On the Fixed-Parameter Tractability of Capacitated Clustering

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Joint work with

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ICALP 2019

(Capacitated) k-median problem

k-median: metric space (V, d)

clients $C \subseteq V$, facilities $\mathbb{F} \subseteq V$

Find set F of k facilities minimizing

$$\sum_{v \in C} \min_{f \in F} d(v, f)$$

Capacitated k-median: facilities have capacities

Find set F of k facilities and assignment of clients to facilities s.t.

- Every facility f is assigned $\leq cap(f)$ clients
- Minimize $\sum_{v \in C} d(v, \text{assignment}(v))$.

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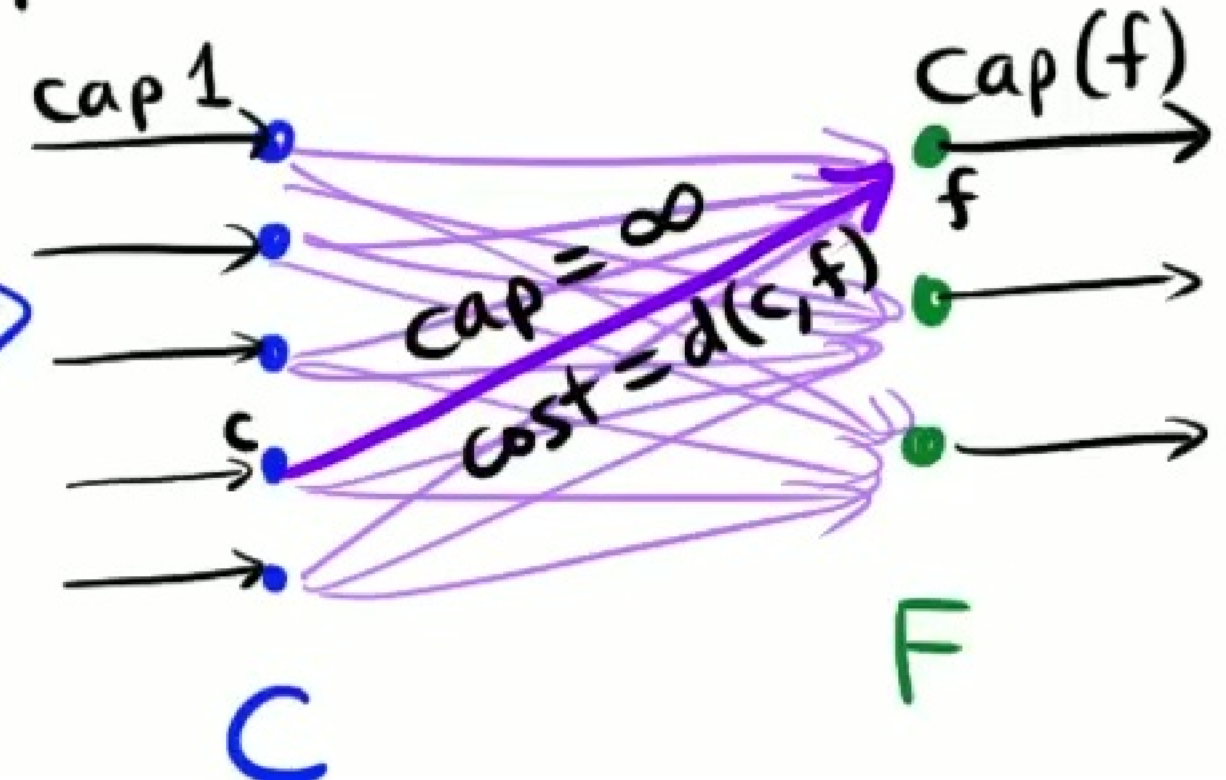

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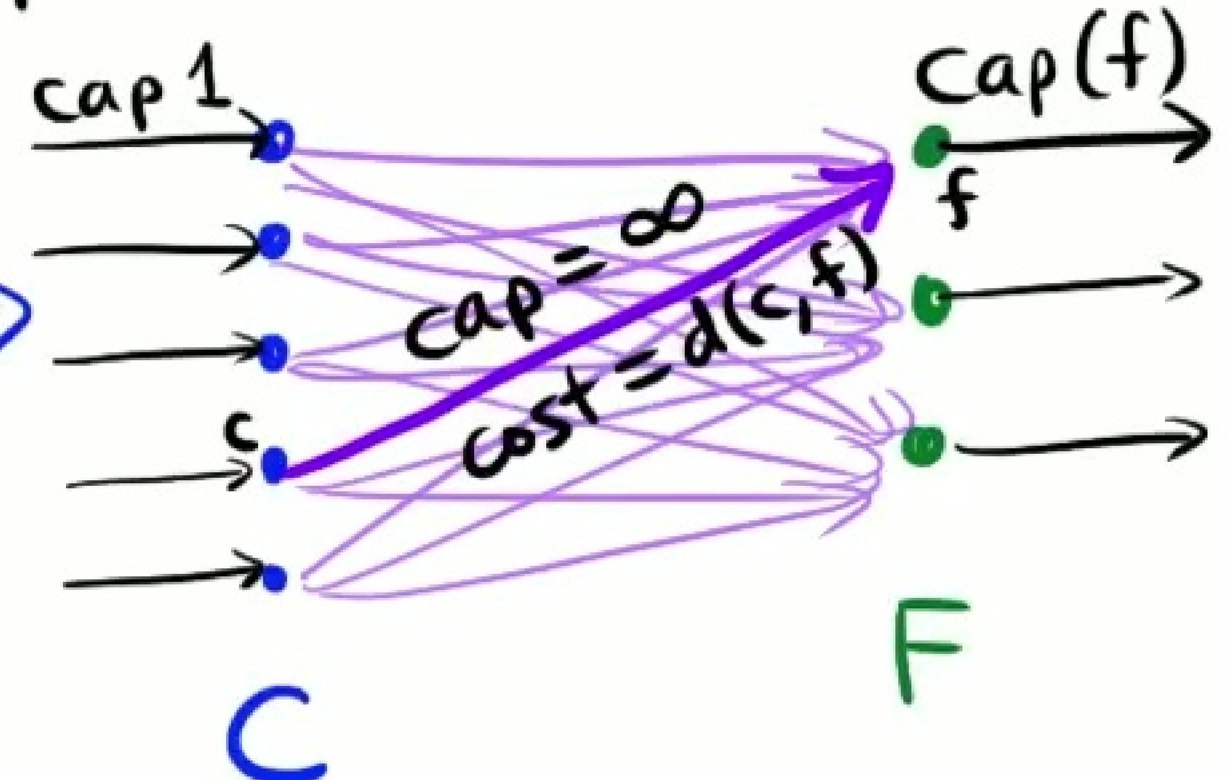
$= \text{MinCostFlow}(C, F)$ (integral)



Prior Work

k-median: ≈ 2.6 apx, $(1+2/e)$-apx hard,
tight $(1+2/e)$ in FPT(k) time ($f(k) \cdot n^{O(1)}$ time)
[CGKLL'19]

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Main Technical Contribution: coreset for capacitated k-median

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Main Technical Contribution: coreset for capacitated k-median
Then, use FPT algo to get $(3+\varepsilon)$-apx

Core-sets

Def[Coreset]: a subset $S \subseteq C$ with weights $w(v): v \in S$
(for cap-k-med) s.t.

$\forall F$ set of k facilities:

$$\mathrm{MinCostFlow}(S, F) \in (1 \pm \varepsilon)\, \mathrm{MinCostFlow}(C, F)$$

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Suffices to solve : $|S|$ many weighted clients
(ideally $|S| \ll |C|$)

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$$\sum_{v \in C} \min_{f \in F} d(v, f) \in (1 \pm \varepsilon) \sum_{v \in C} \min_{f \in F} d(v, f)$$

Thm[Chen; Feldman-Langberg]: $\exists$ coreset for k-median size $\mathrm{poly}(k \log n\, \varepsilon^{-1})$.

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This work: $\exists$ coreset for cap-k-med (and cap-k-means) size $\text{poly}(k \log n\, \varepsilon^{-1})$.

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- Random sampling
- For each F $(|F|=k)$,

$$\Pr\left[\sum_{v\in S} w(v)\, d(v,F) \in (1\pm\varepsilon)\sum_{v\in C} d(v,F)\right] \gg 1-n^{-k}$$

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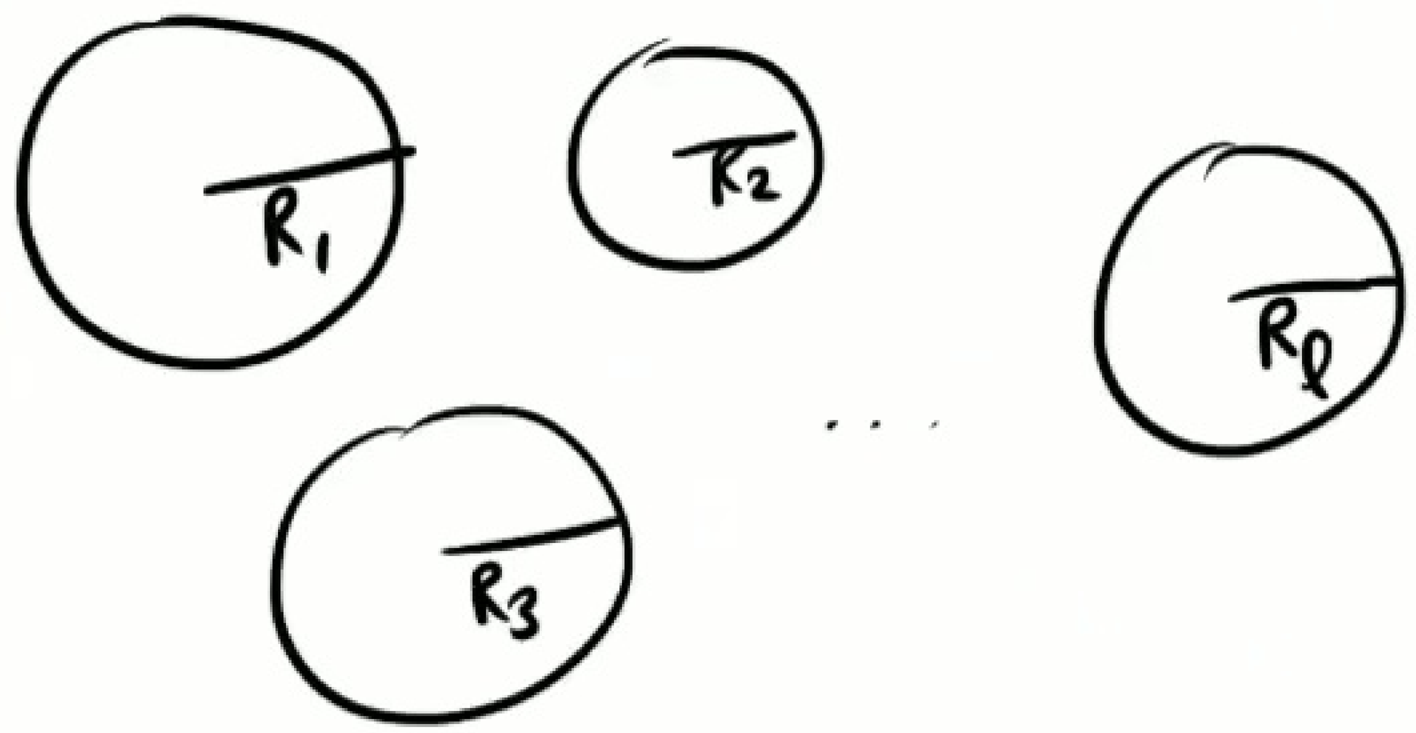

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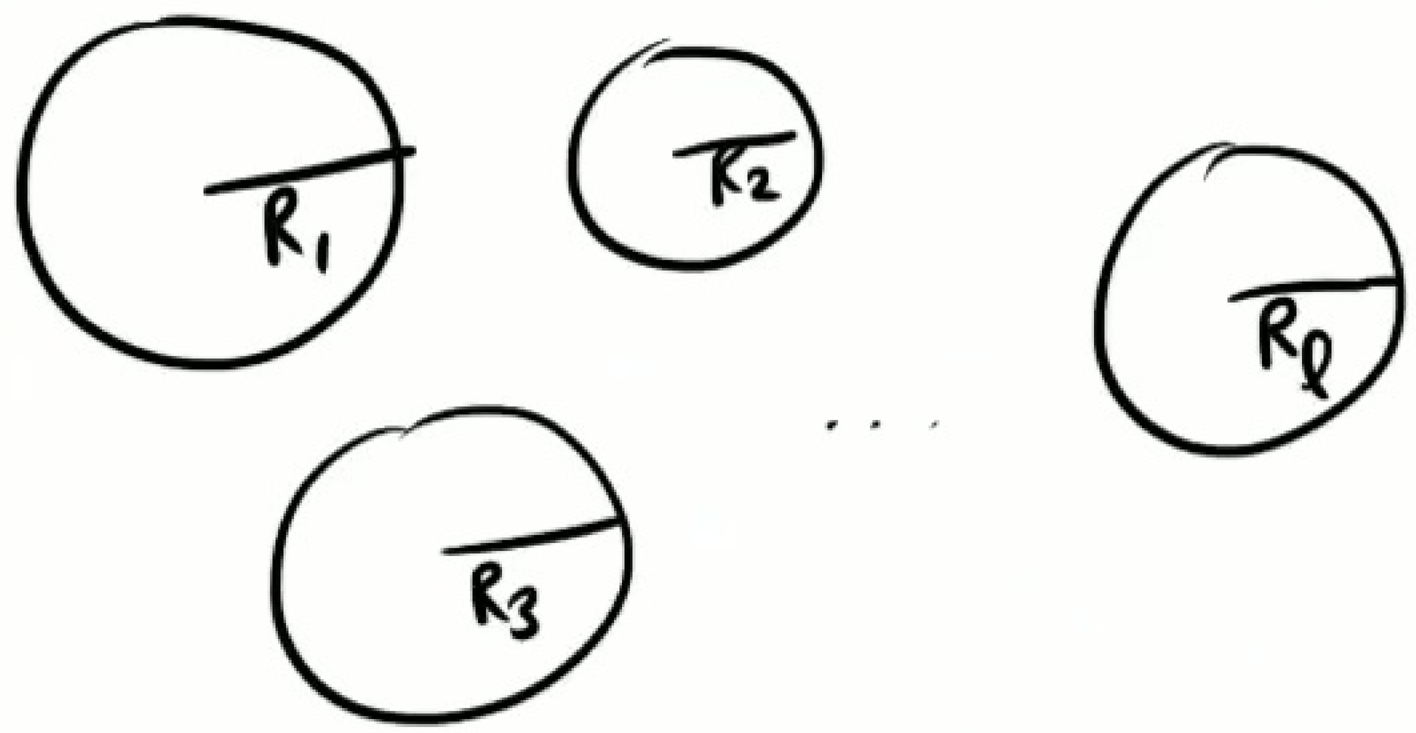

