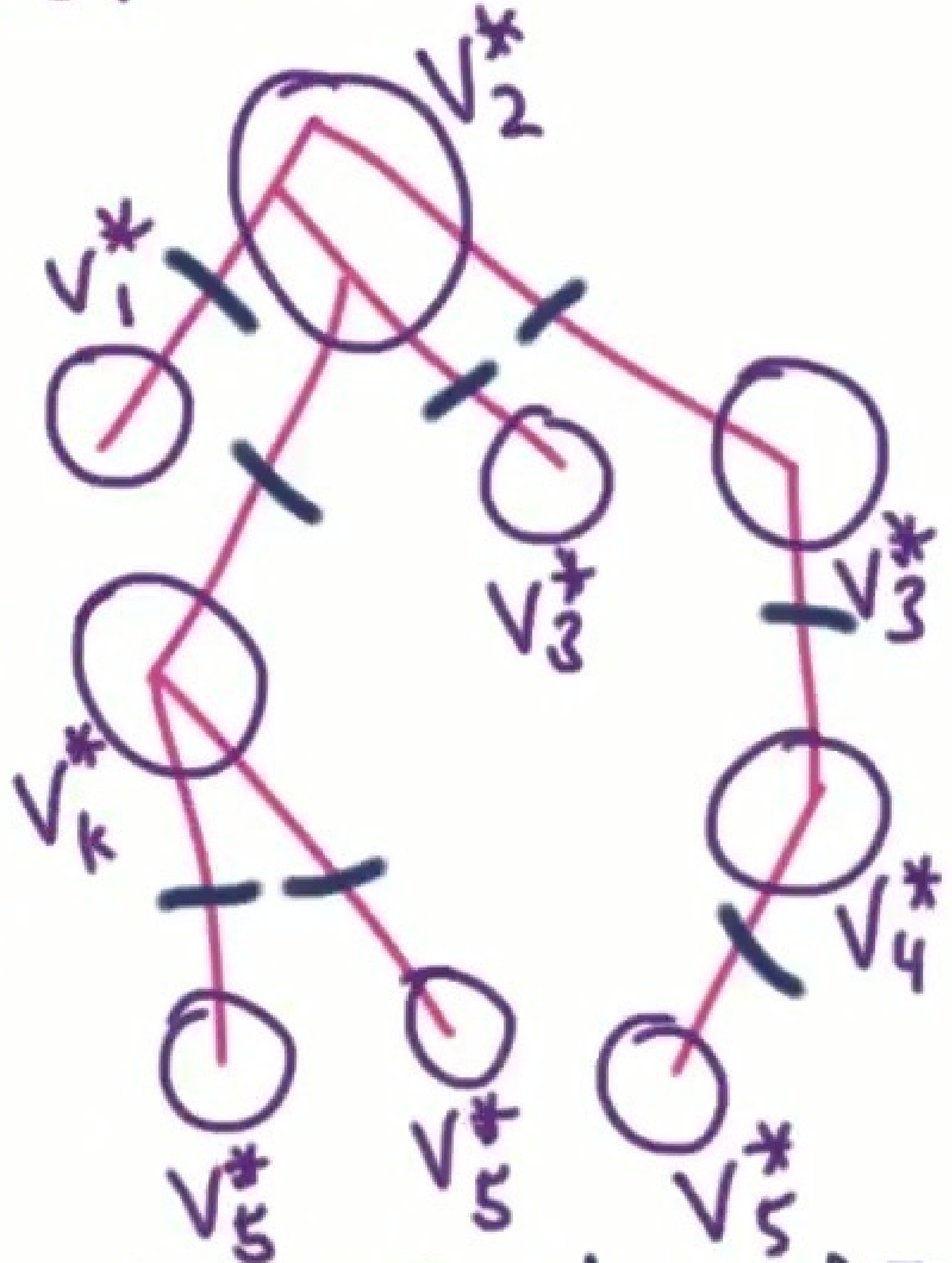

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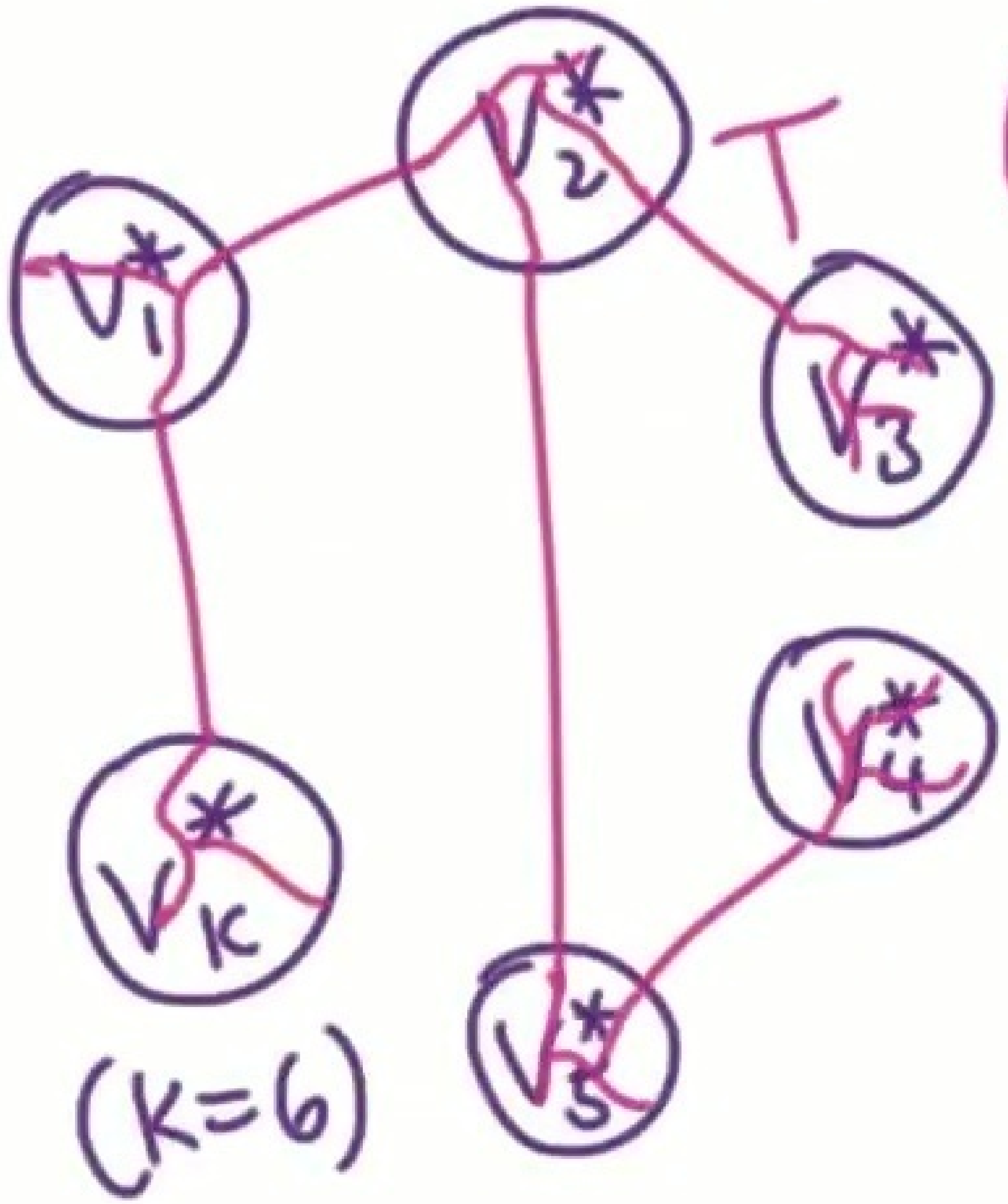

