Fair Cuts:
Motivation, Definition, and Applications

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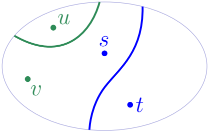

Used in divide-and-conquer algorithms for Gomory-Hu tree (all-pairs mincut): "dividing" on s–t mincut does not destroy u–v mincut

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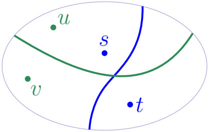

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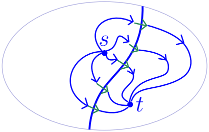

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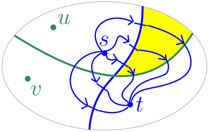

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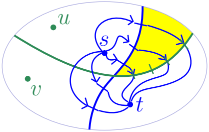

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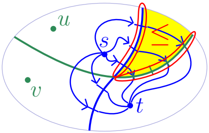

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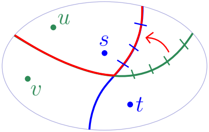

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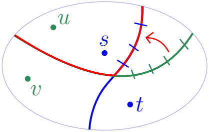

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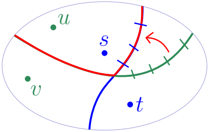

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  - We have uncrossed the u–v mincut with the s–t mincut.

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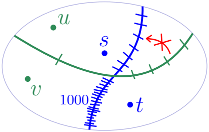

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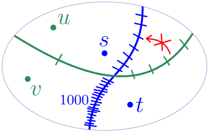

  - Main obstacle to obtaining approximate Gomory-Hu tree (all-pairs mincut) from approximate s–t mincut

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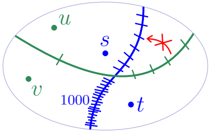

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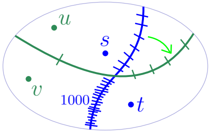

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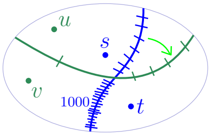

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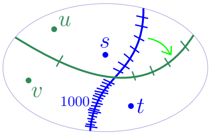

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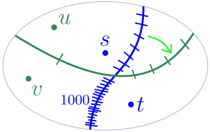

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- To formally define $(1+\epsilon)$-fair, we switch to flow-based perspective again

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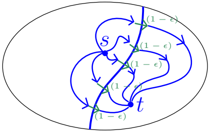

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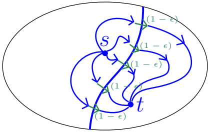

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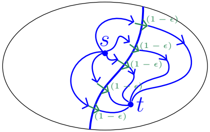

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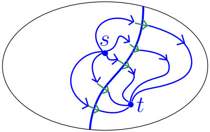

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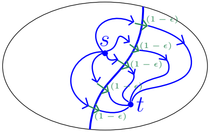

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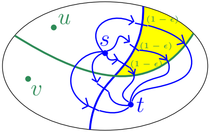

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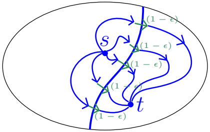

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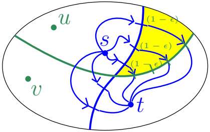

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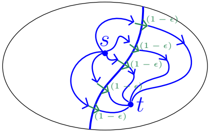

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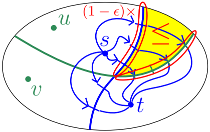

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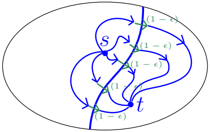

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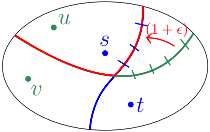

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    - Following known reductions: approximate Steiner mincut, approximate Gomory-Hu tree (all-pairs mincut) in $\tilde{O}(m)$ time, also parallel

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  - Can compute all minimum isolating cuts (one for each terminal) in about max-flow time
  - What about approximate minimum isolating cuts in approximate max-flow time?
    - Naïve approach fails because can't uncross
    - Fair cuts: approximate minimum isolating cuts in $\tilde{O}(m)$ time
    - Following known reductions: approximate Steiner mincut, approximate Gomory-Hu tree (all-pairs mincut) in $\tilde{O}(m)$ time, also parallel
- Expander pruning in expander decomposition

Fair Cuts: Applications

- Theorem: can compute $(1+\epsilon)$-approximate fair cuts in $\tilde{O}(m/\epsilon^3)$ time.
  - Follows $(1+\epsilon)$-approximate mincut algorithm [Sherman'13]
- Minimum Isolating Cuts: a useful primitive for graph cut algorithms
  - Given terminals $T \subset V$, the minimum isolating cut at terminal $t \in T$ is the minimum cut separating t from all other terminals.
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- Expander pruning in expander decomposition
  - First $\tilde{O}(m)$ time ϕ-expander decomposition algorithm for all values of ϕ