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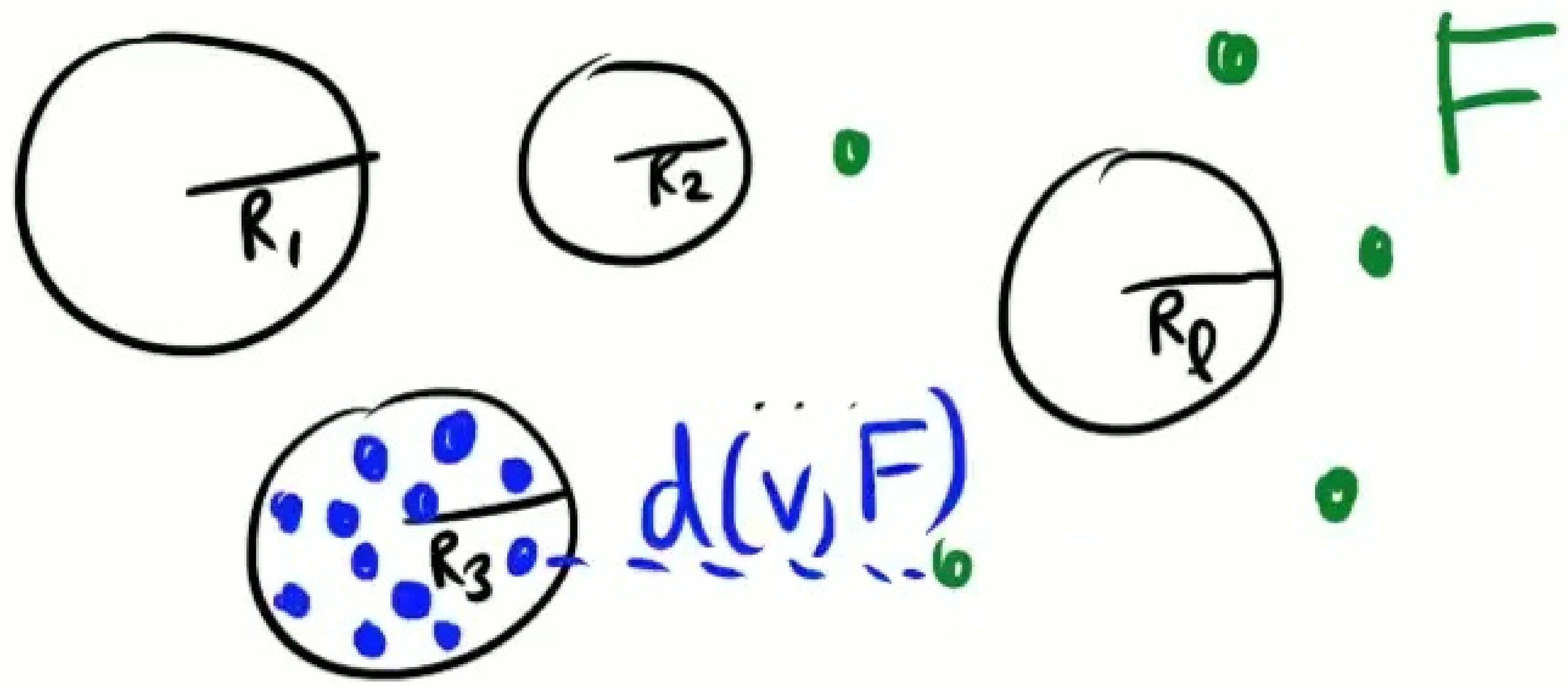

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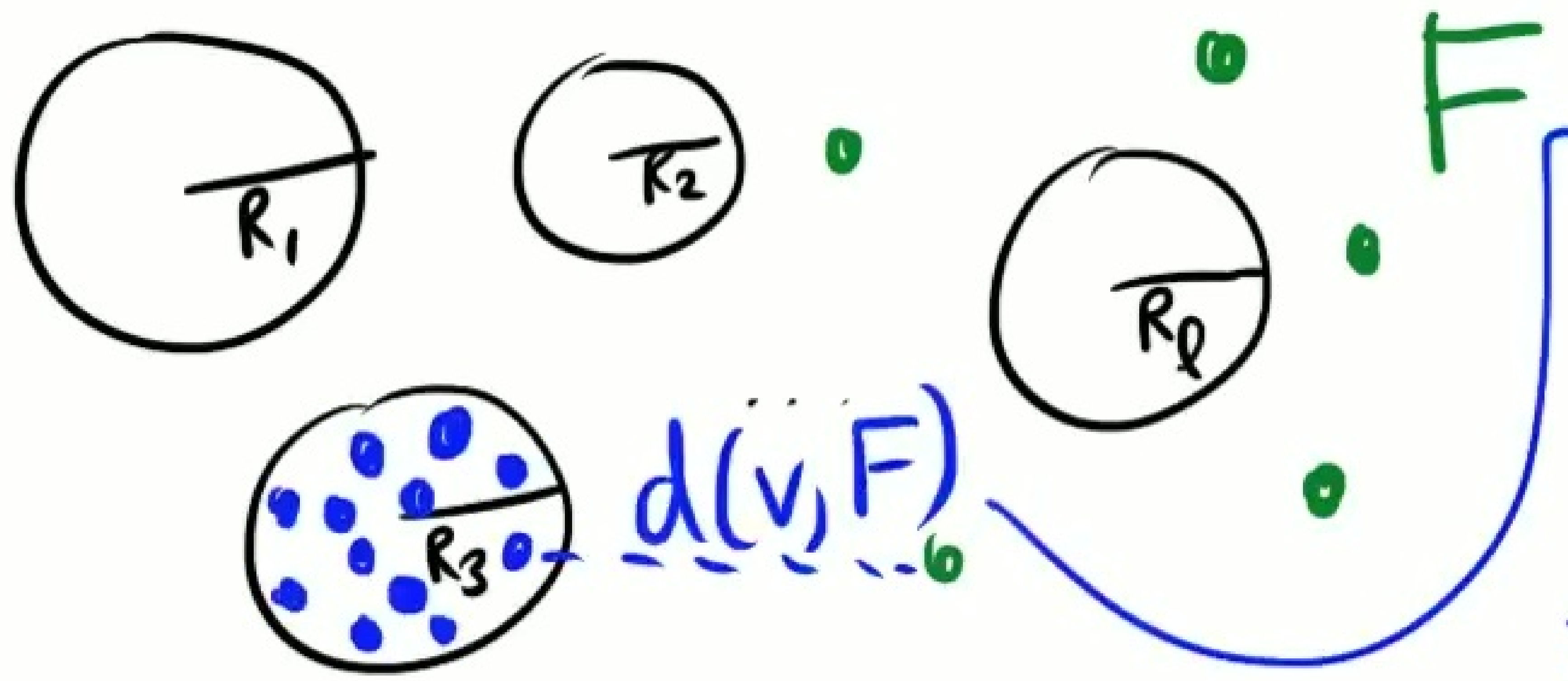

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Chernoff bound:

$\underset{v \in S}{\mathrm{avg}}(d(v,F)) \in \underset{v \in C_i}{\mathrm{avg}}(d(v,F)) + \varepsilon R_i$

w.p. $\geq 1 - n^{-k}$

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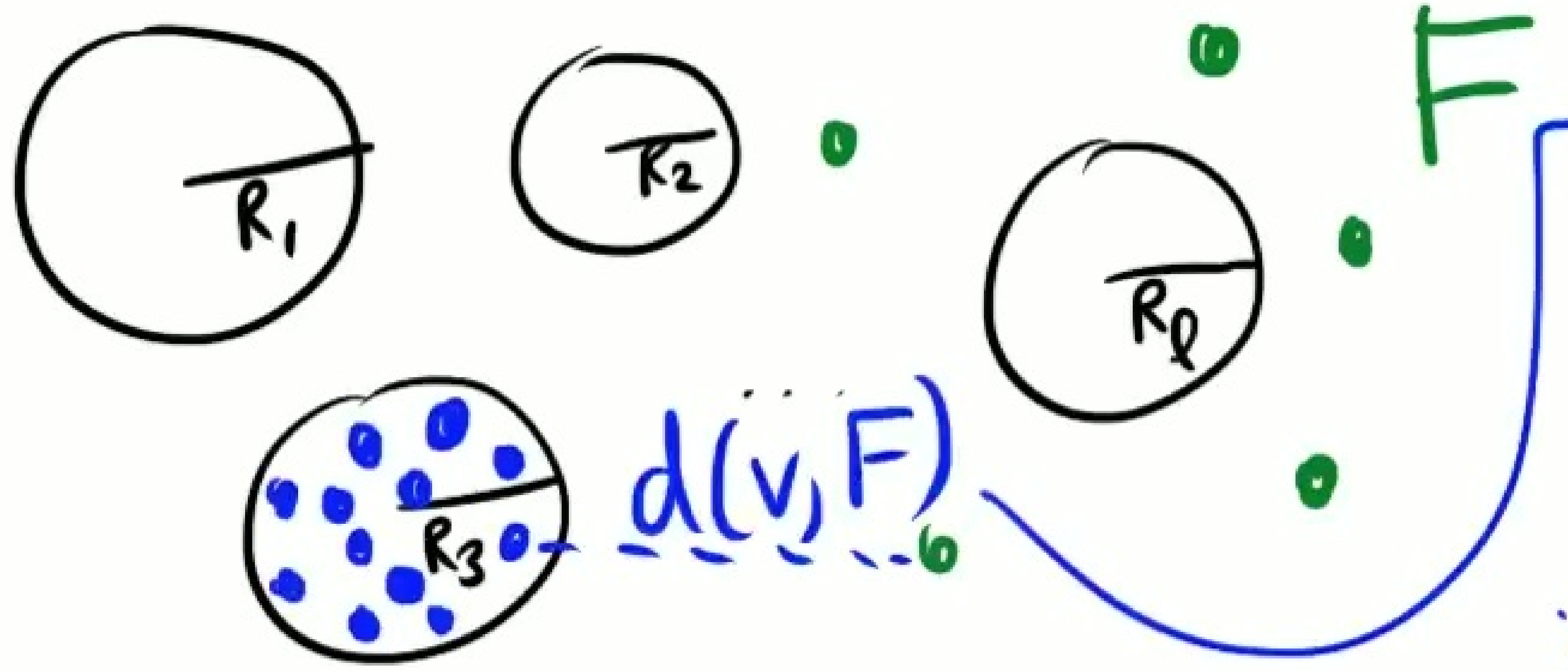

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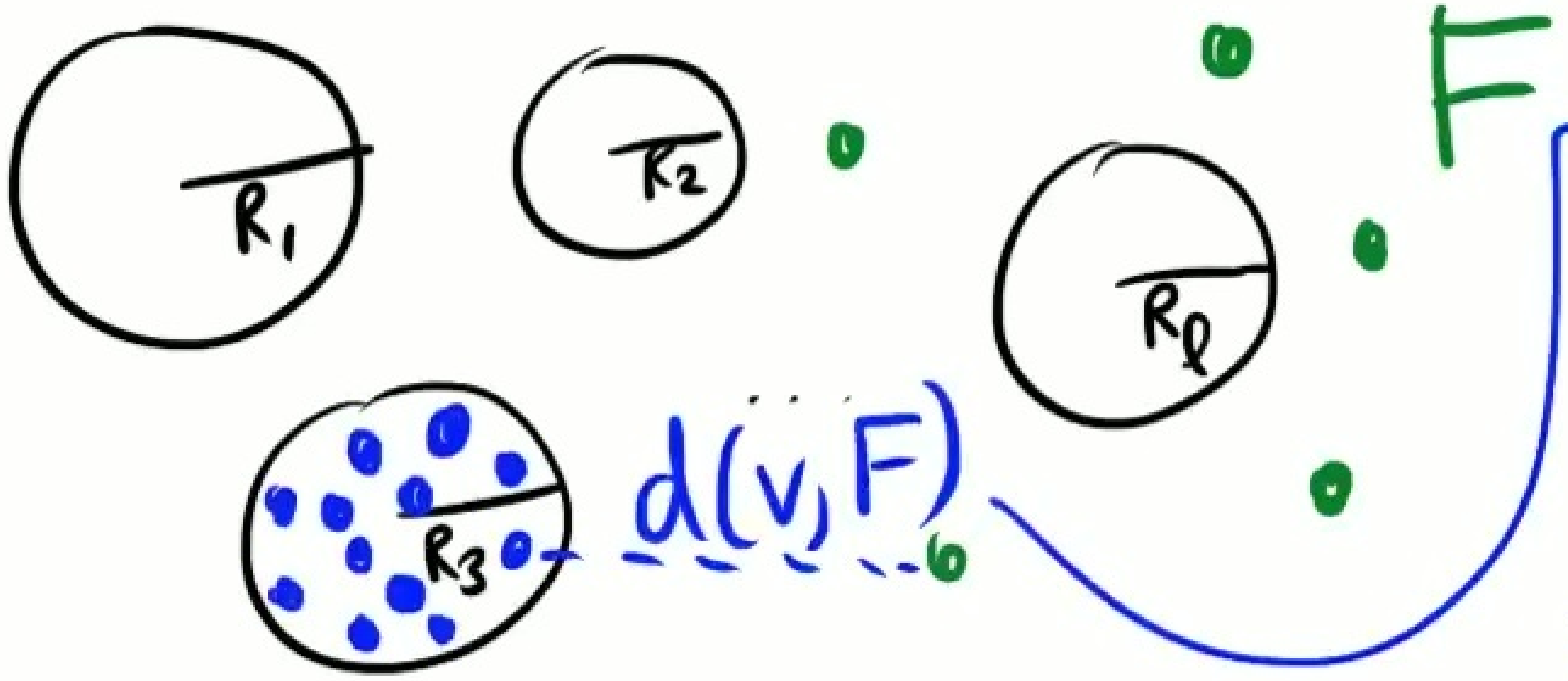

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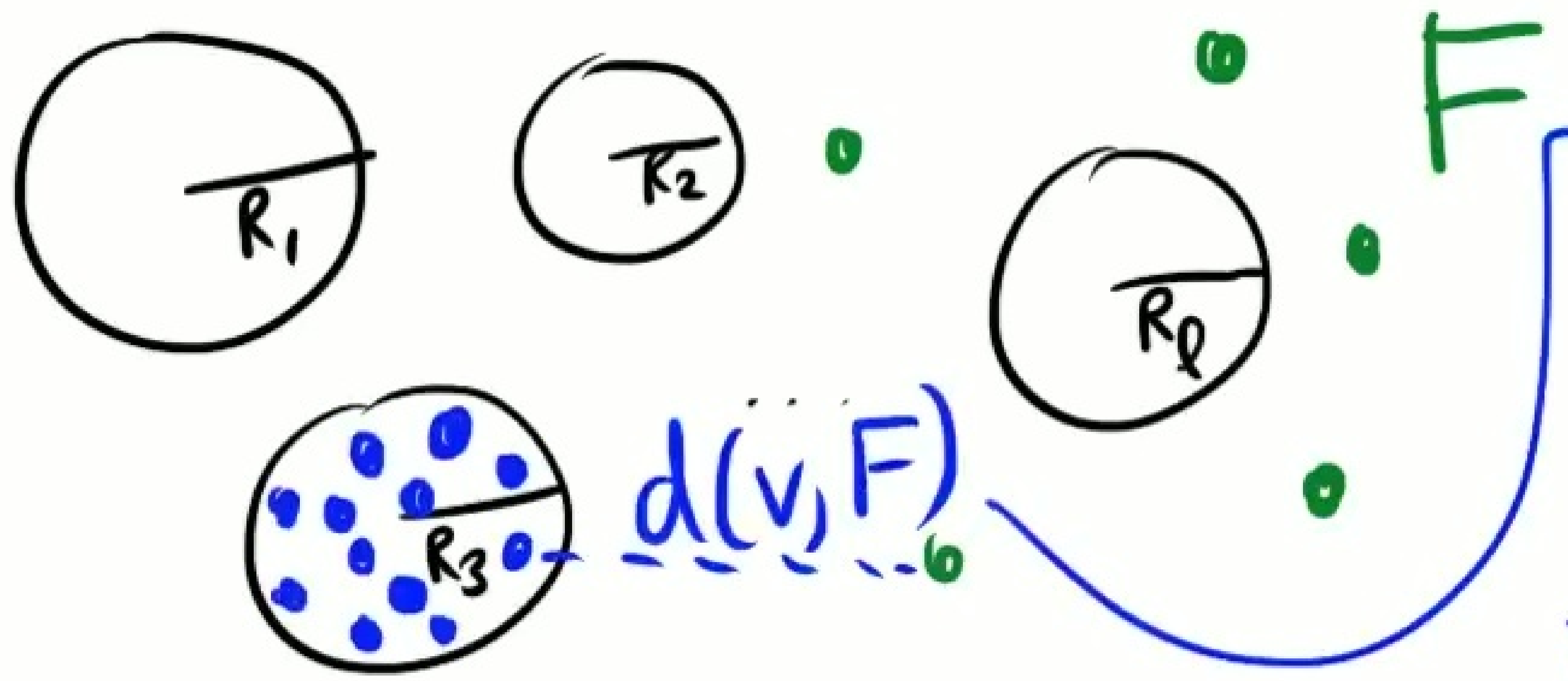