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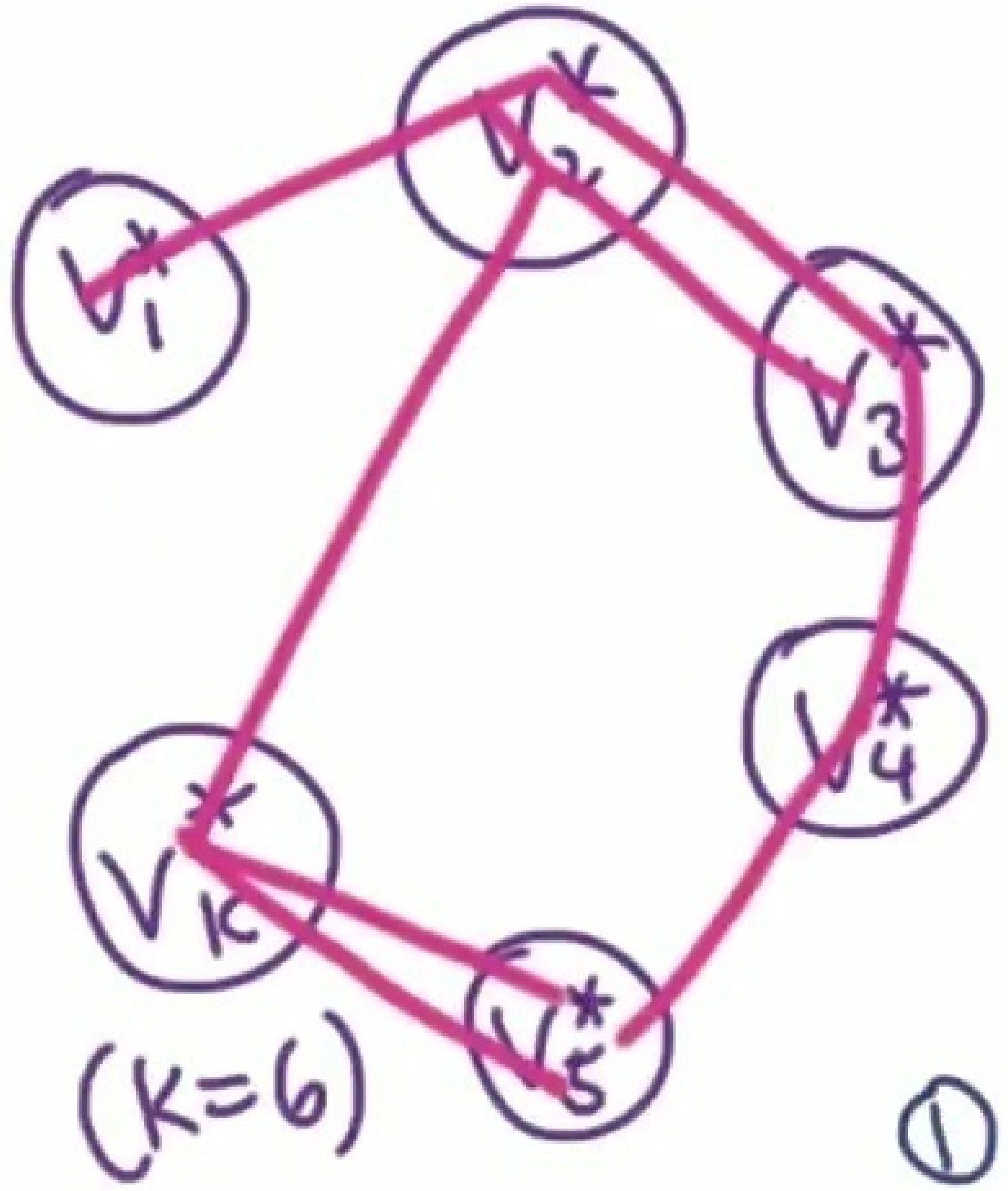
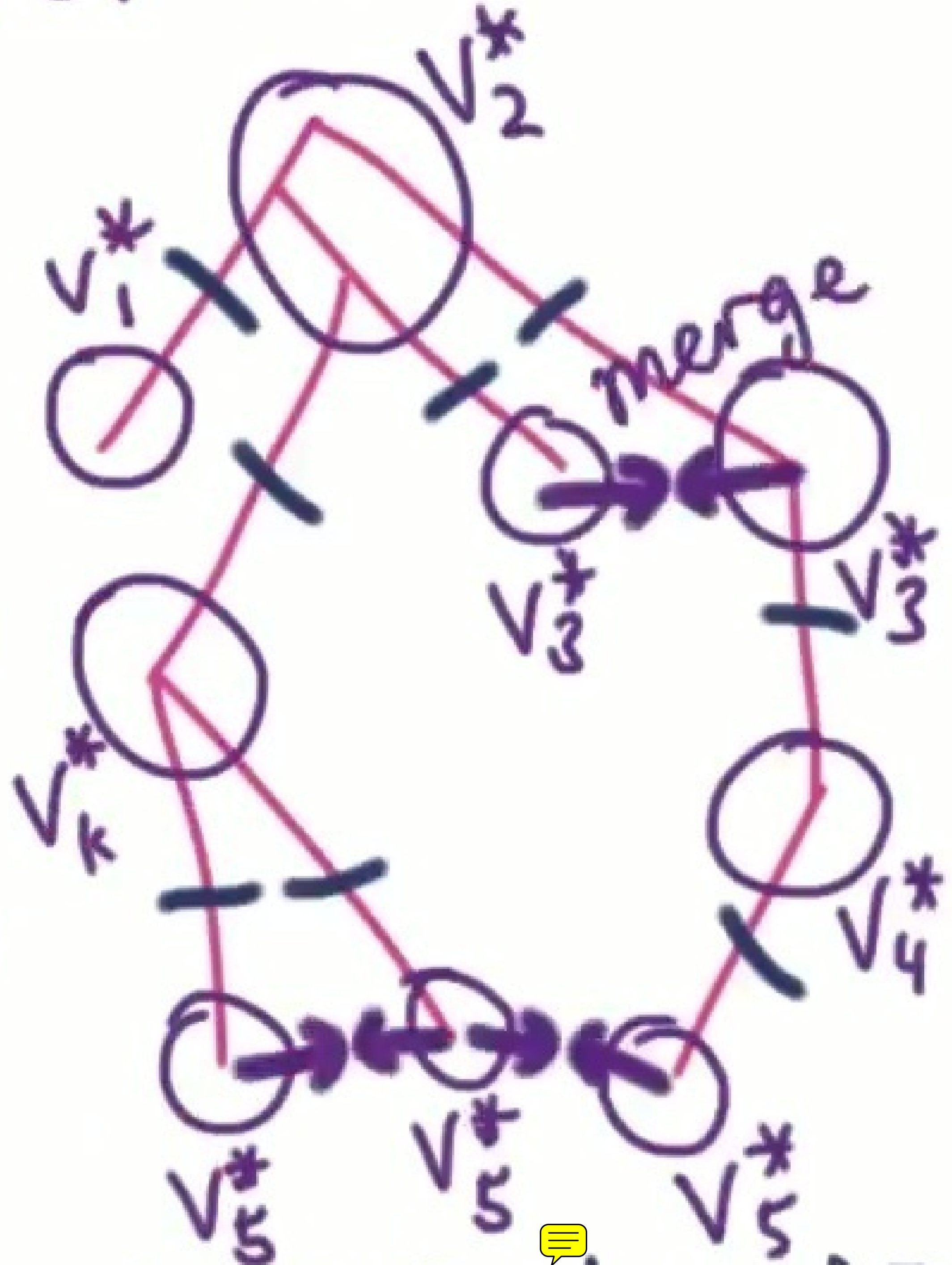


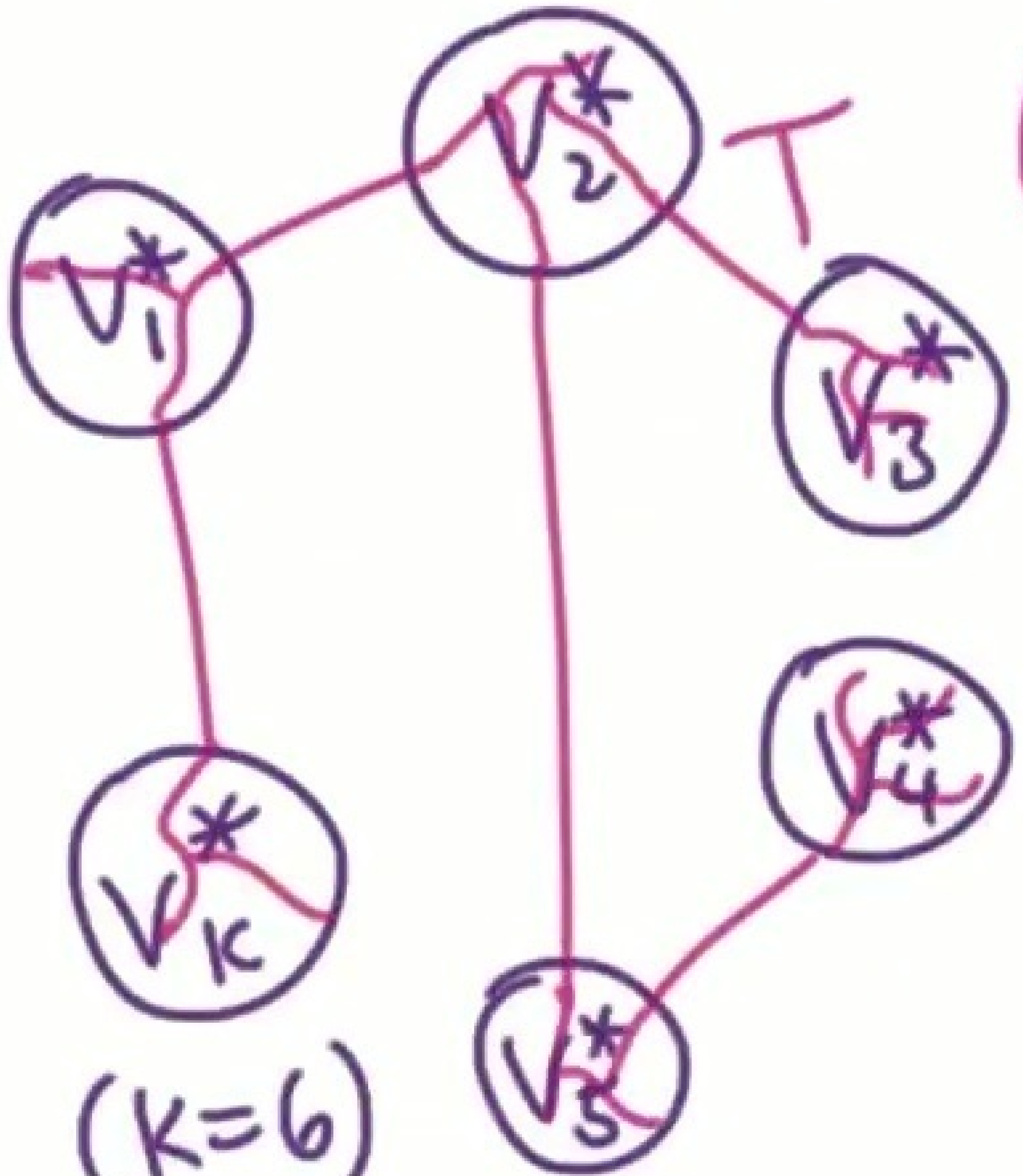


① cut $\le 2k-2$ edges of T
② merge comps

Matrix Multiplication

Exact $n^{(1+\omega/3)k}$
Thorup's tree packing
Reduction to (k-1)-resp.
$\rightarrow$ k-clique-like mtx.mult.
Hardness $n^{(\omega/3)k}$

Easy case: ① (k-1)-respects tree

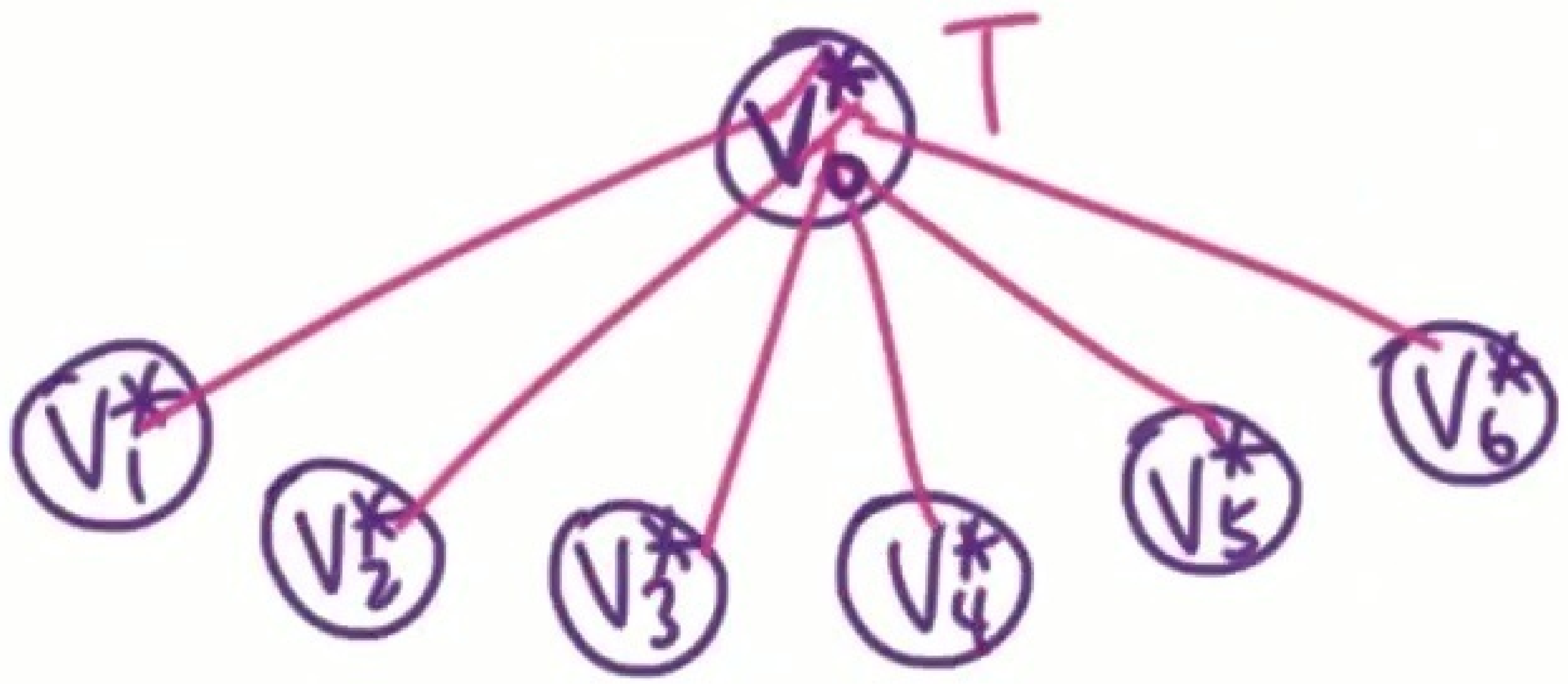


(k-1)-respects
(best possible)

Matrix Multiplication

Easy case: ① $(k-1)$-respects tree

② T connects V_i^* in a star



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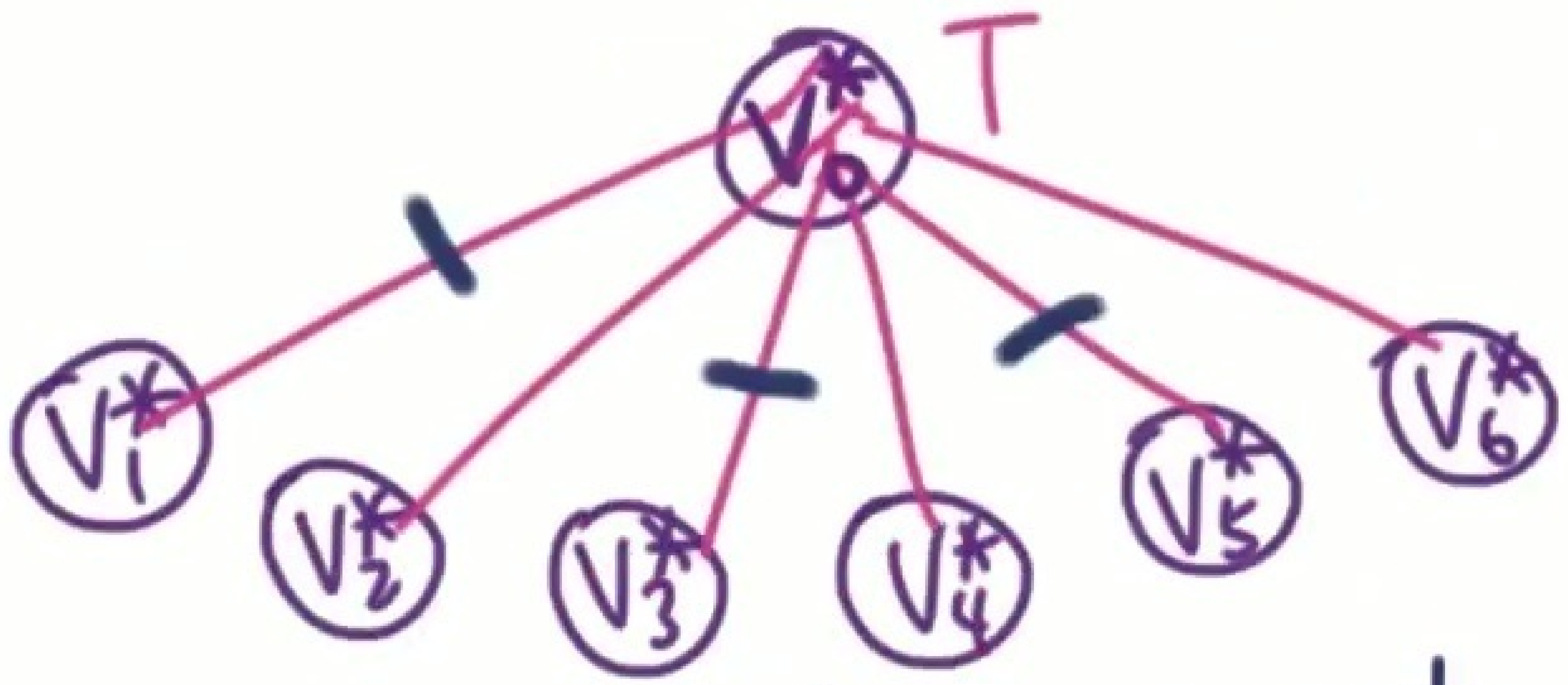

k-cut: isolate (k-1) components V_i^* to minimize k-cut between $V_{i_1}^*, V_{i_2}^*, \ldots, V_{i_{k-1}}^*$ and $\overline{V_{i_1}^* \cup \cdots \cup V_{i_{k-1}}^*}$

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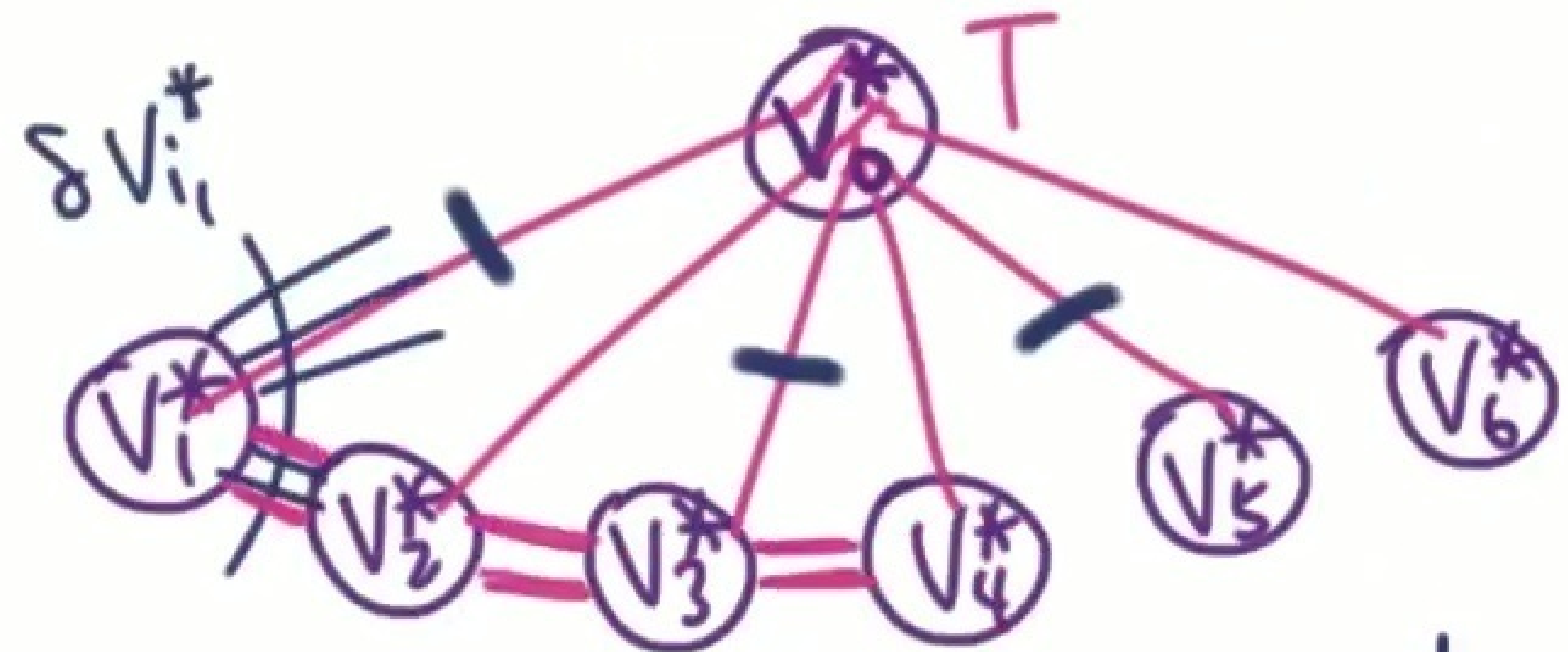

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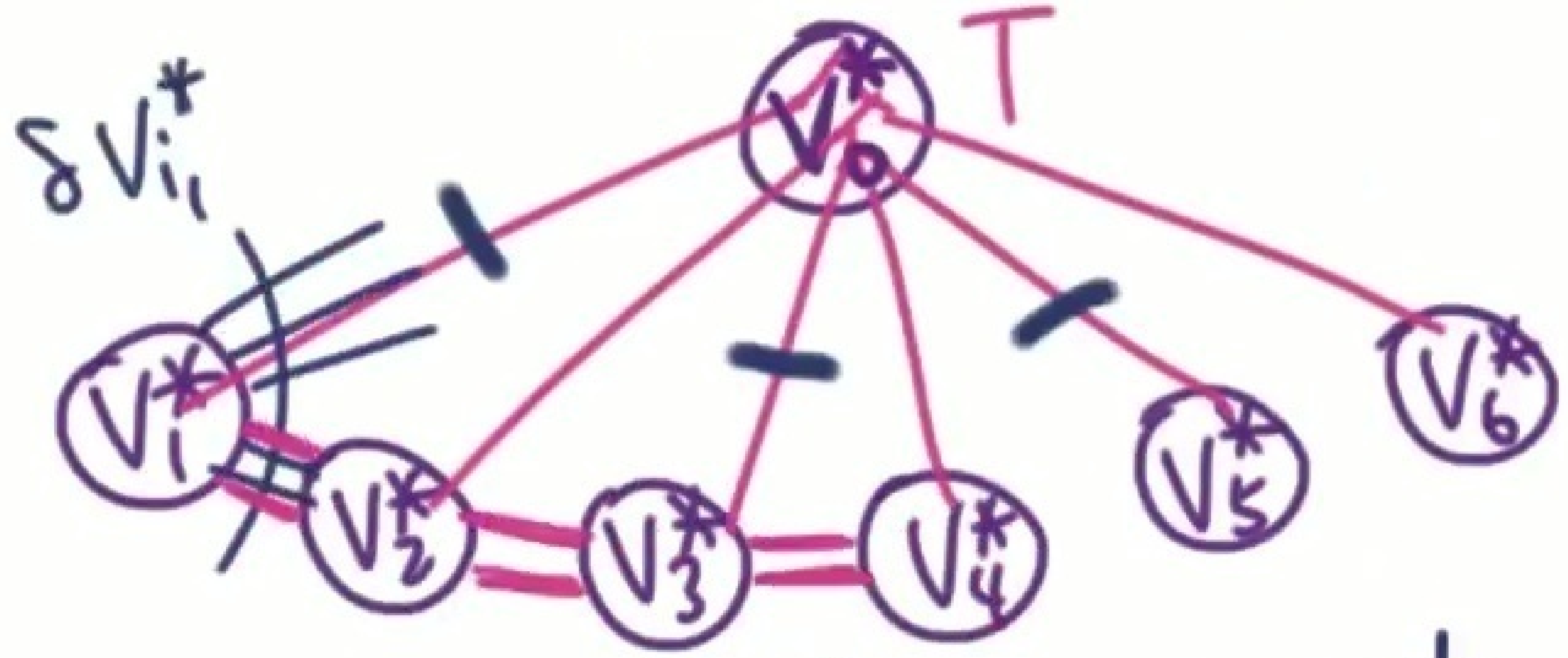