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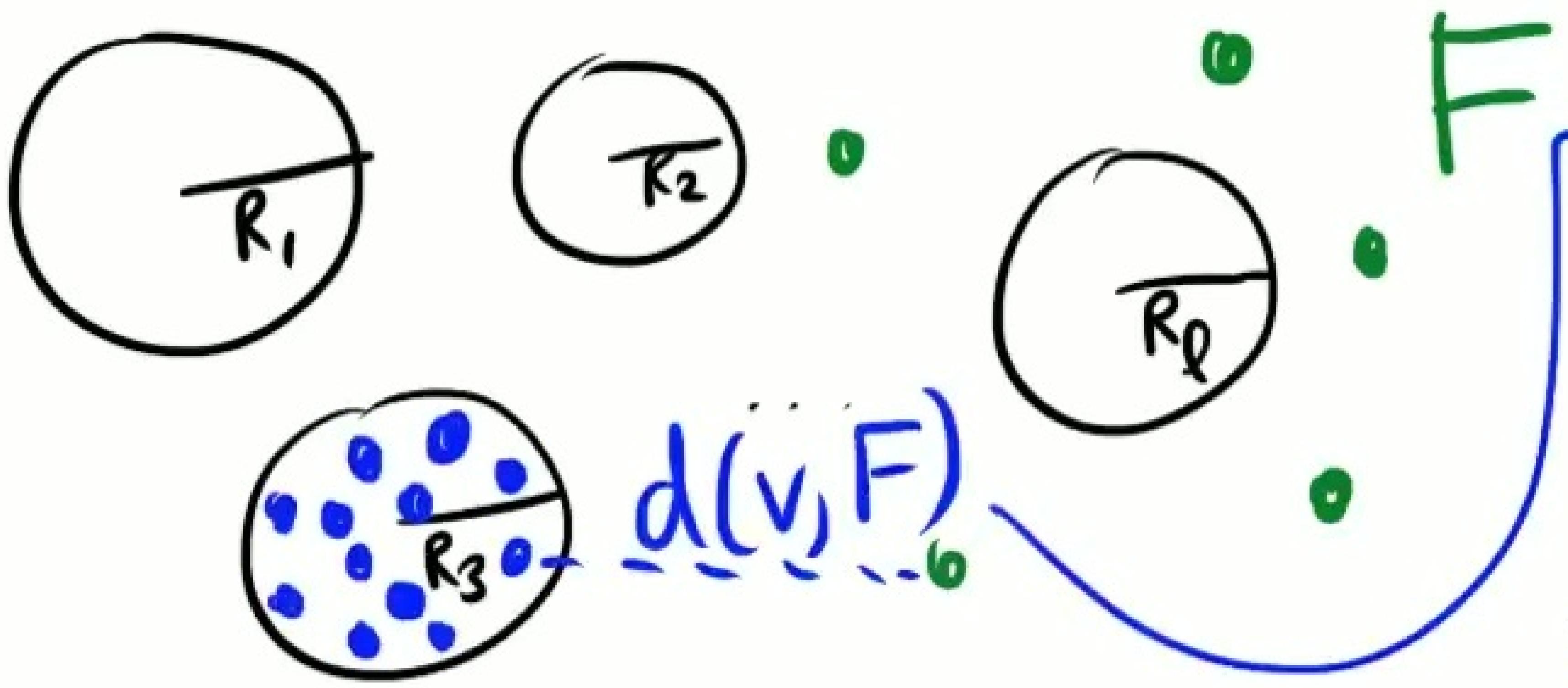

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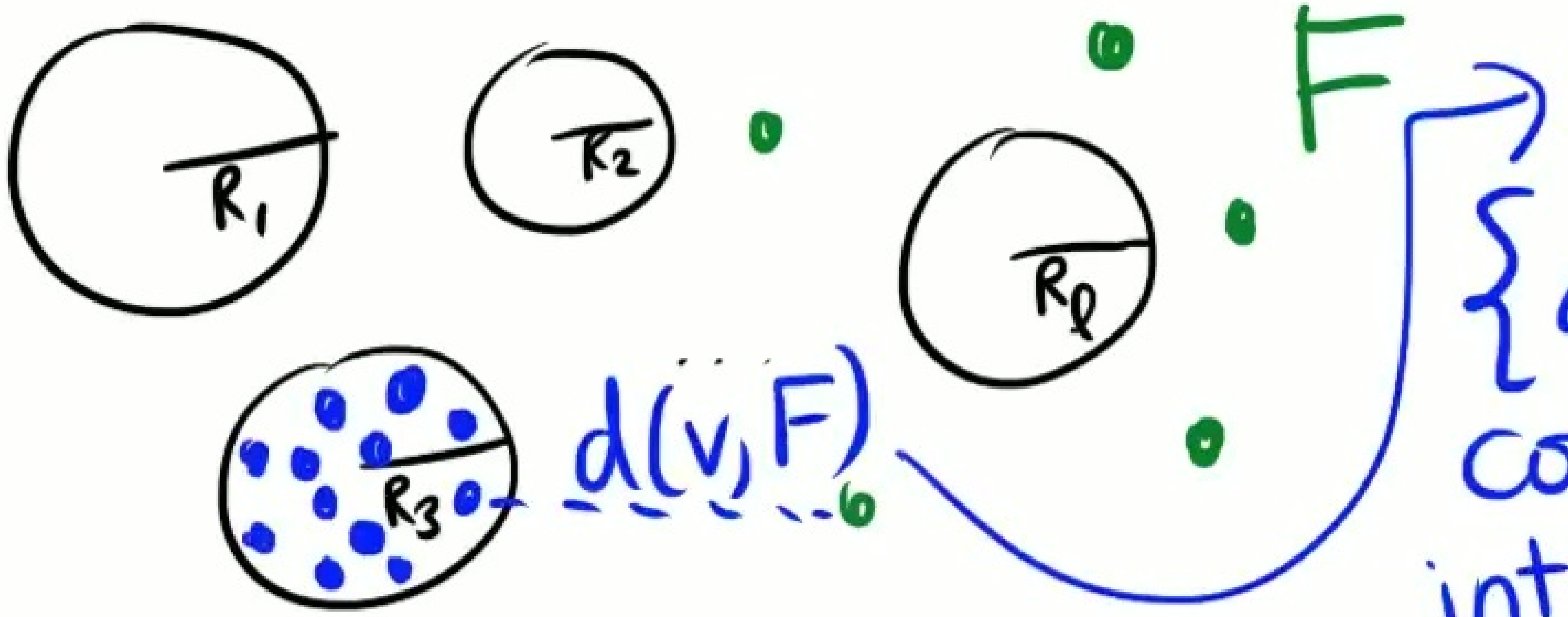

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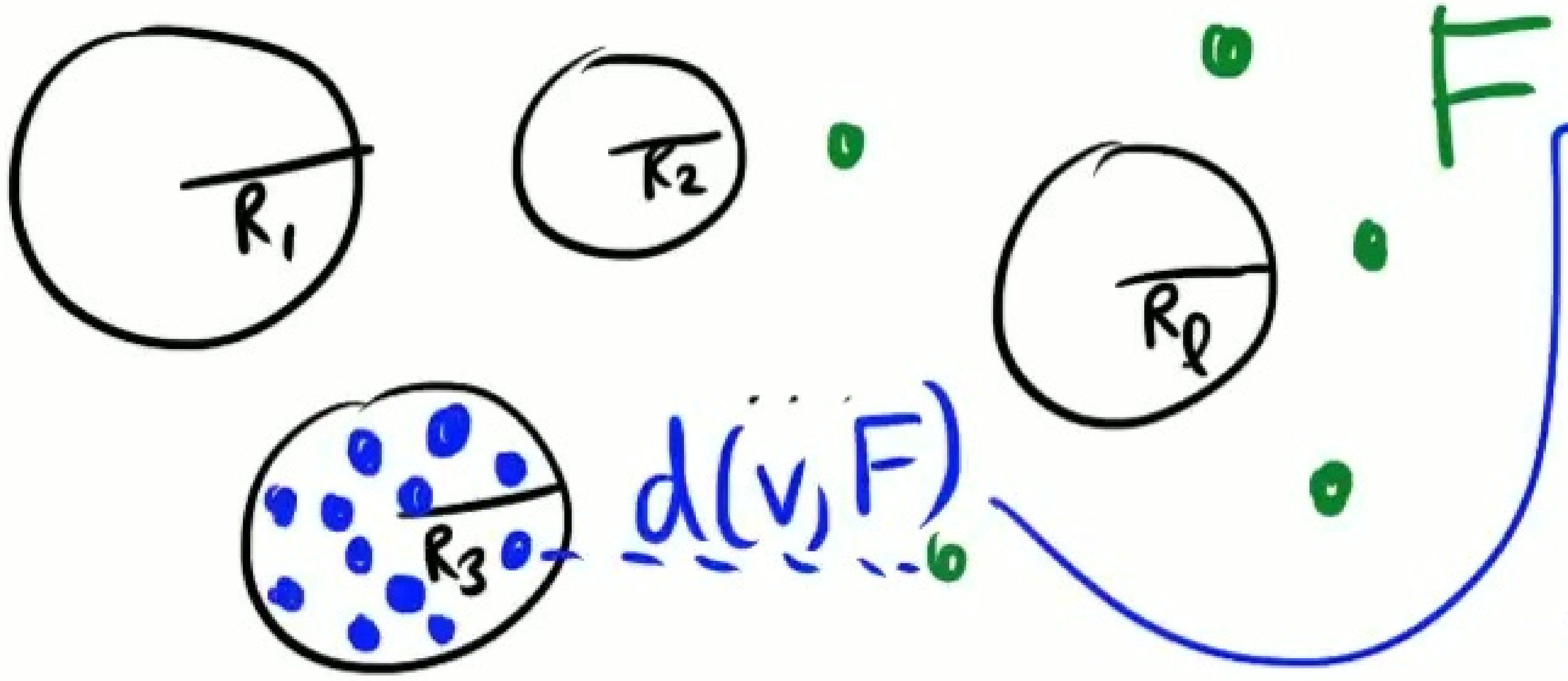

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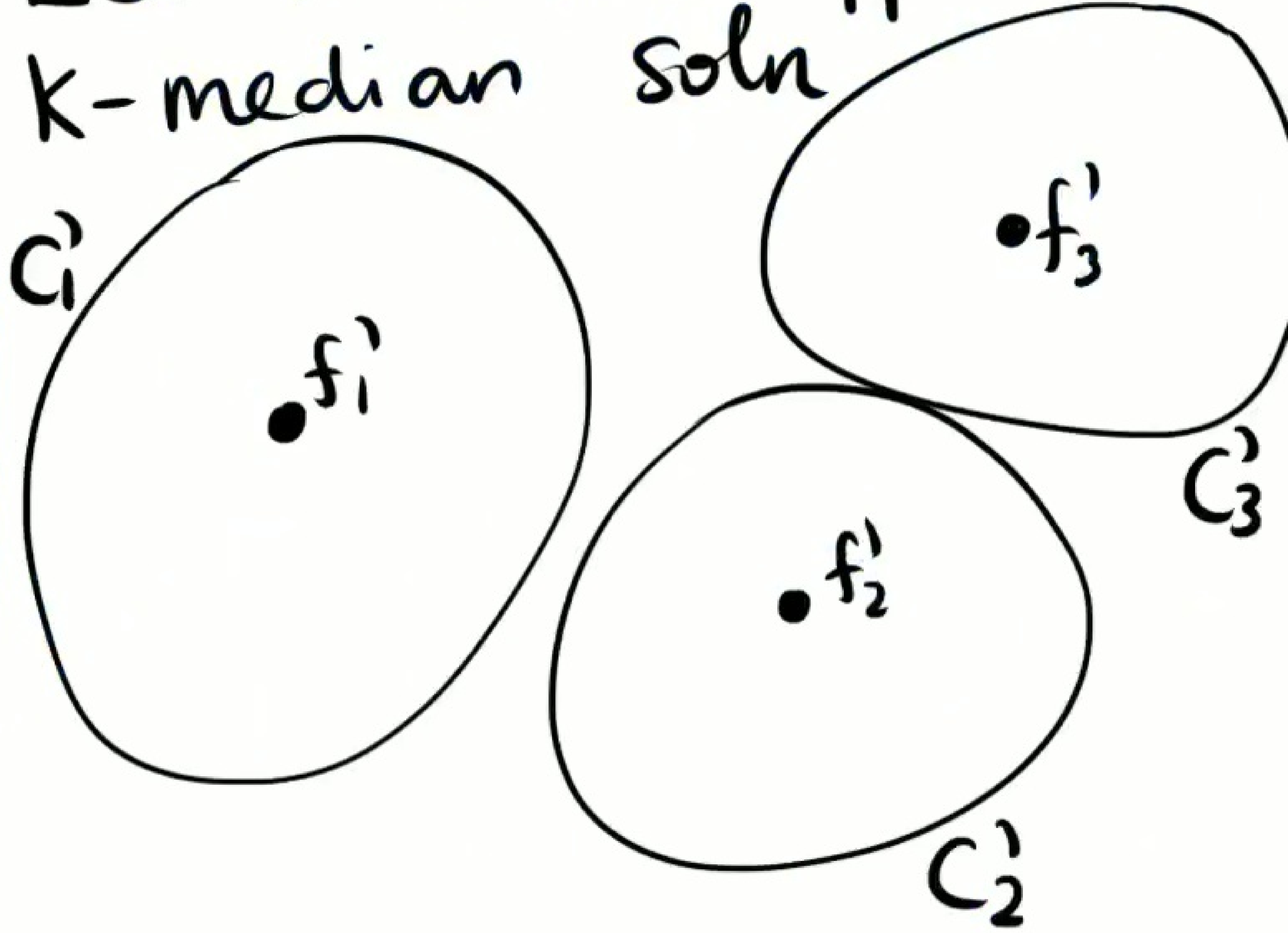

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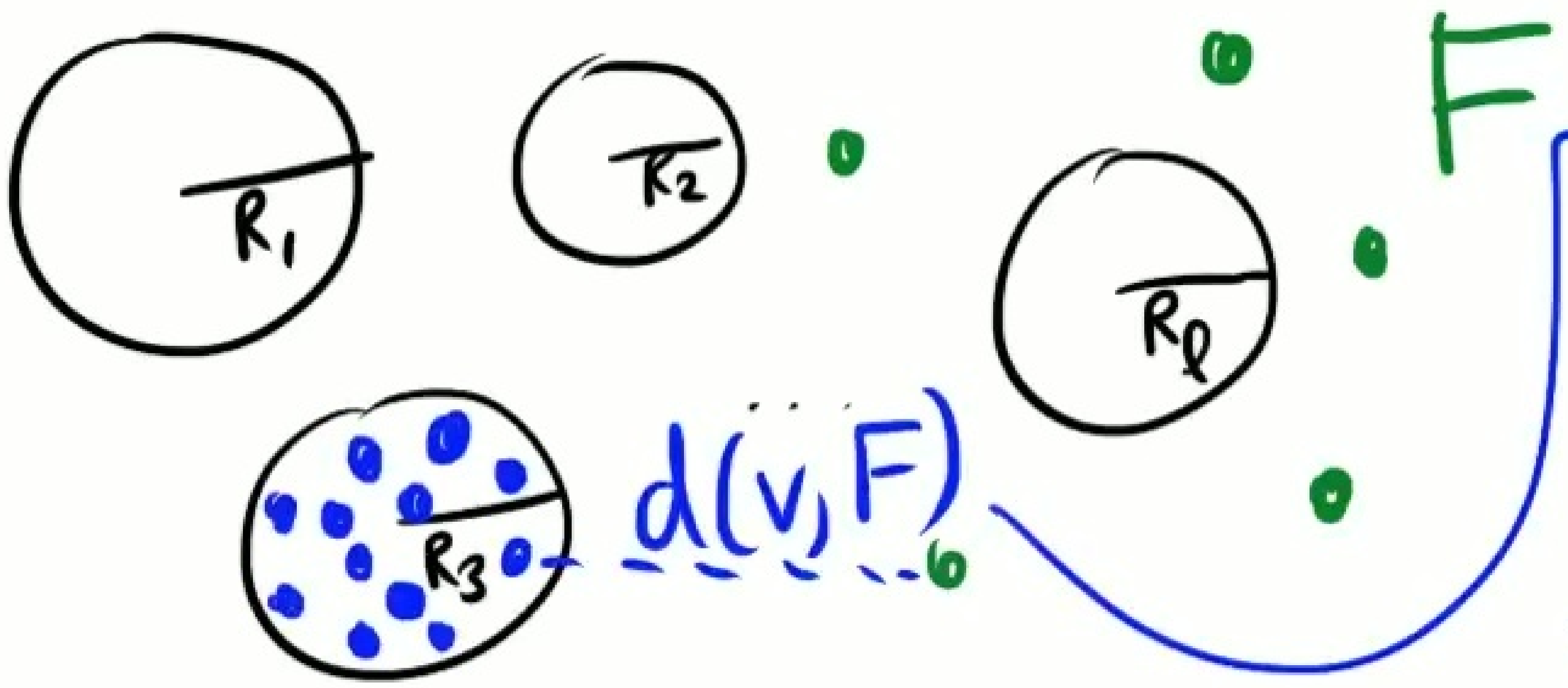

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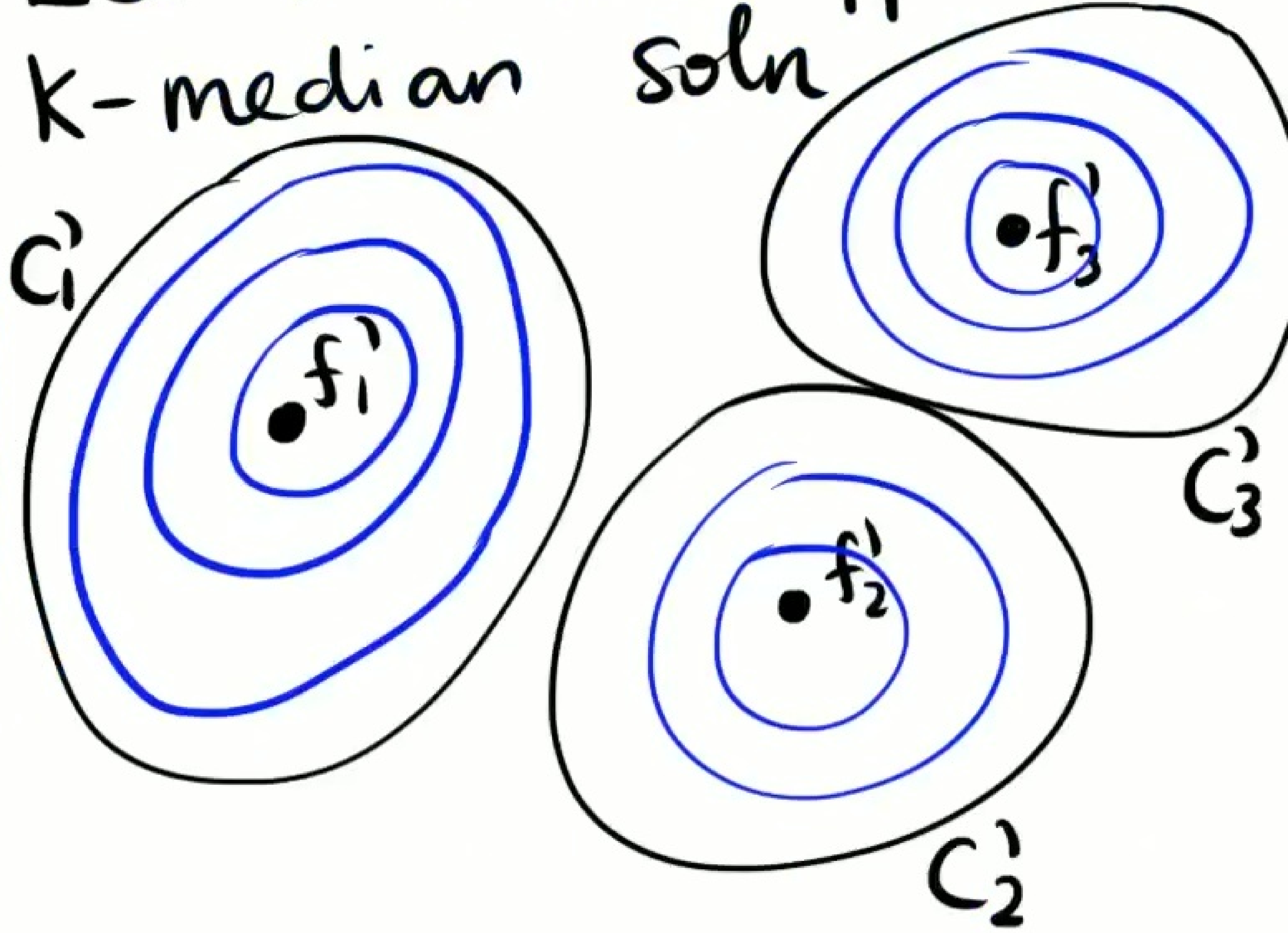

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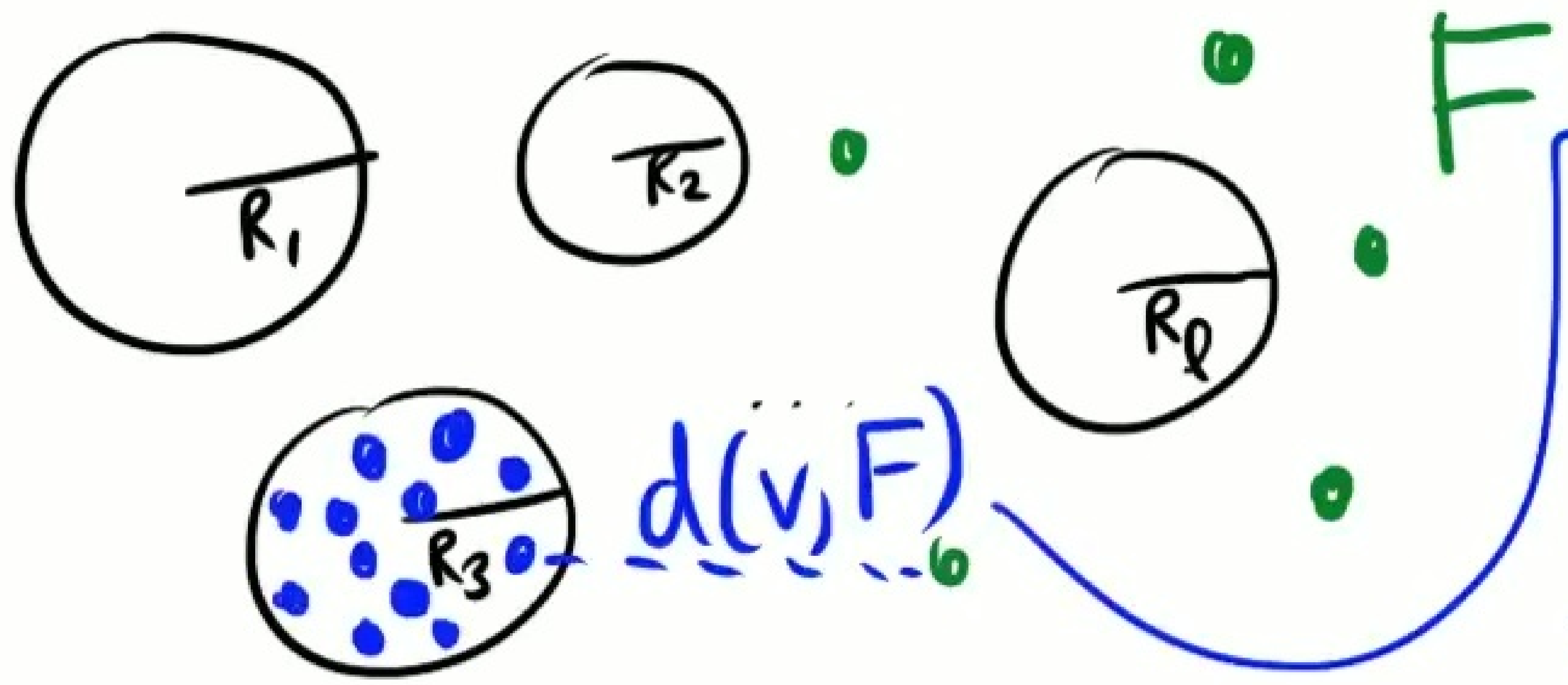

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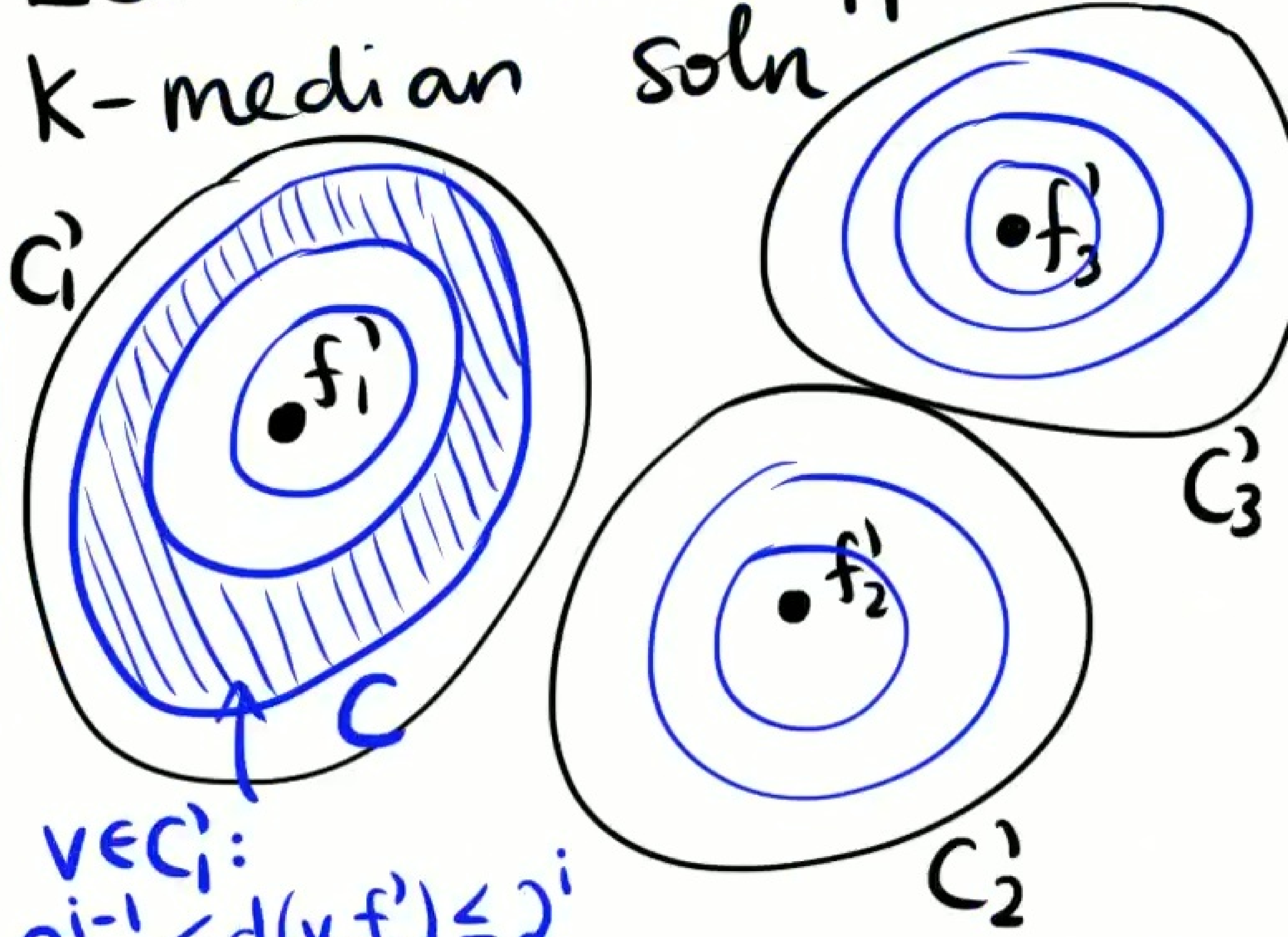

$v \in C_i$:
$2^{i-1} \le d(v,f_i') \le 2^i$

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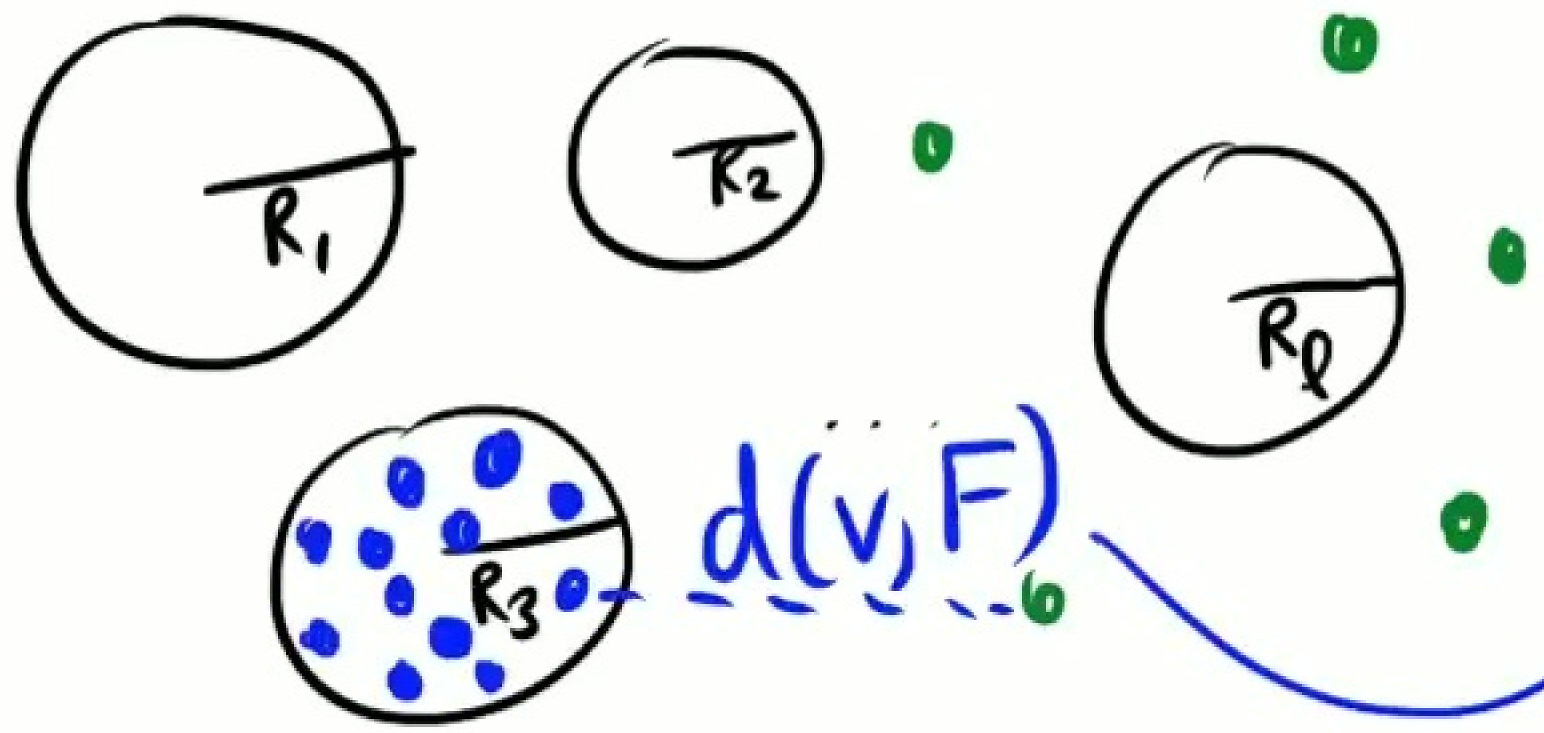