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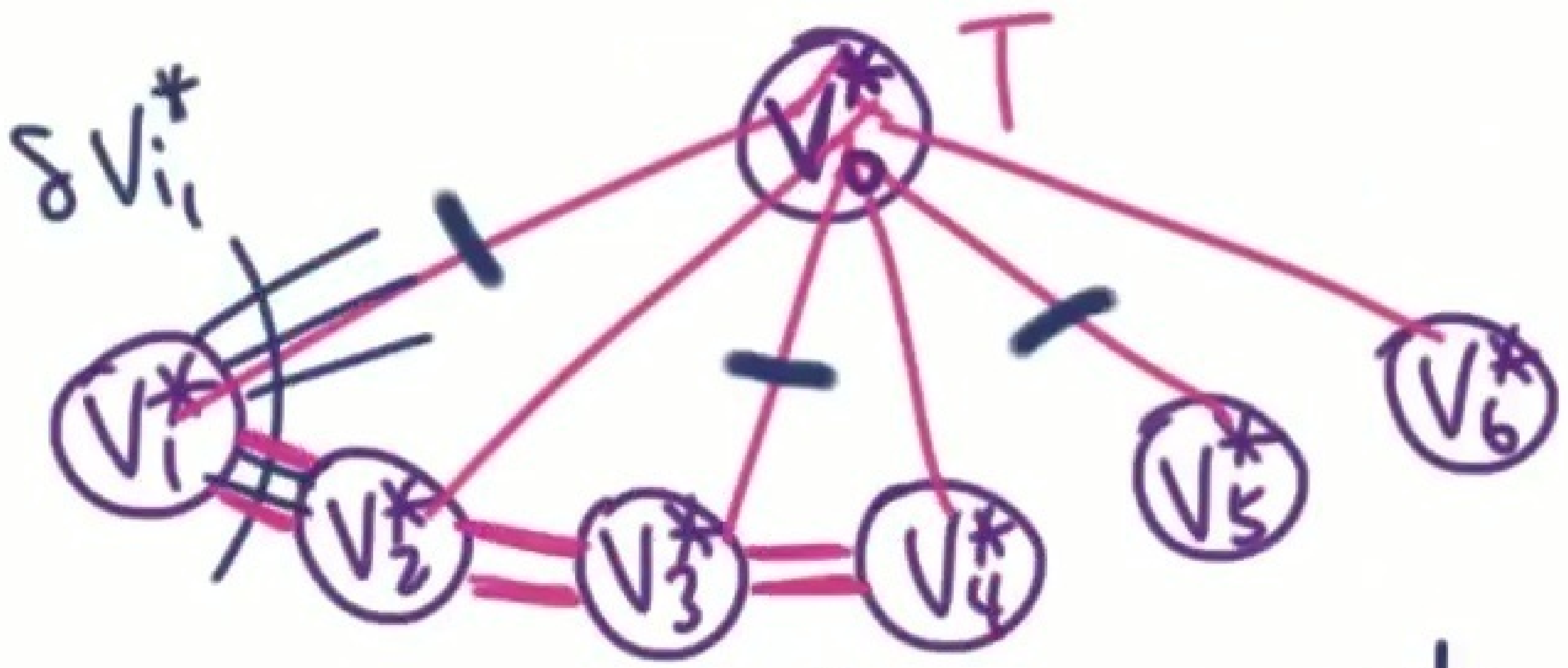

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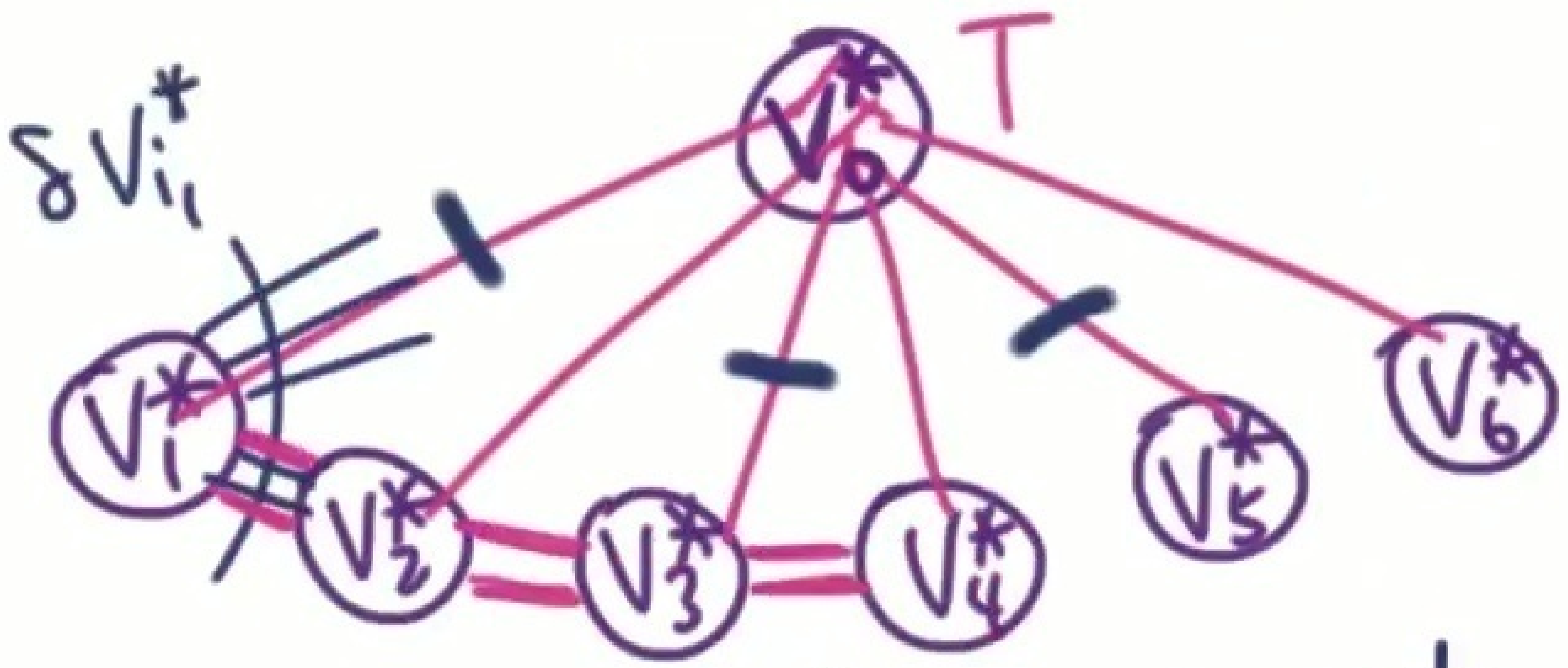

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- Unweighted graph: node weights $w(\delta V_i^*) \in [n^2]$, edge weights $-w(E[V_i^*, V_j^*]) \in [-n^2]$
- Can solve node+edge weighted k-clique, integer weights in $[-W, W]$, in $O(W \cdot n^{(\omega/3)k})$

(k-1)-respecting tree

Medium case: • (k-1)-respects tree, still want $n^{(\omega/3)k}$

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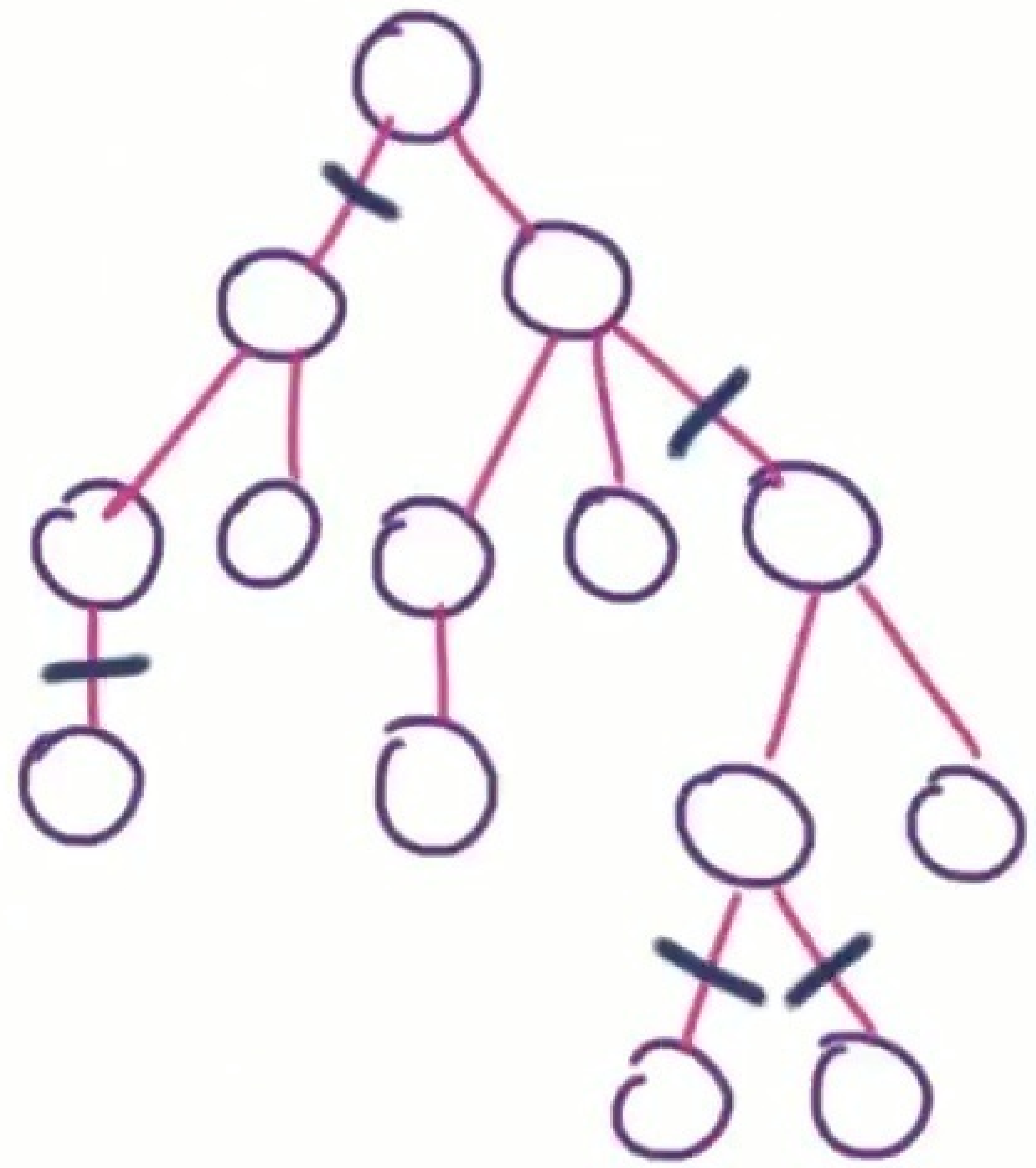