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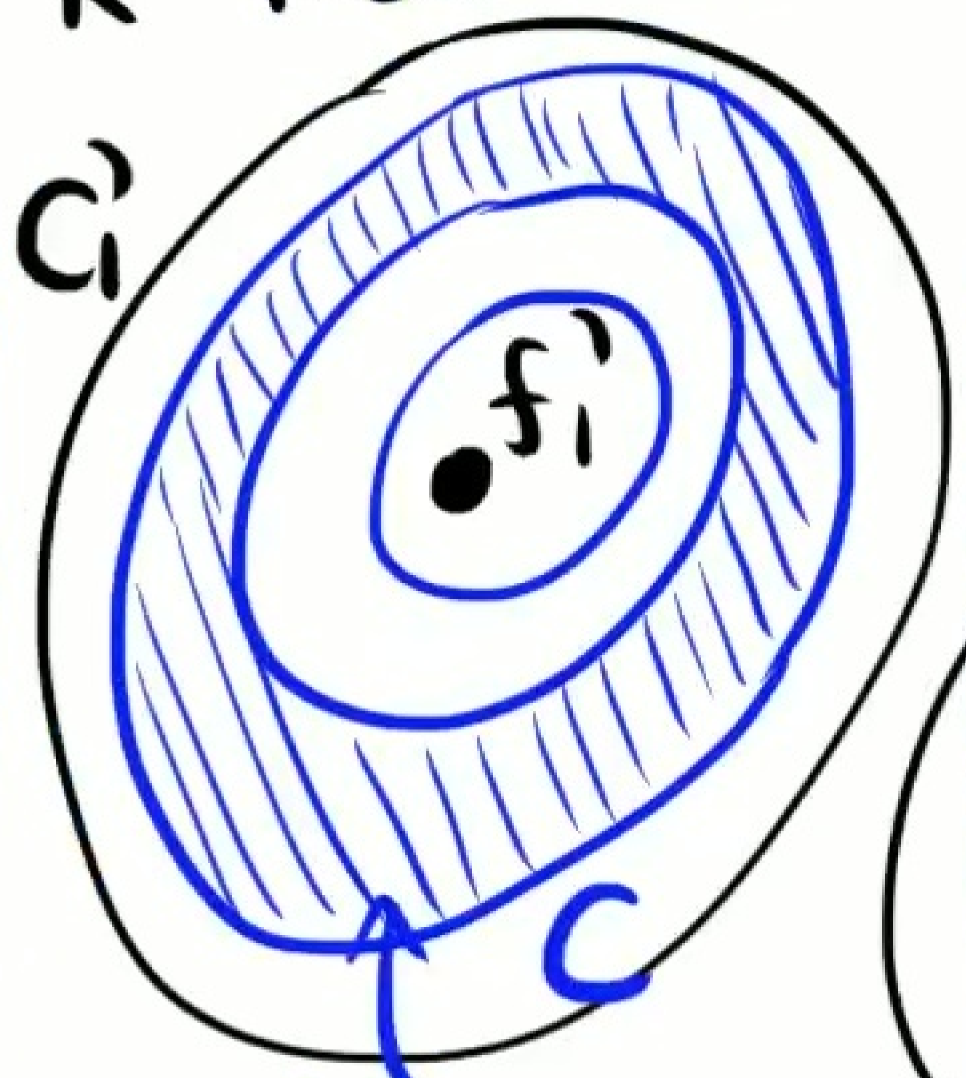

$v\in C_i$:
$2^{i-1} \le d(v,f_i') \le 2^i$

$\to R \le 2\cdot 2^i$
error $\le |C|\cdot\varepsilon\cdot 2^{i+1}$

- Within each cluster C_i:
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Total error:

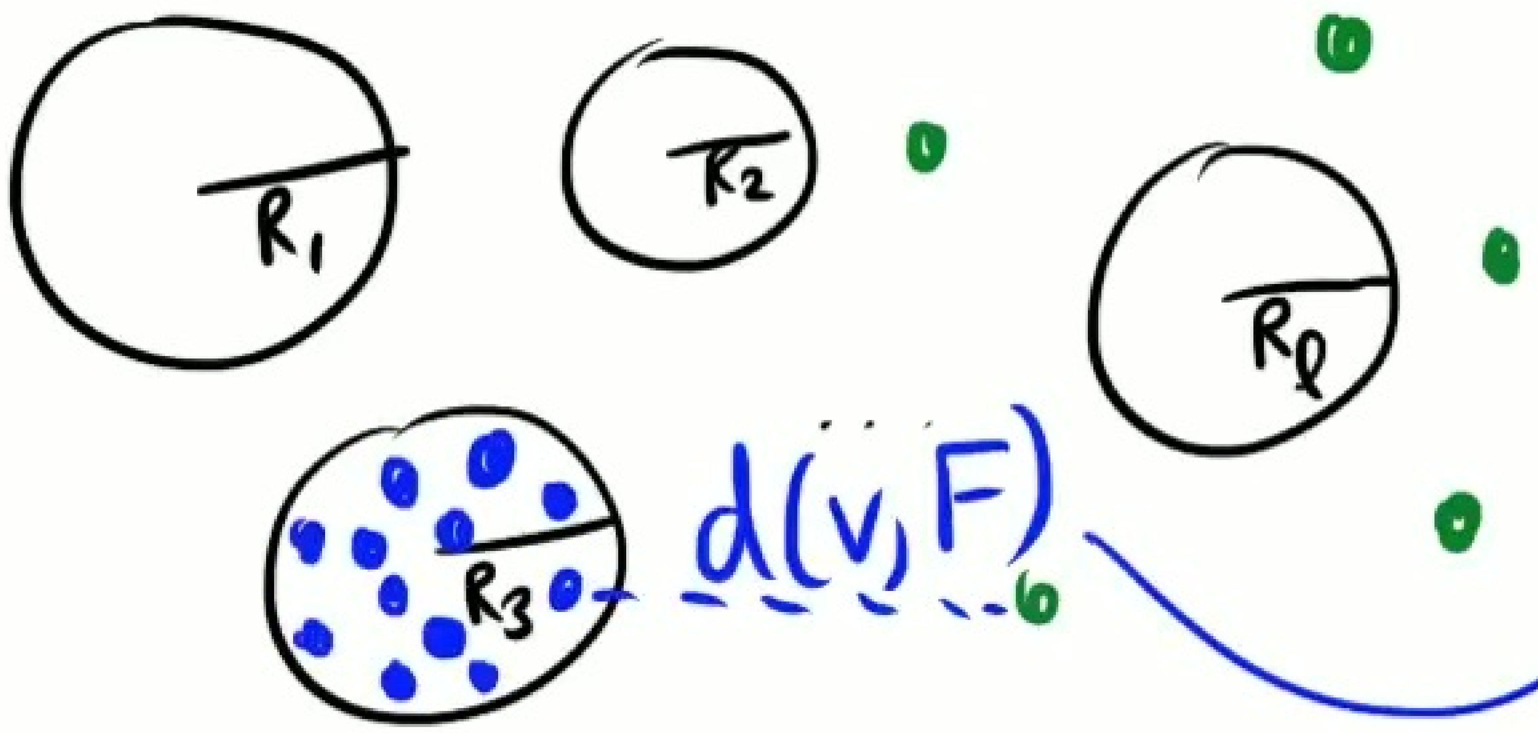
$\sum_{C_i} |C_i|\cdot\varepsilon R_i = \varepsilon \underbrace{\sum_{C_i} |C_i|\cdot R_i}_{=O(OPT)?}$

Chen's Coreset Algorithm

- Random sampling
- For each F ($|F|=k$),
  $\Pr\left[\sum_{v\in S} w(v)\, d(v,F) \in (1\pm\varepsilon)\sum_{v\in C} d(v,F)\right] \gg 1-n^{-k}$

  Union bound over all F

- Suppose can cluster C into ℓ clusters



Claim:
$\{d(v,F) : v$
contained in
interval size $O(R_i)$

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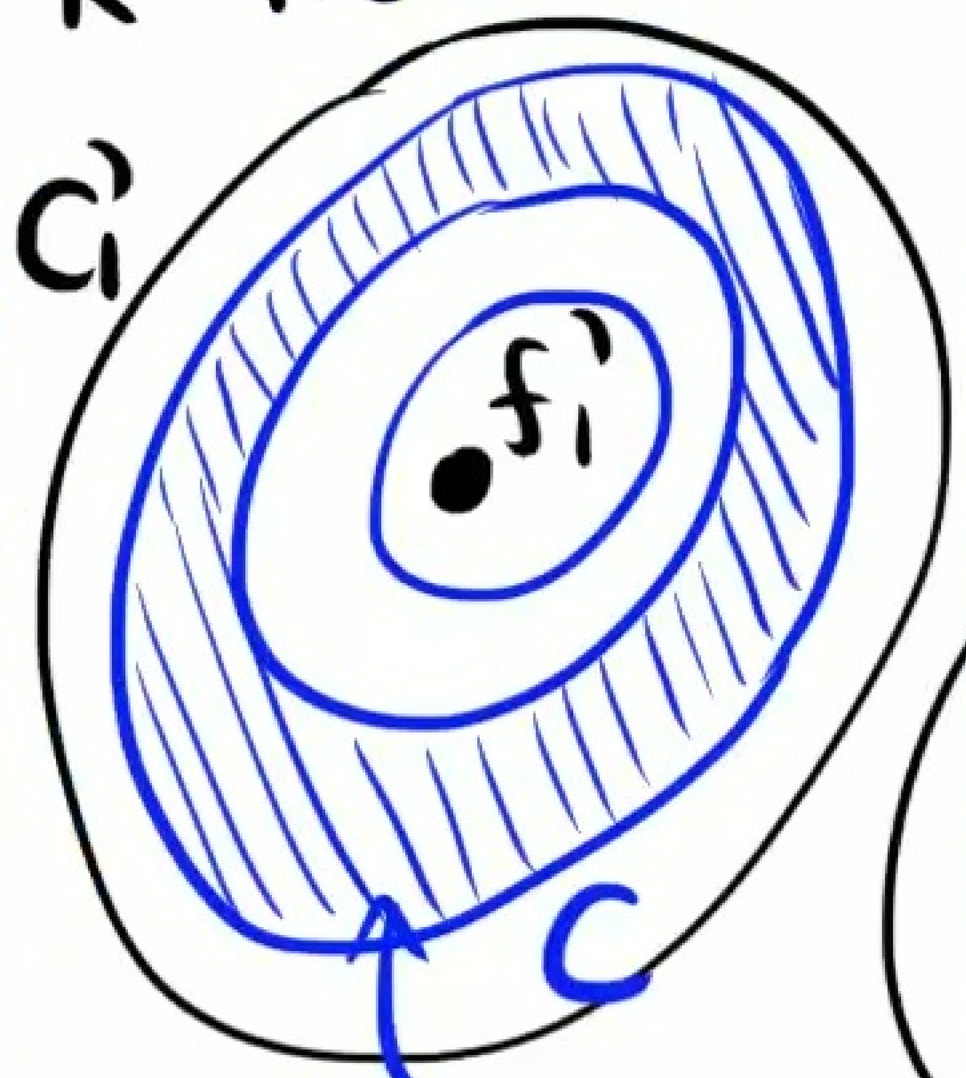

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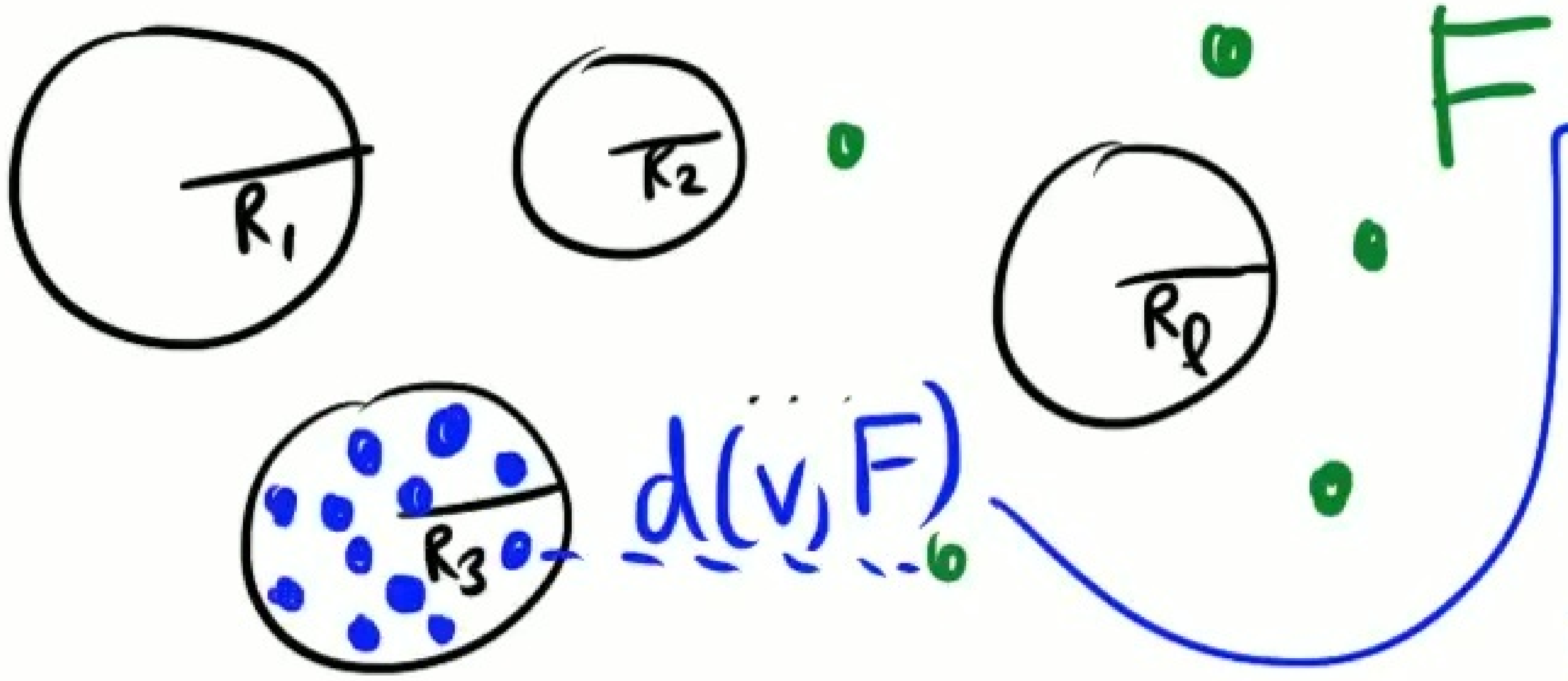

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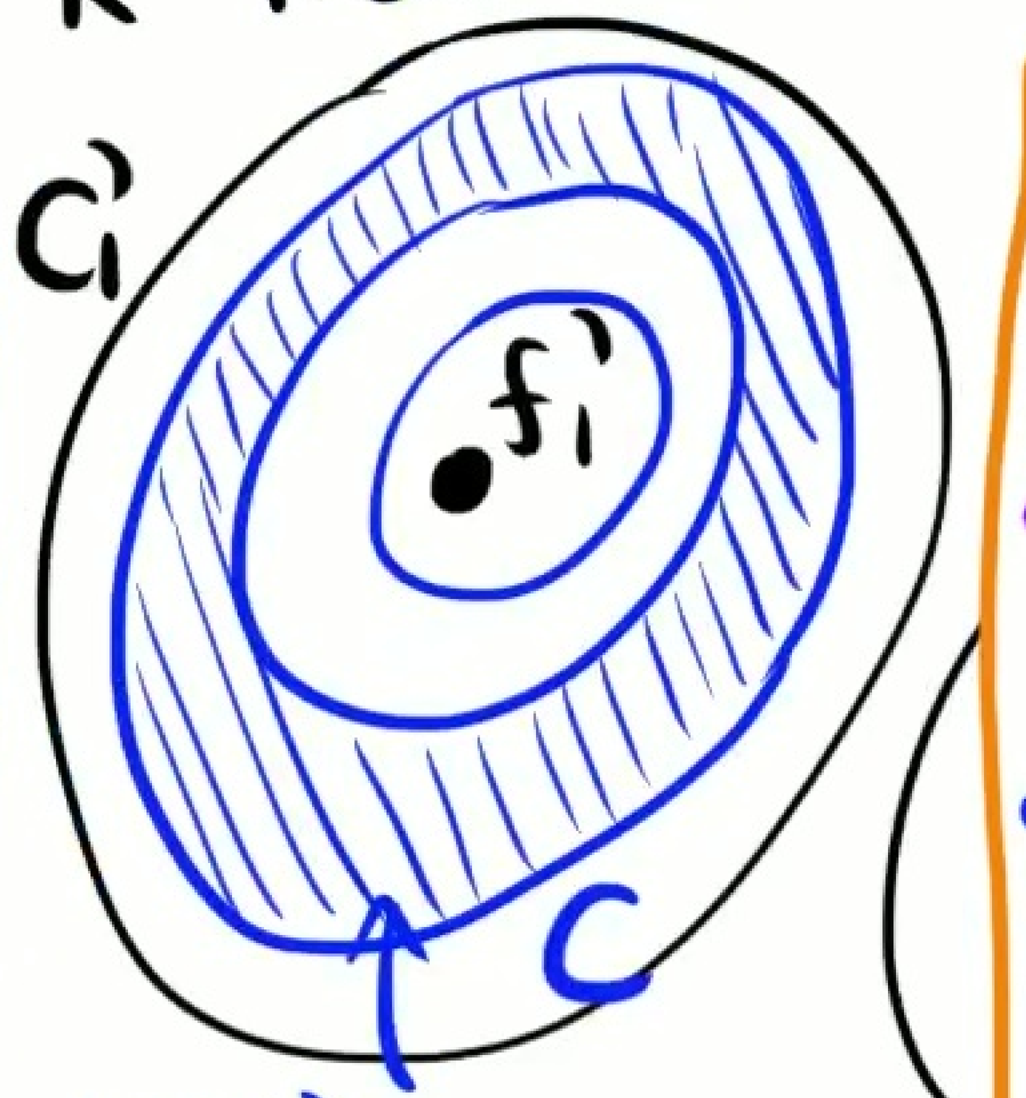

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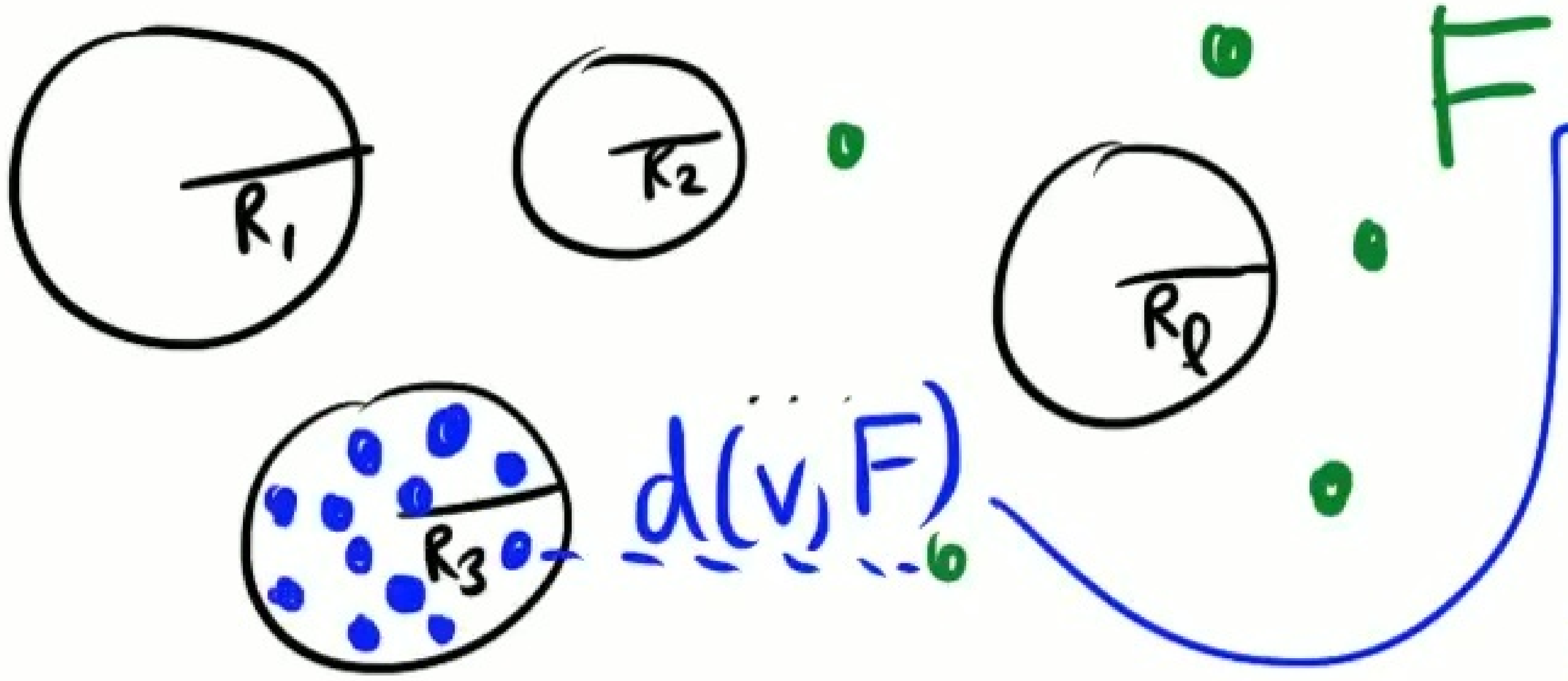

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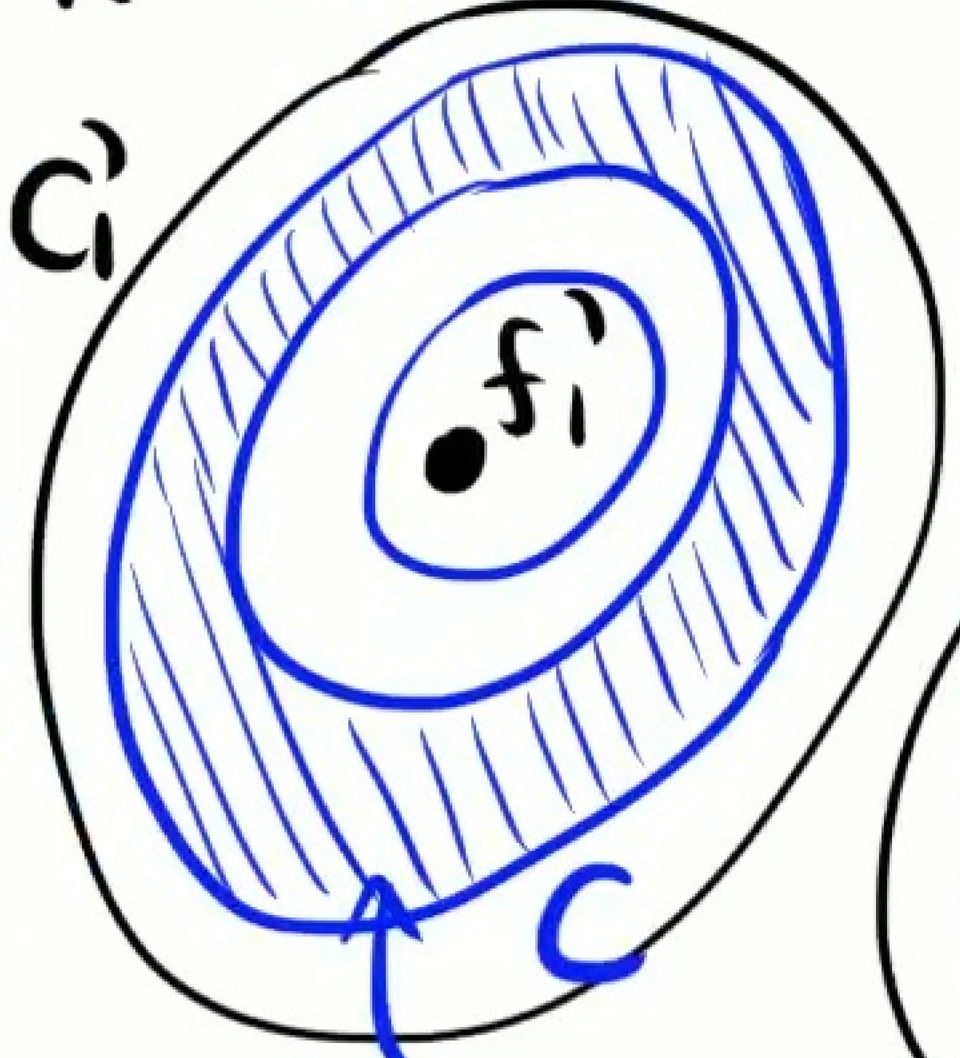

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- Contribution to soln $\ge |C|\cdot 2^{i-1}$
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  $\le \sum_C |C|\,\varepsilon\, 2^{i+1}$
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  $\le 4\varepsilon(\text{soln}) \le 12\varepsilon\,\text{OPT}$

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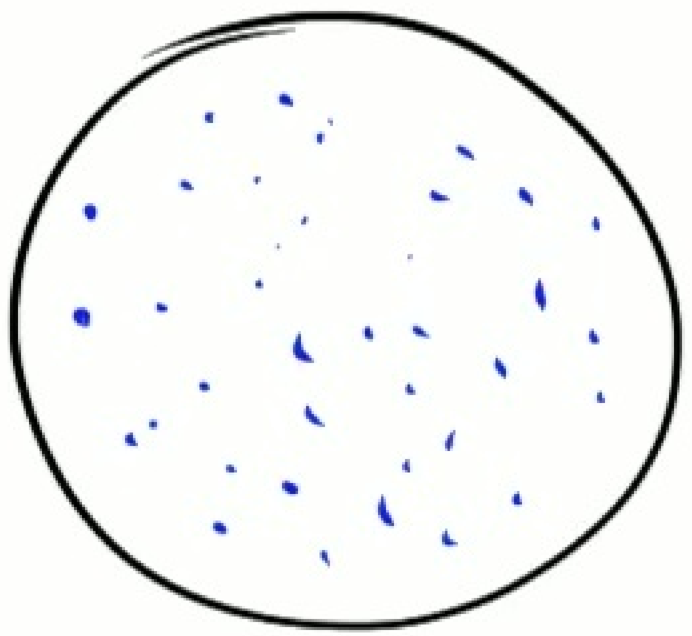
Extending to Capacitated

- This talk: single cluster
-



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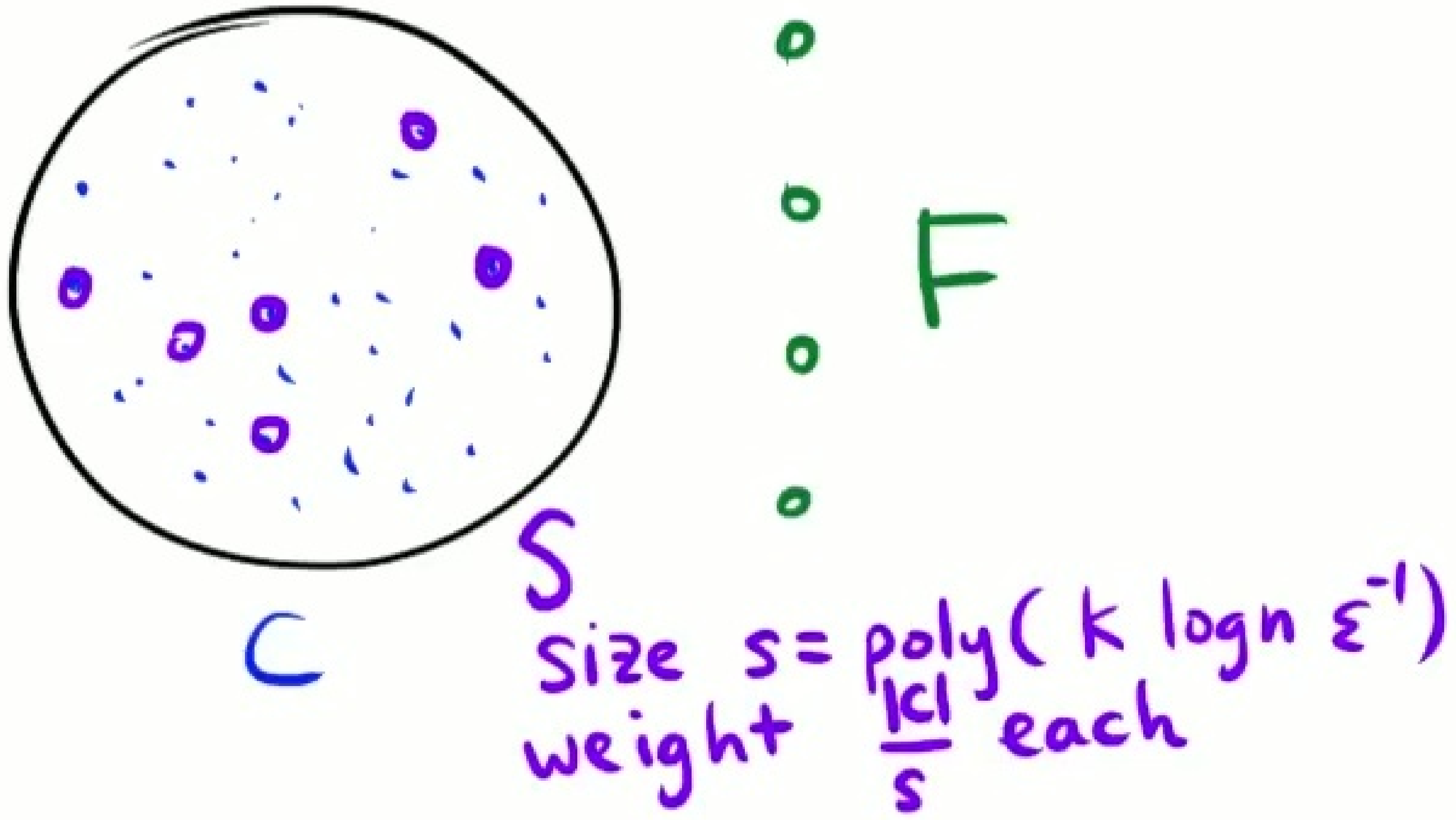
- 

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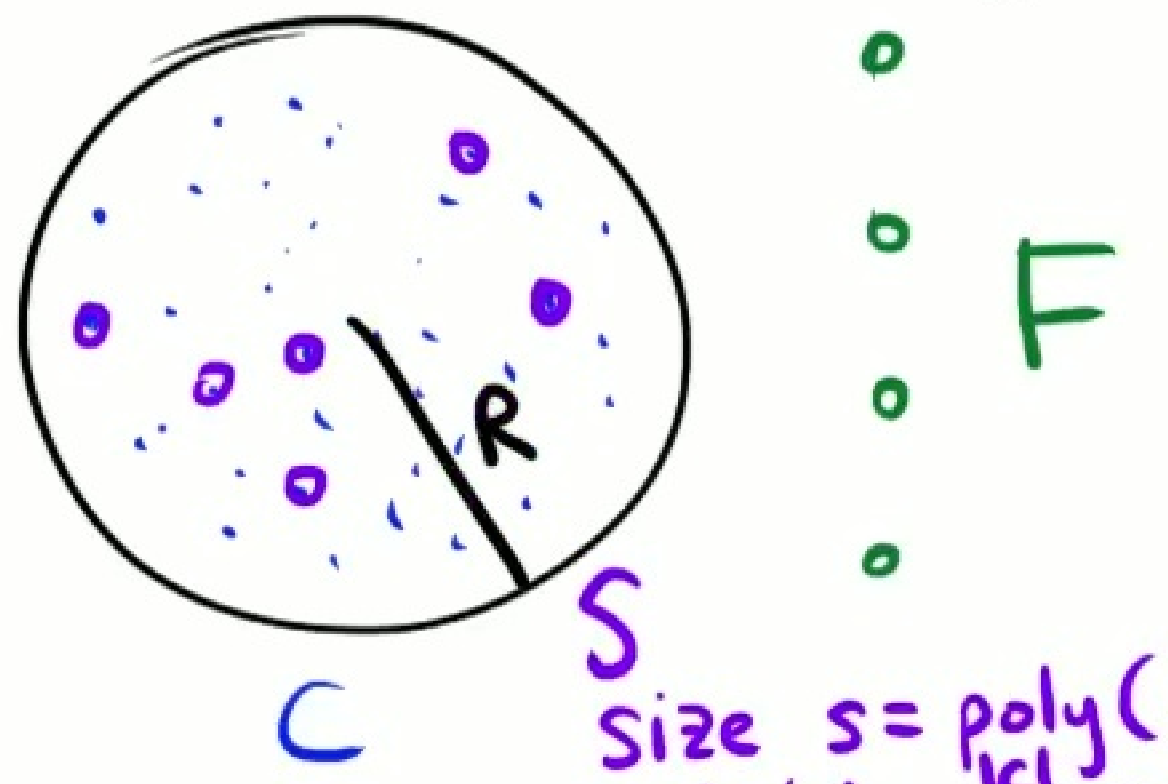
- 

Size $s = \text{poly}(k \log n\, \varepsilon^{-1})$
weight $\frac{|C|}{s}$ each

To show:

$\text{MinCostFlow}(C,F) \in \text{MinCostFlow}(S,F) \pm \varepsilon |C| R$

w.p. $\geq 1 - n^{-k}$.