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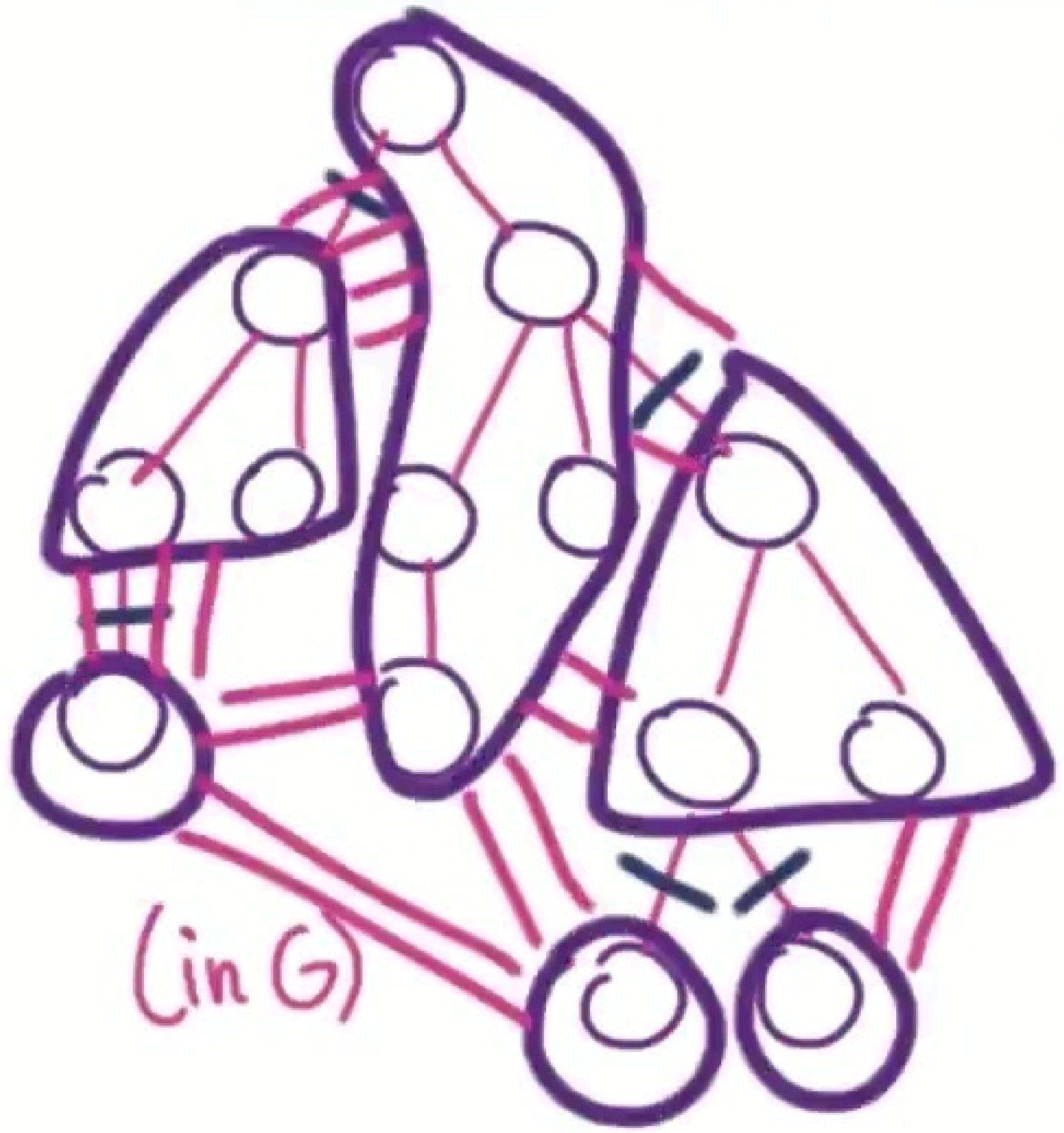

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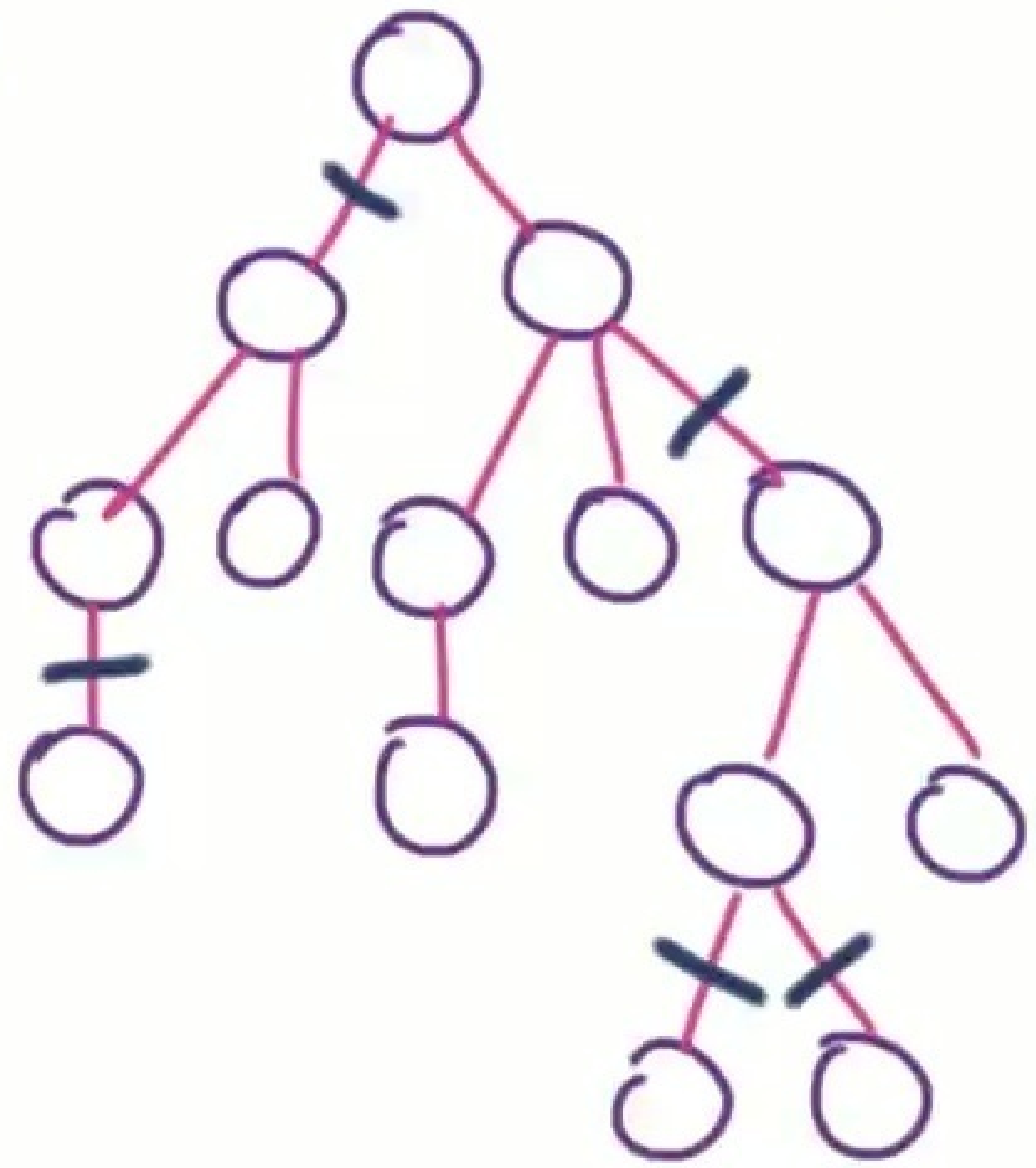

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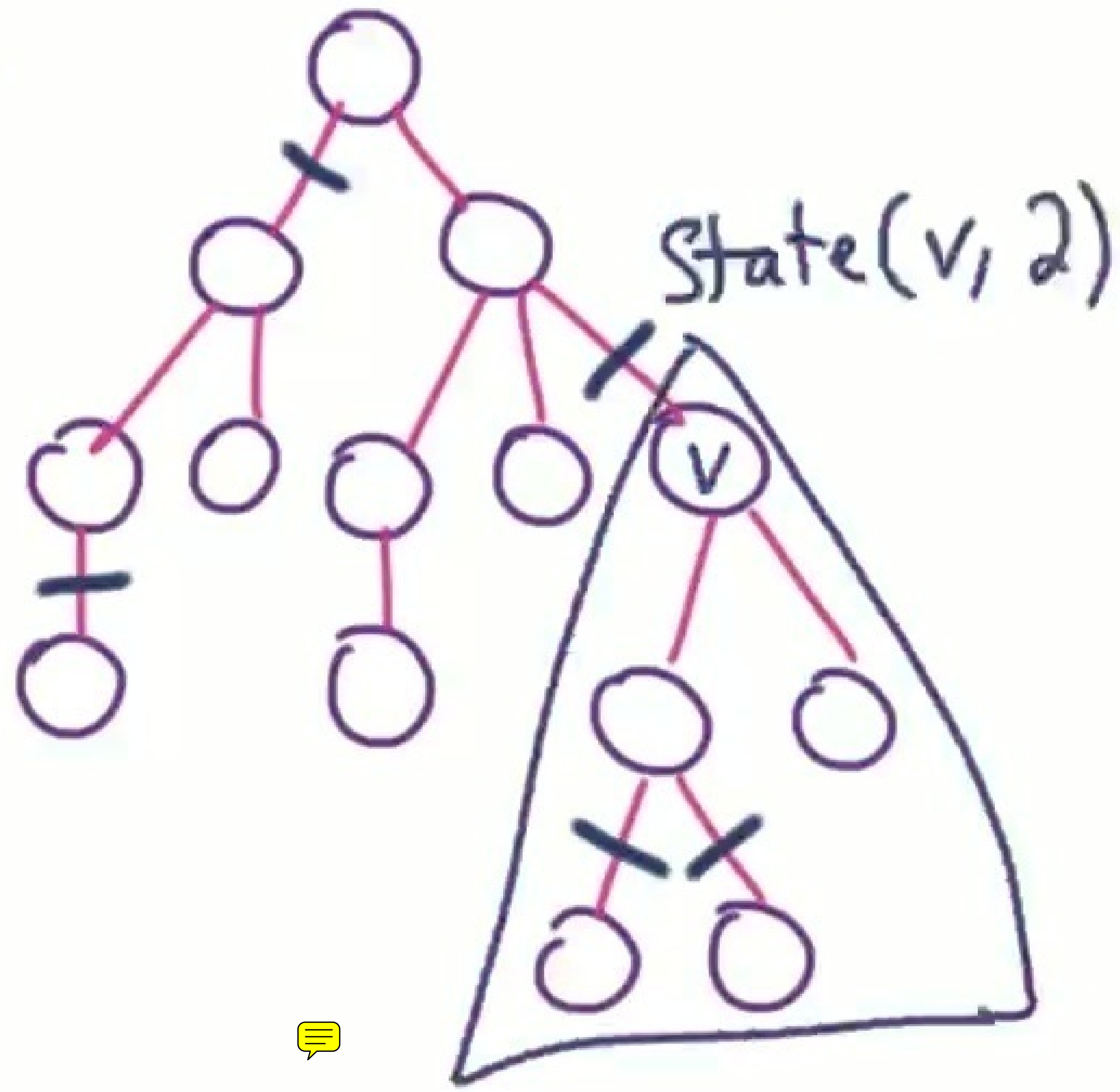

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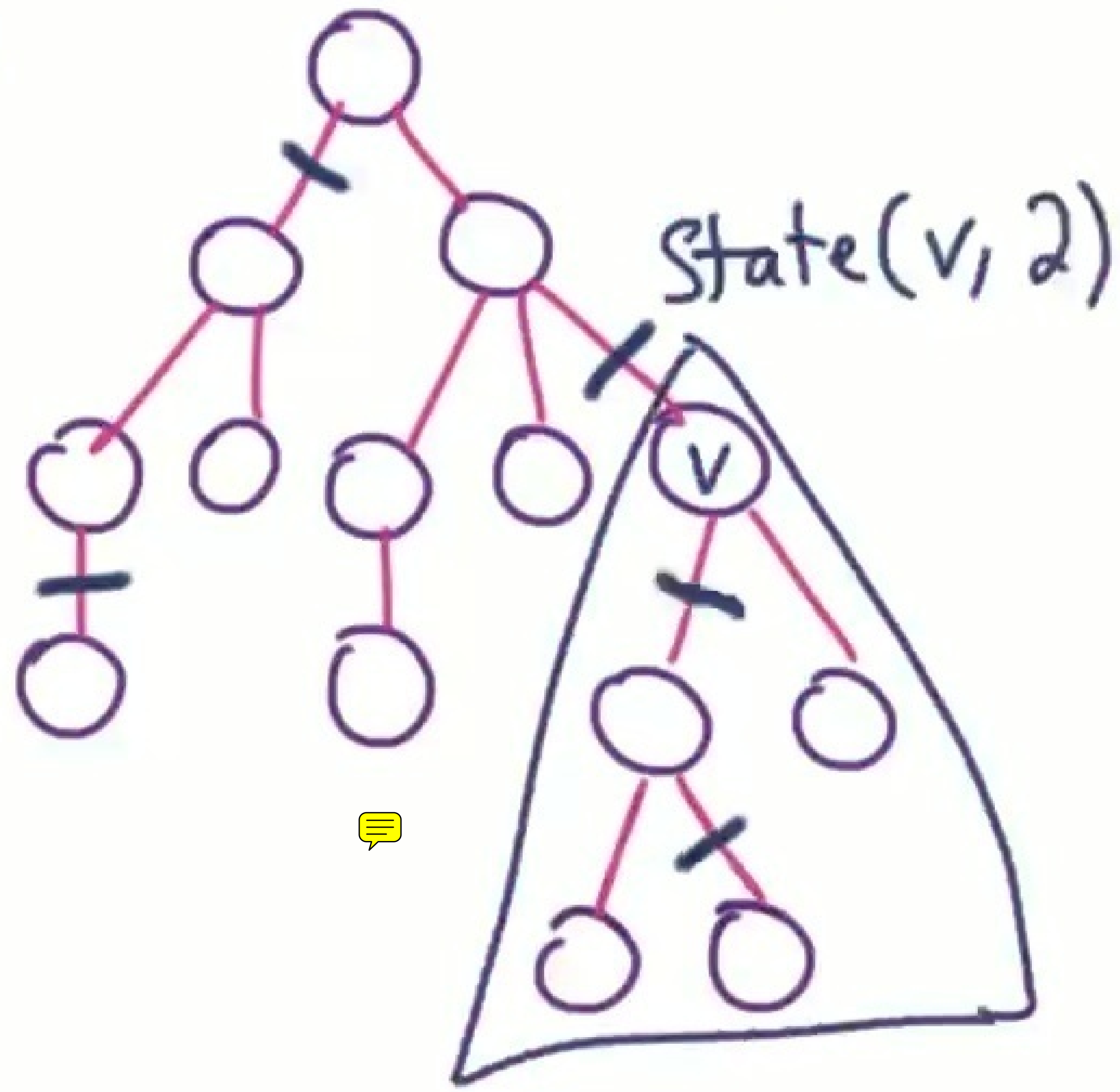

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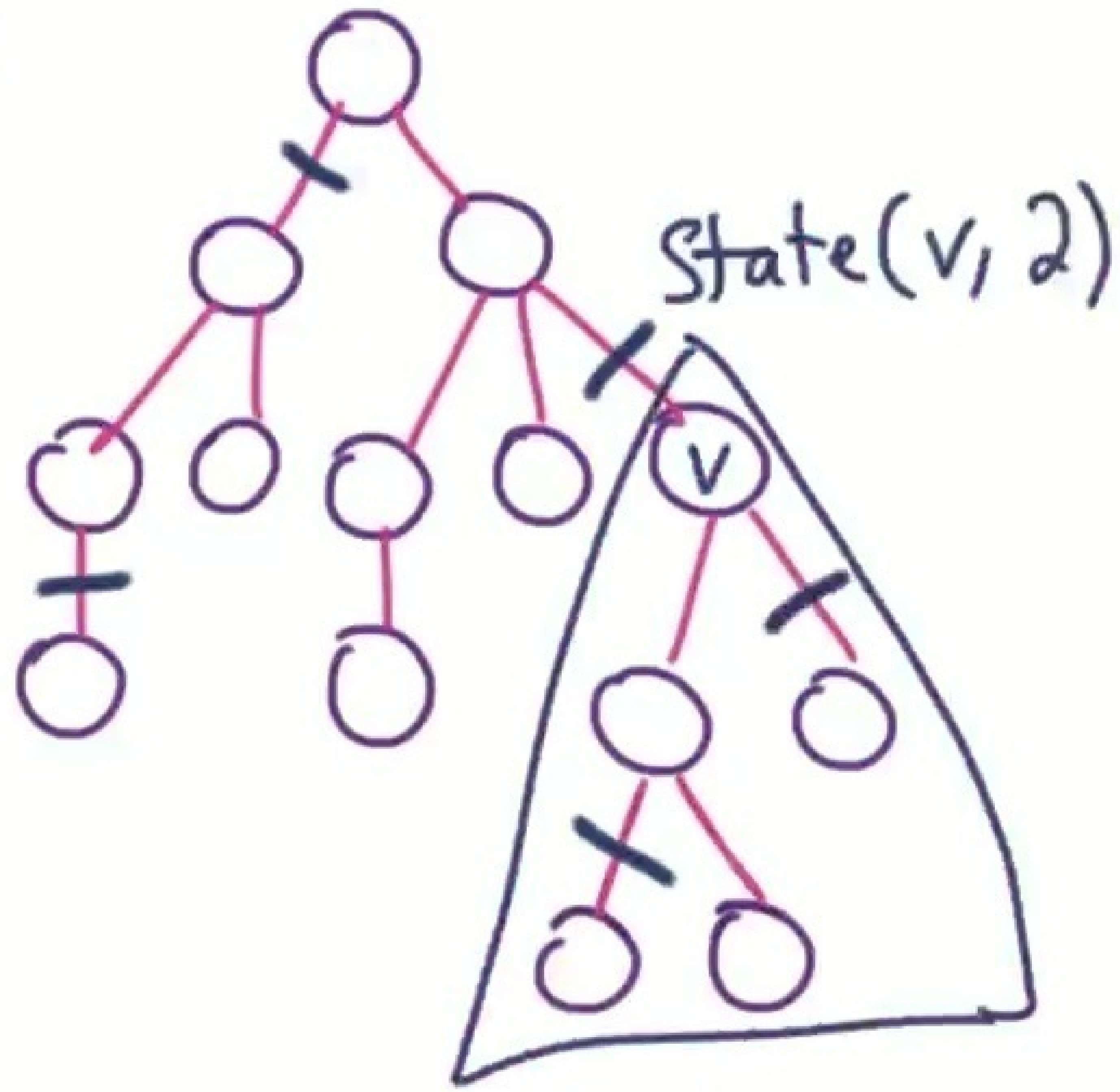

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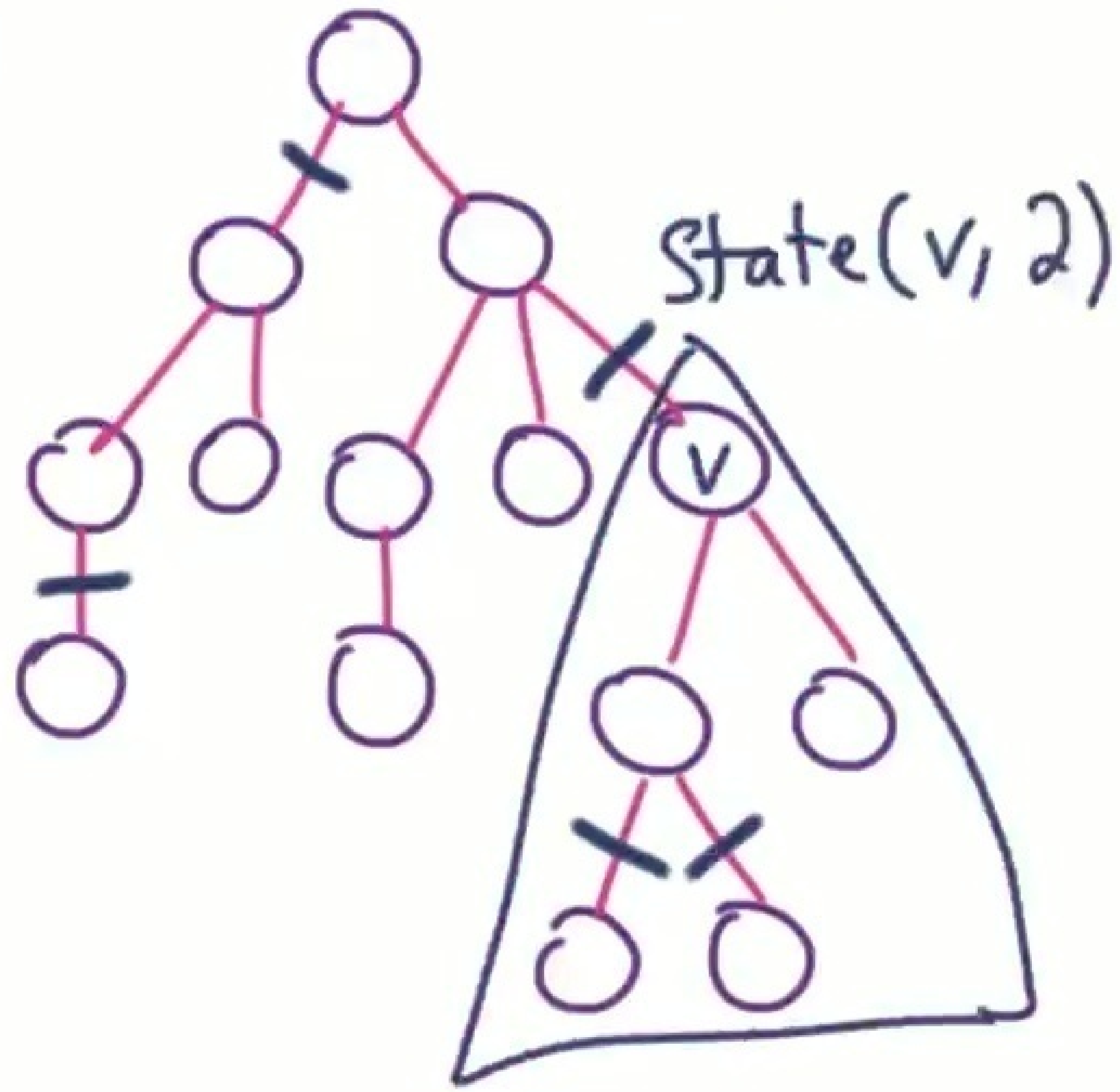
Intuition: if delete v's parent, then v's subtree is independent instance