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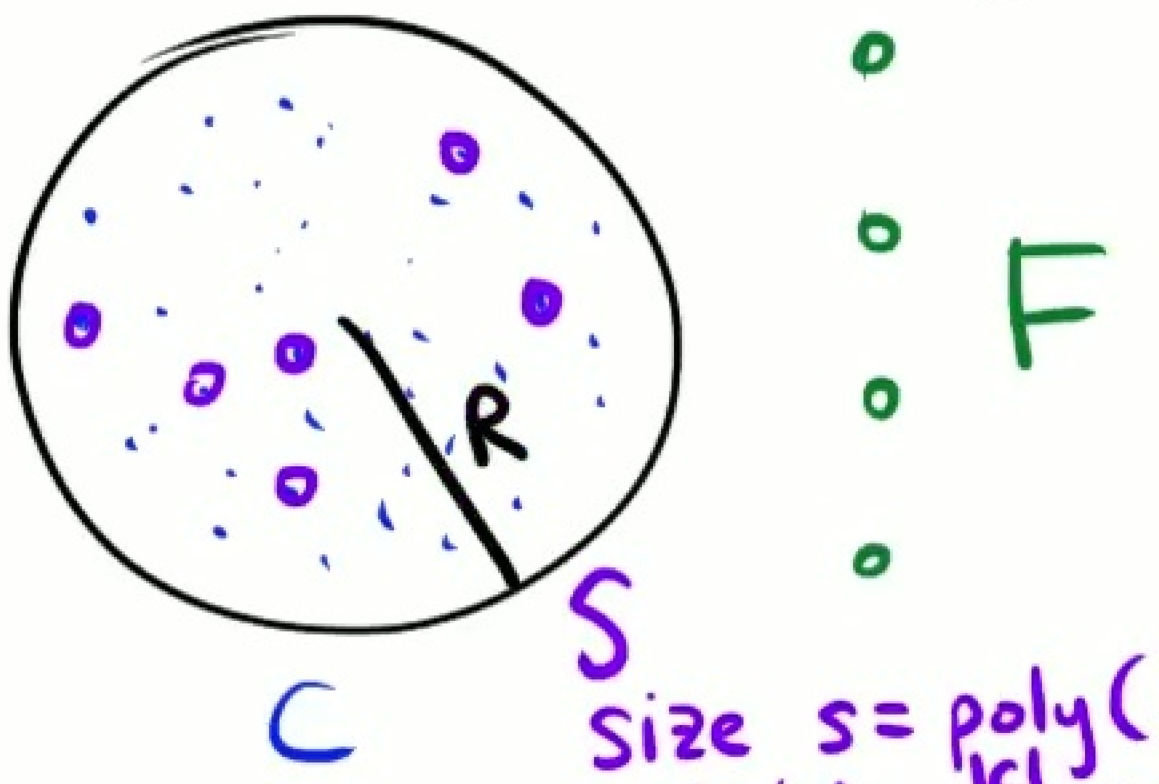
- 

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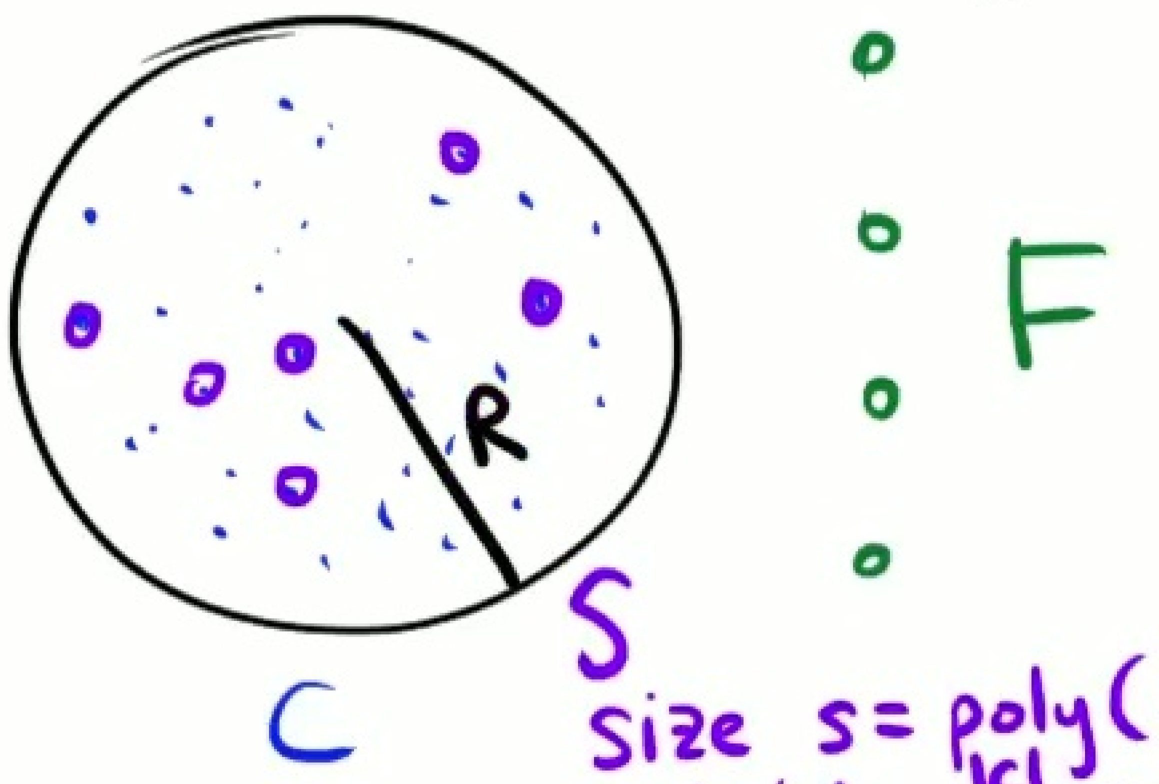
- 

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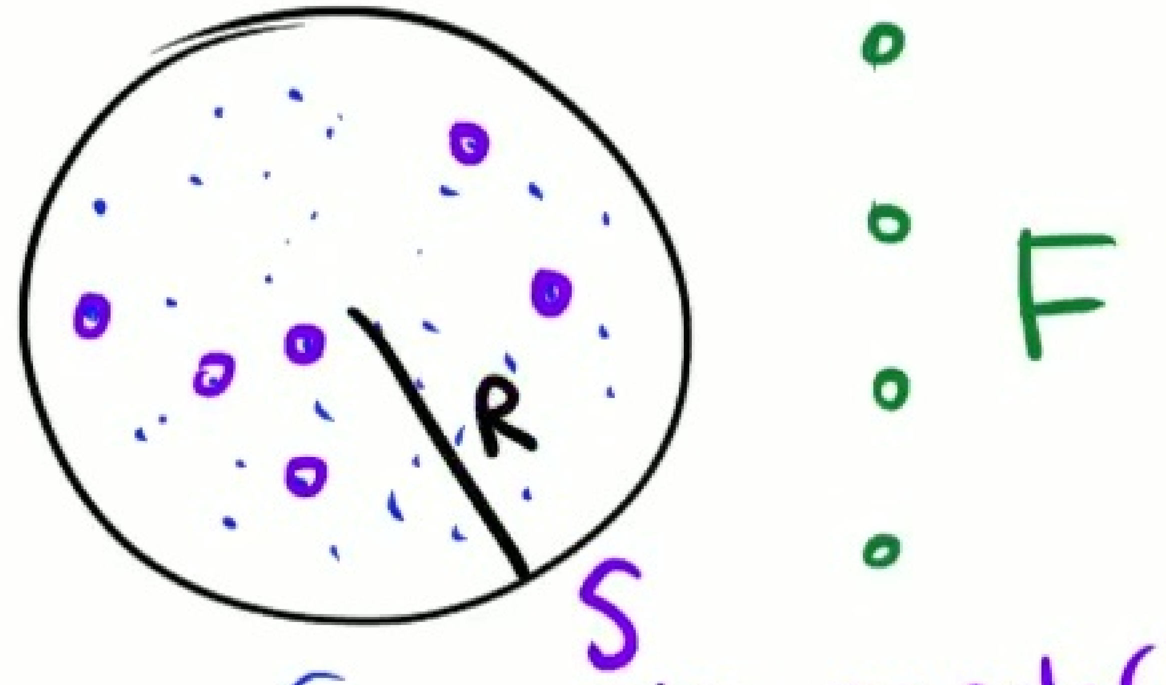
- 

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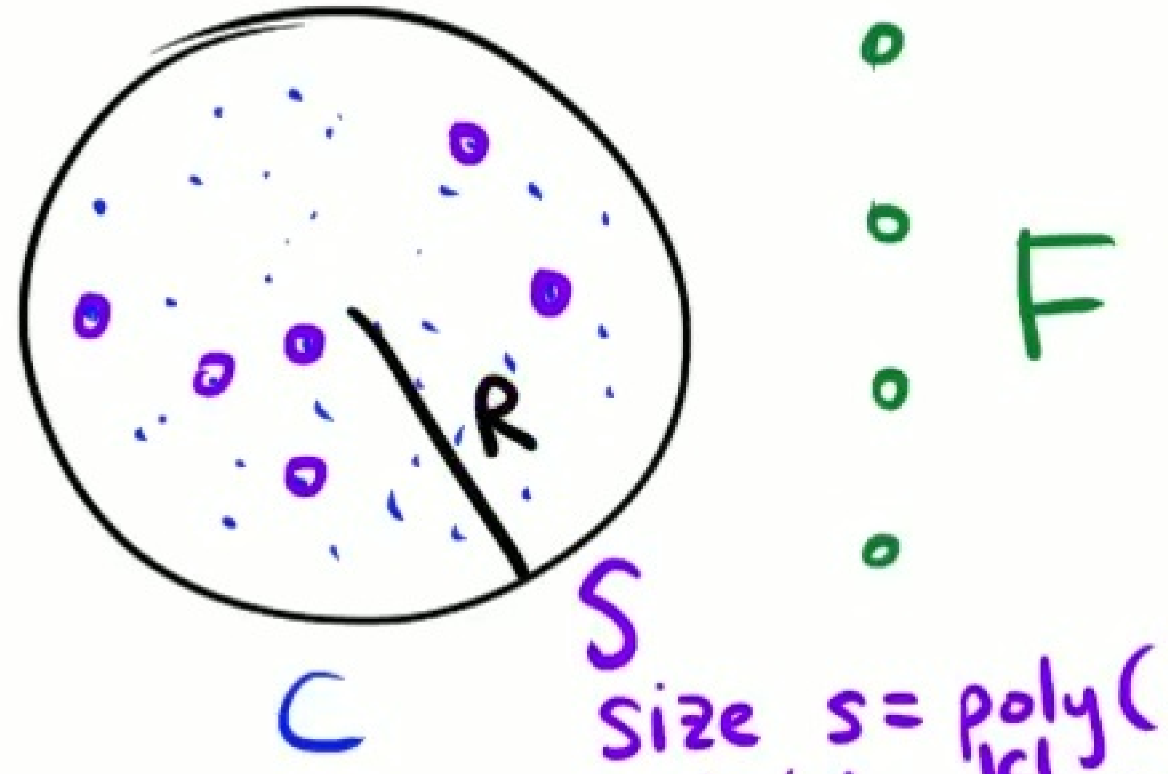
- 

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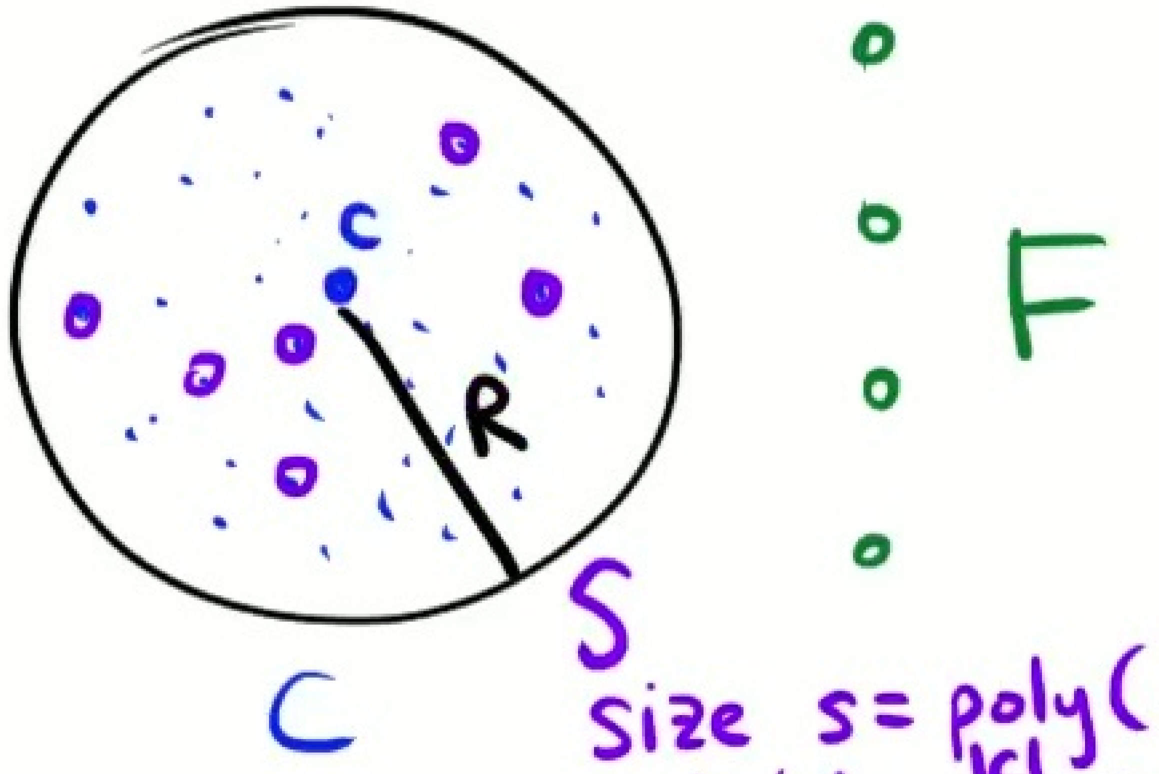
- 