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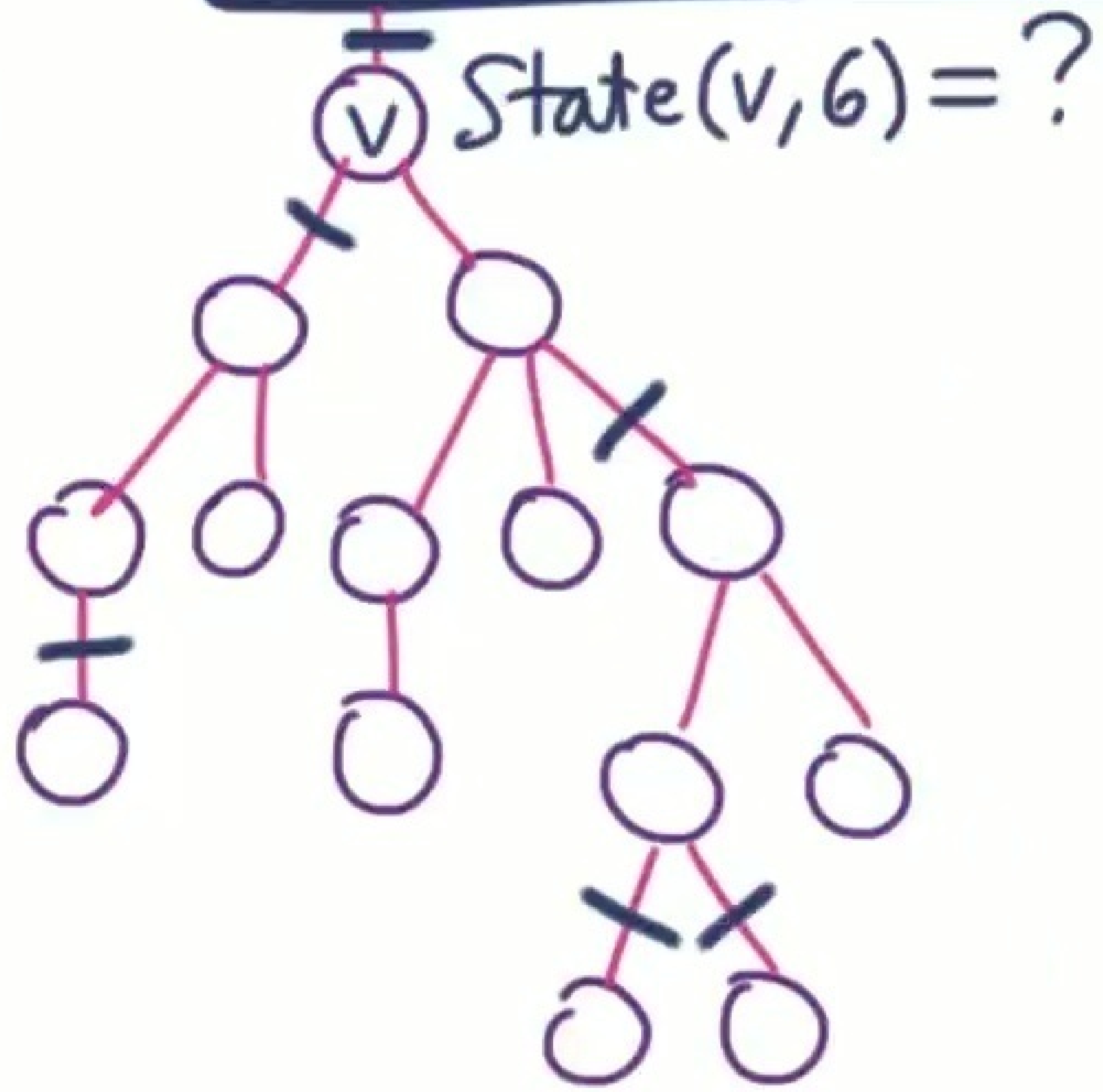
Computing State(v,s):



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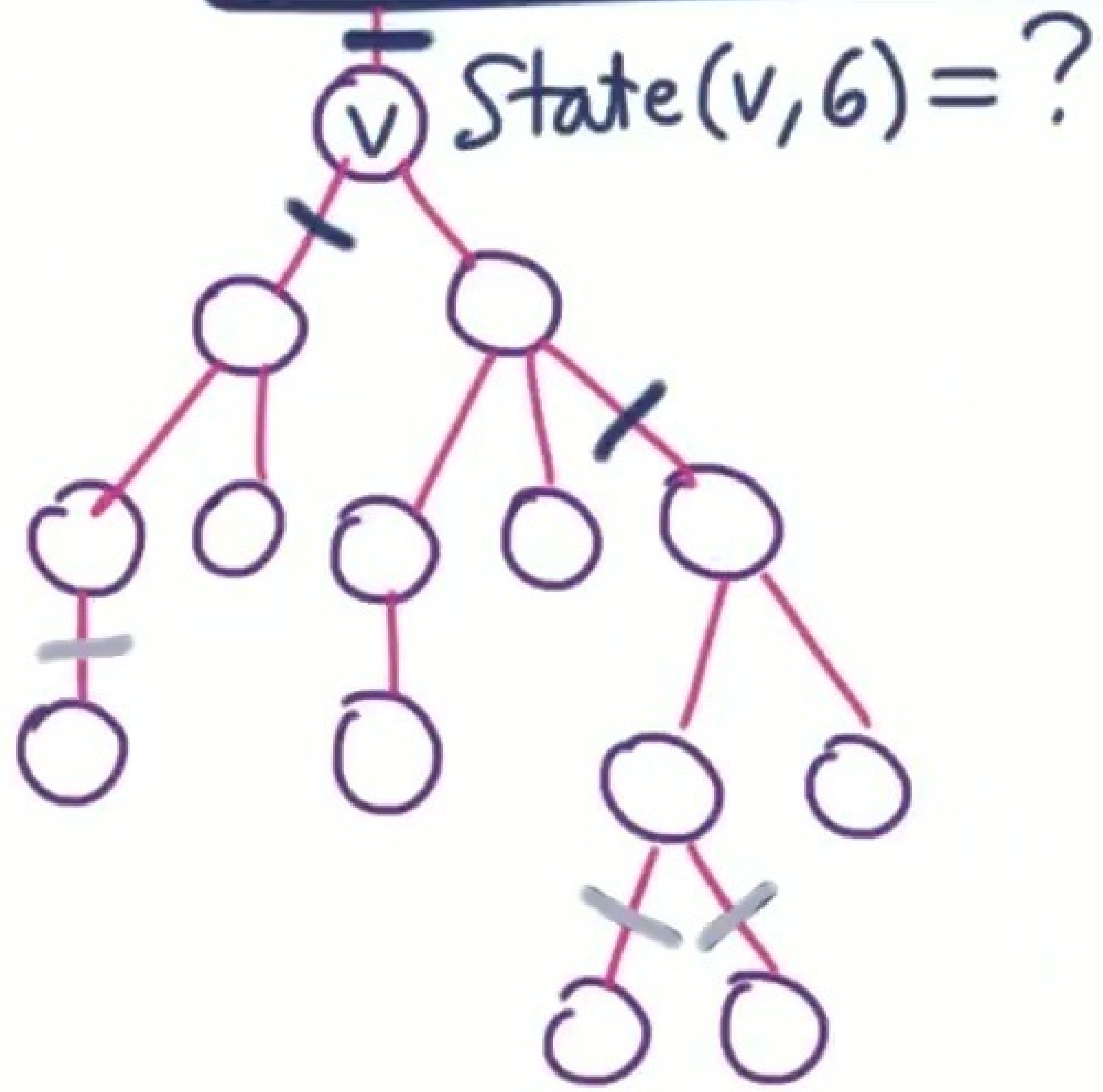
Computing State(v,s):
• focus on "maximal" edges



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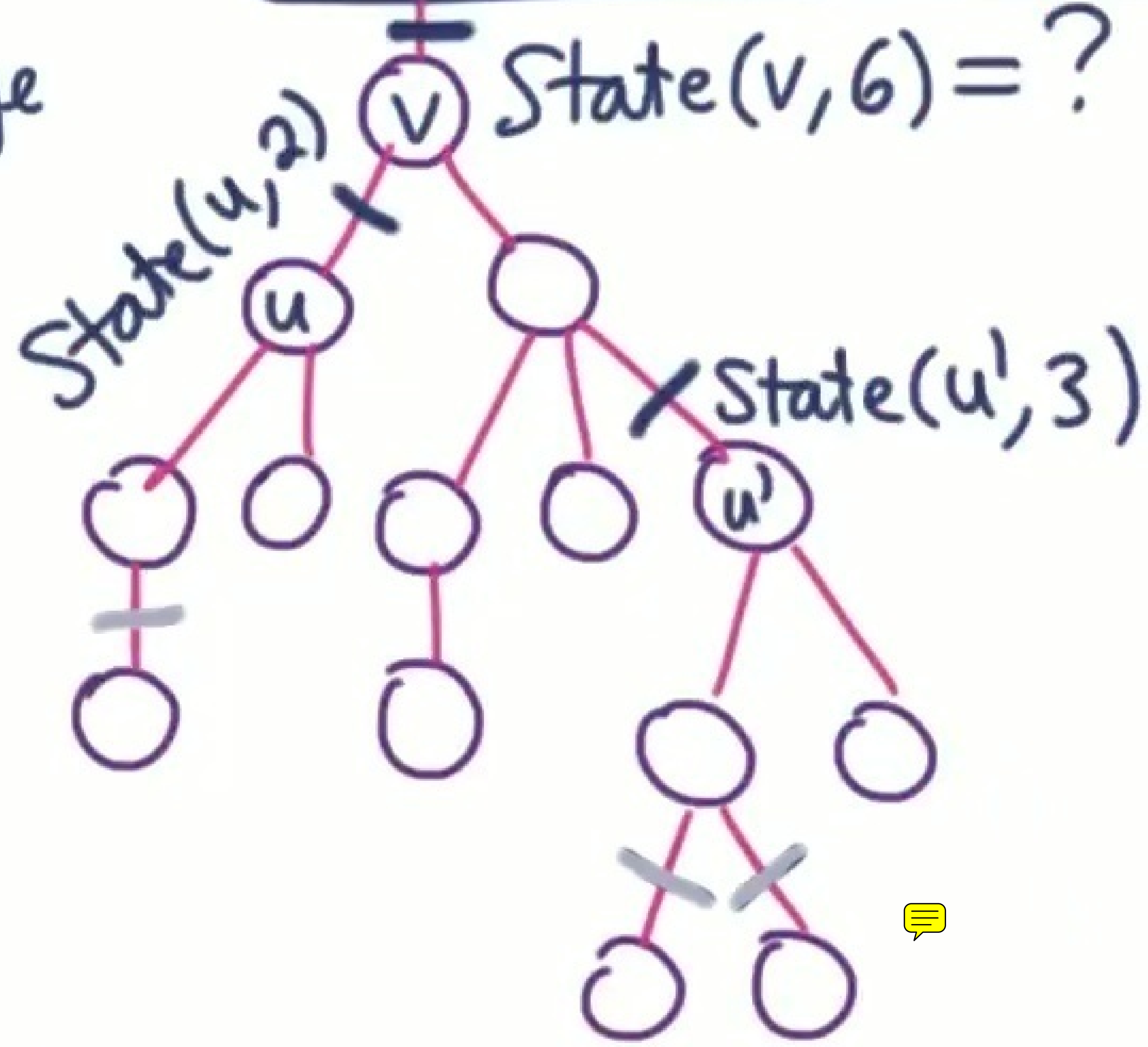

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Computing State(v,s):
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- for each u, ℓ: node (u, ℓ) with node weight State(u, ℓ)

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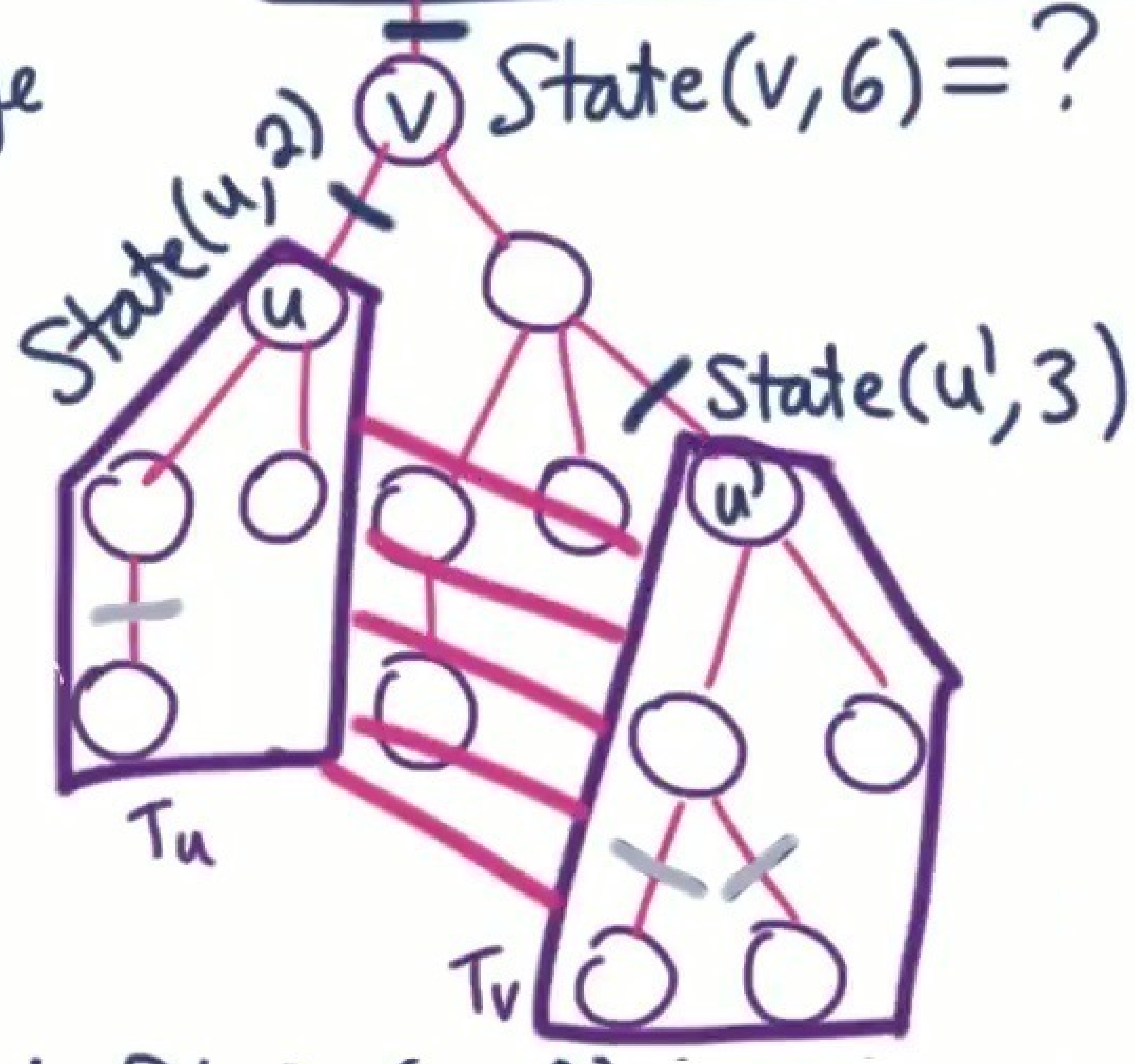

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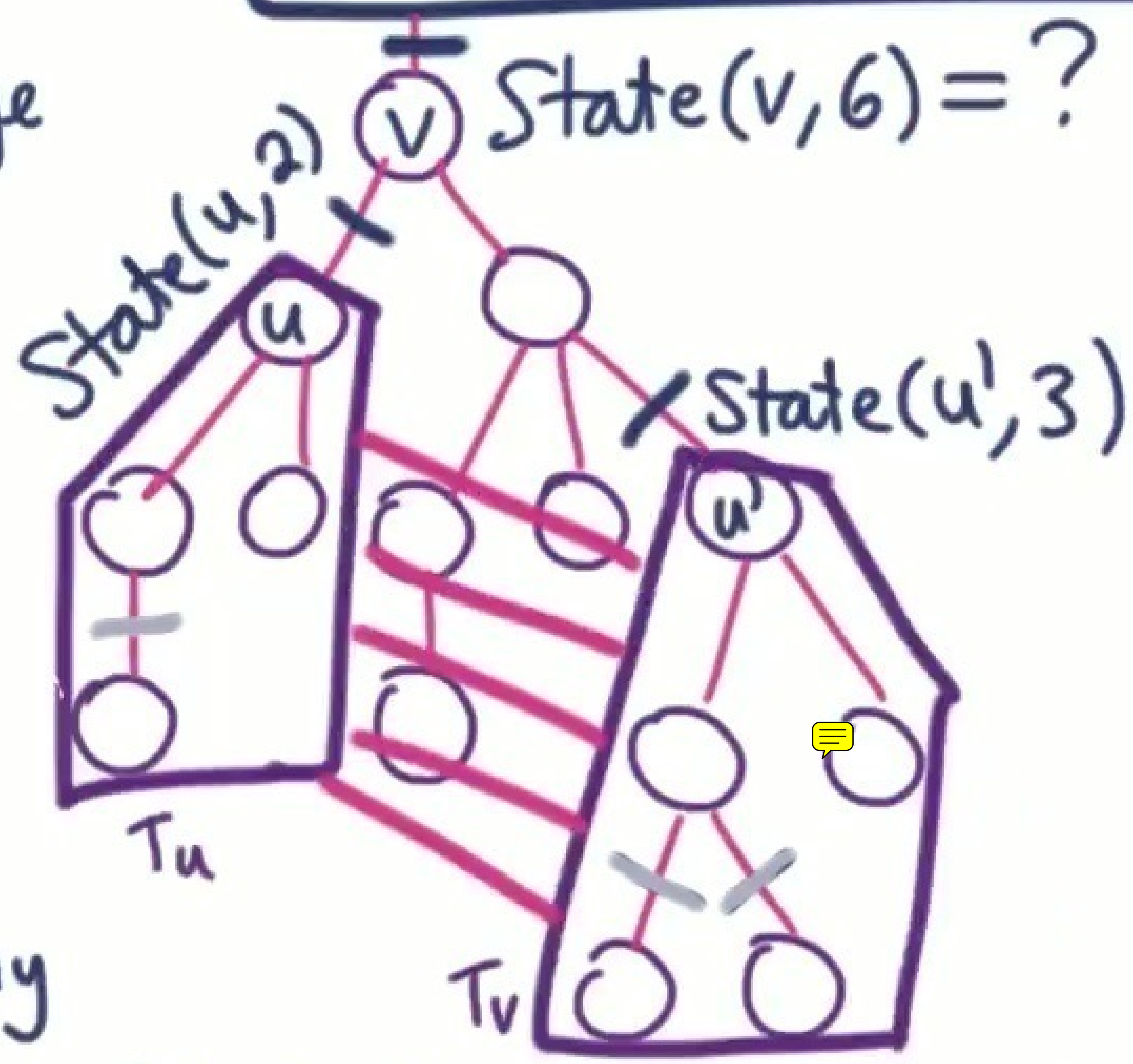

Computing State(v,s):

- focus on "maximal" edges, suppose r many
- for each u, ℓ: node (u,ℓ) with node weight State(u, ℓ)
- Edge weight between (u,ℓ), (v,ℓ'): $-w(E[T_u, T_v])$
- Find min node+edge weight r-clique $(u_1,\ell_1)\ldots(u_r,\ell_r)$ s.t. $\begin{cases} u_i\text{'s are incomparable} \\ \sum \ell_i = s-1 \end{cases}$

$2k-2 \rightarrow (k-1)$-respecting

Exact $n^{(1+\omega/3)k}$

Thorup's tree packing

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Hardness $n^{(\omega/3)k}$

Thm: Given a $(2k-2)$-respecting tree, can compute $f(k)n^k$ many trees s.t. one of them is 1-respecting, w.h.p.

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$\Rightarrow f(k)n^{k+o(1)}$ trees, one of which is $(k-1)$-resp.

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- poly(n) trees, one of which is $(2k-2)$-resp.
  $\Rightarrow f(k)n^{k+o(1)}$ trees, one of which is $(k-1)$-resp.
- $n^{(\omega/3)k}$ time per tree, $f(k)n^{(1+\omega/3)k+o(1)}$ total

$(1+\varepsilon)$-approx k-cut

Exact $n^{(1+\omega/3)k}$

Thorup's tree packing

Reduction to (k-1)-resp.

k-clique-like mtx.mult.

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Thm: Given (k-1)-respecting tree, can find $(1+\varepsilon)$-apx k-cut in $f(k,\varepsilon)\,\mathrm{poly}(n)$ (FPT) time

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- Exact is impossible in FPT (W[1]-hard)

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- Exact is impossible in FPT (W[1]-hard)
- Uses approximation algo techniques (ε-nets) combined with FPT techniques (color-coding)

Apply thm on each of $f(k)n^{k+O(1)}$ trees

$\Longrightarrow$ $(1+\varepsilon)$-apx in $f(k,\varepsilon)\,n^{k+O(1)}$ time

Open problems

Exact $n^{(1+\omega/3)k}$

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- Faster exact algo? $\begin{cases} \text{upper bound } n^{(2\omega/3)k} \\ \text{lower bound } n^{(\omega/3)k} \end{cases}$

  "fine-grained complexity" of k-cut?

Open problems

- Faster exact algo?
  - upper bound $n^{(2\omega/3)k}$
  - lower bound $n^{(\omega/3)k}$

  "fine-grained complexity" of k-cut?

- Faster combinatorial exact algo?

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- Faster <u>combinatorial</u> exact algo?
  - [GLL'18, unpublished] $f(k)\, n^{1.99k}$ time

    $\Longrightarrow f(k)\, n^{1.99k}$ extremal # min k-cuts

- Lower bound n^k (combinatorial k-clique)

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- Better approximation?
  - $(1+\varepsilon)$-apx in $f(k,\varepsilon)\,\mathrm{poly}(n)$ time?