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total demand $|C|$

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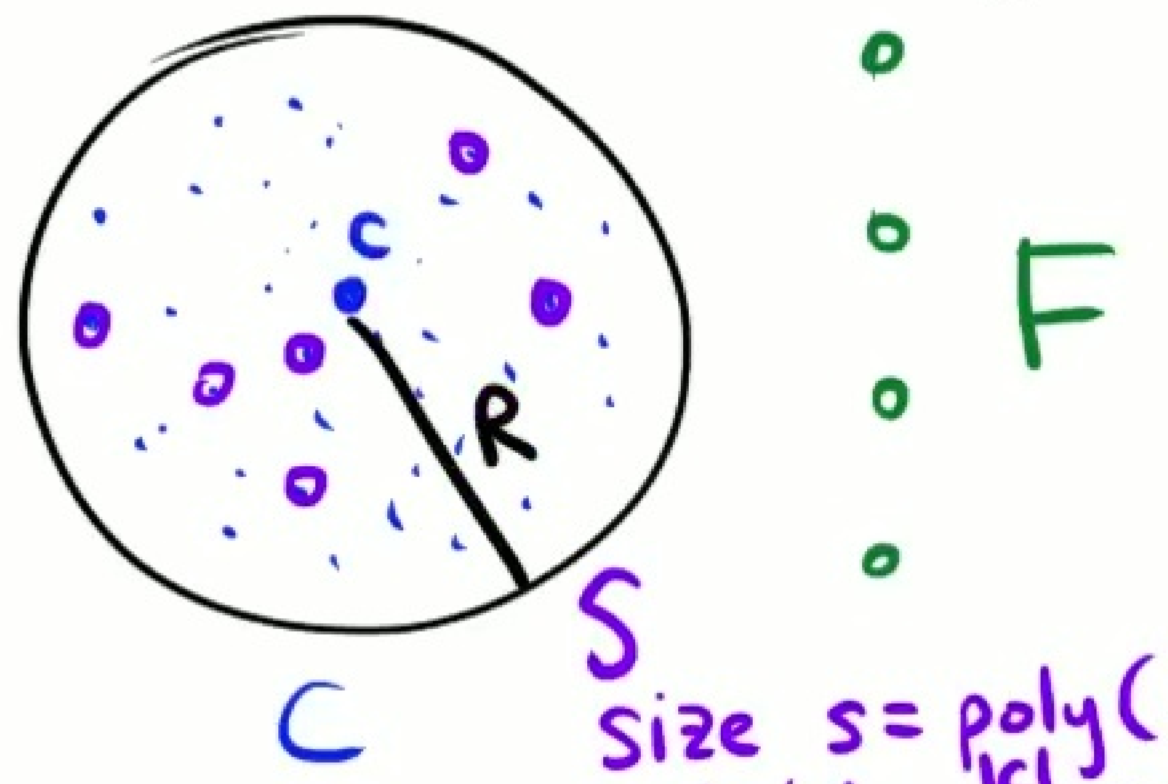
- 

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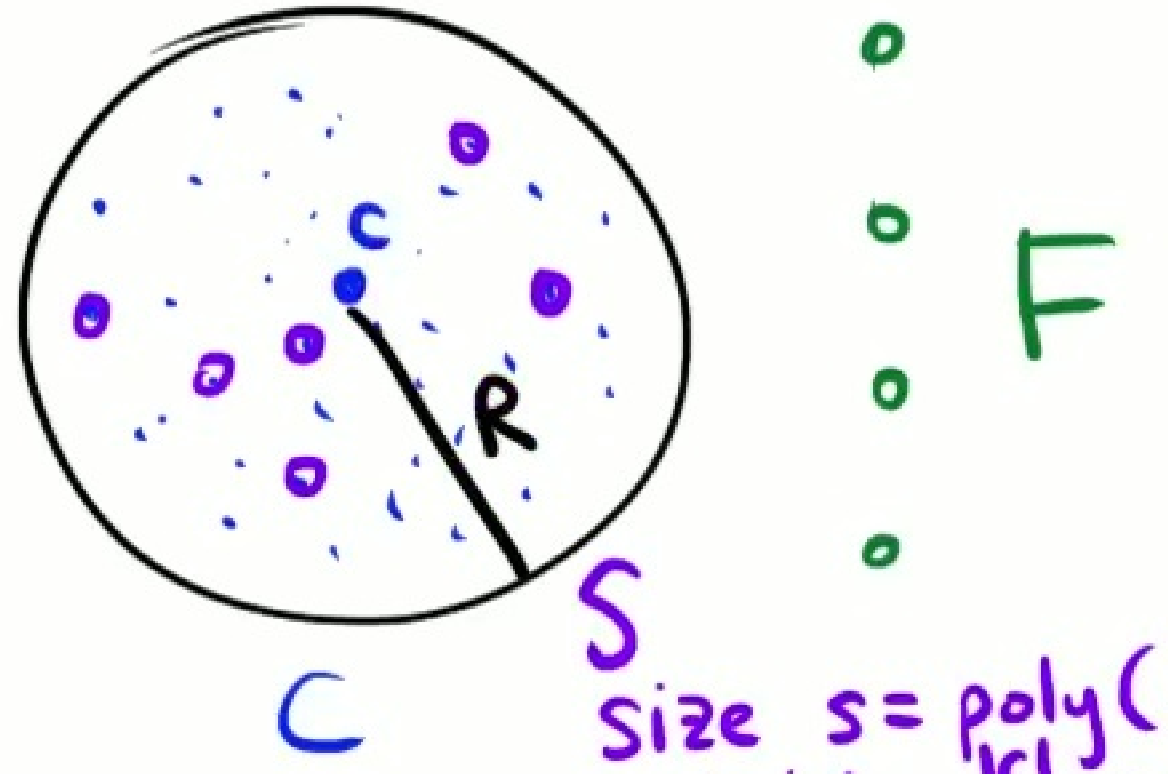
- 

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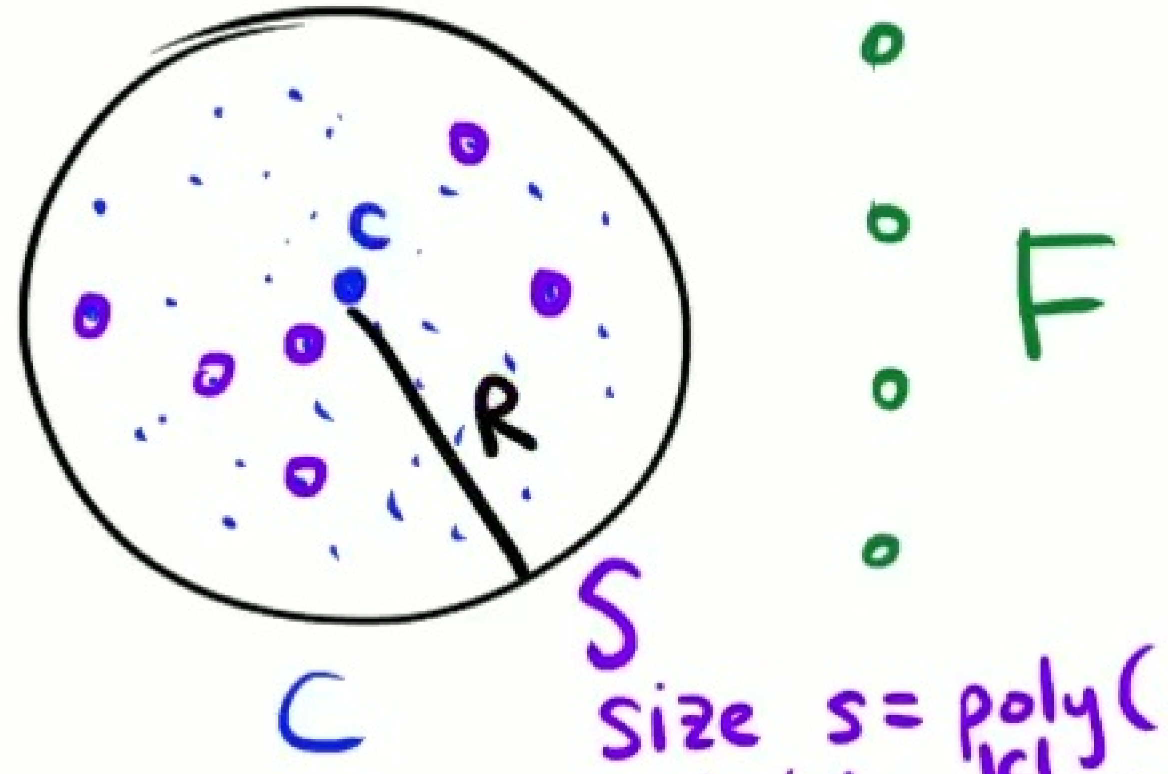
- 

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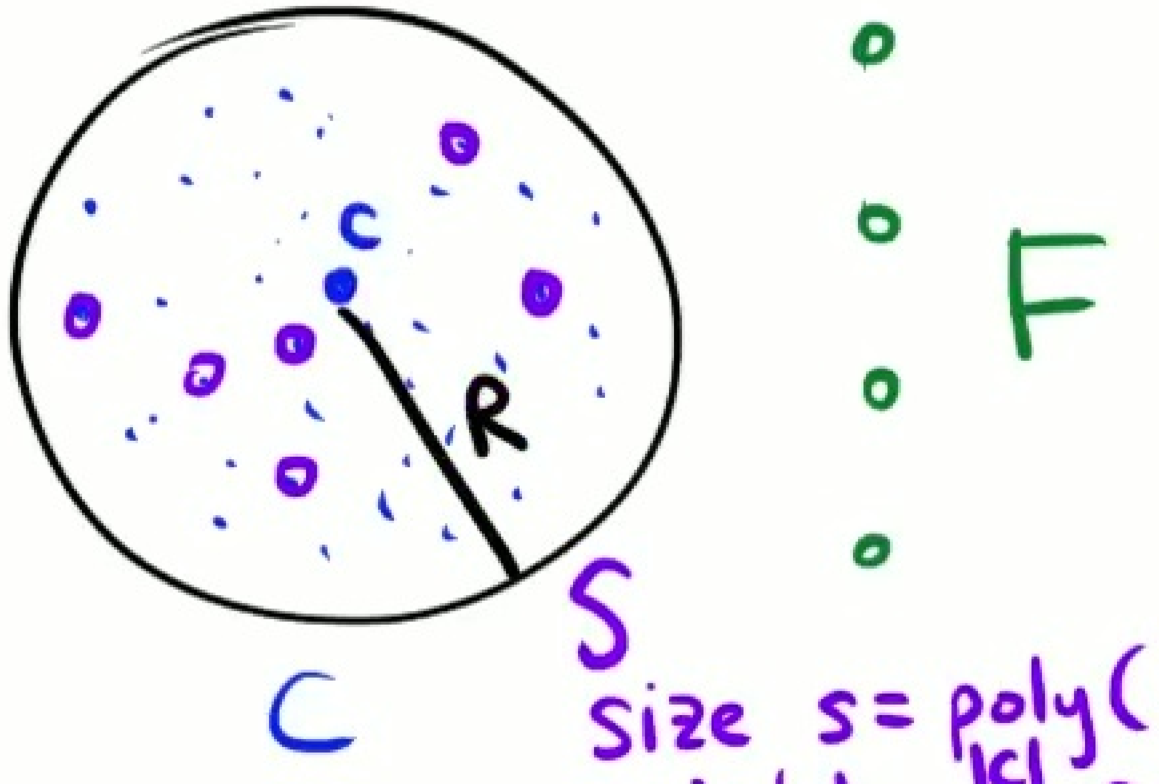
- 

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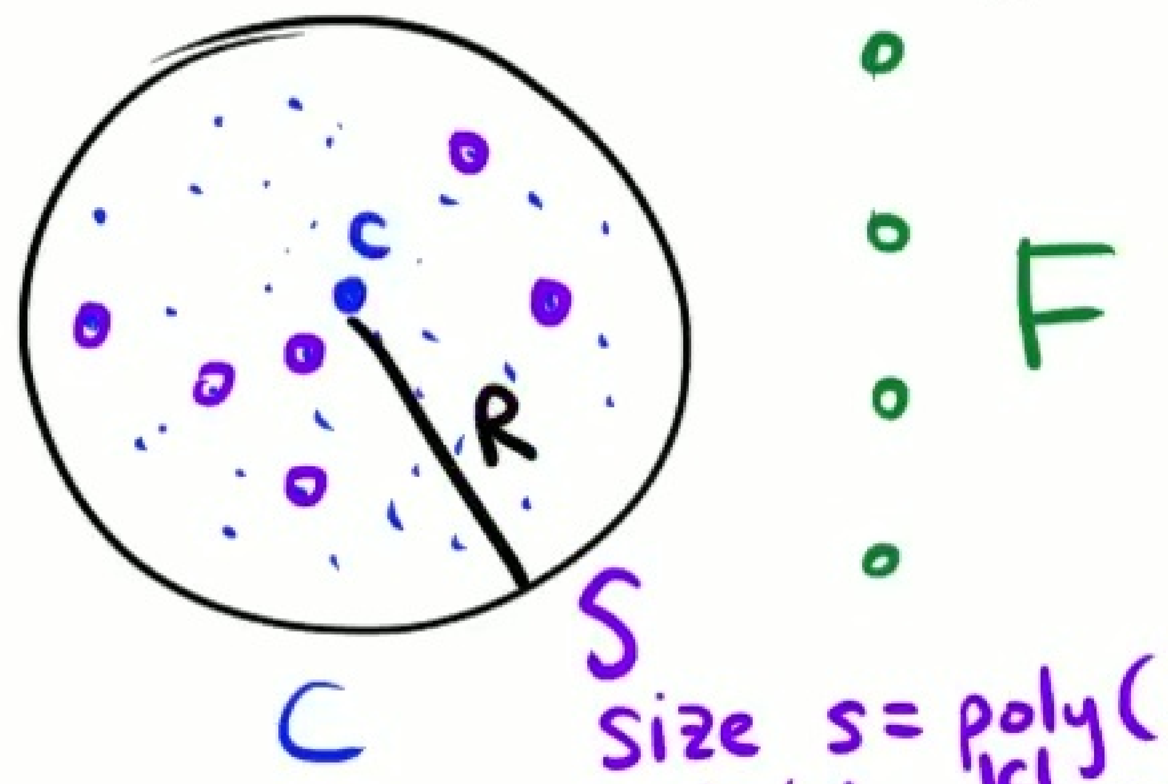
- 

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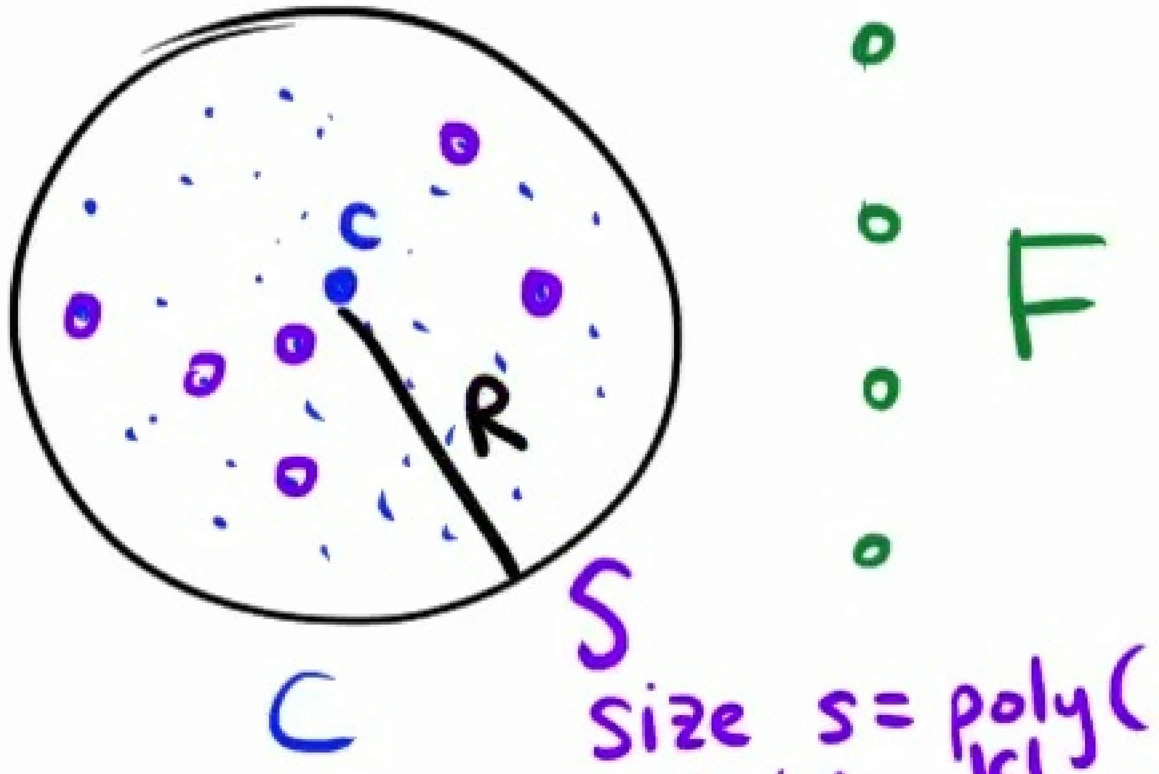
- 

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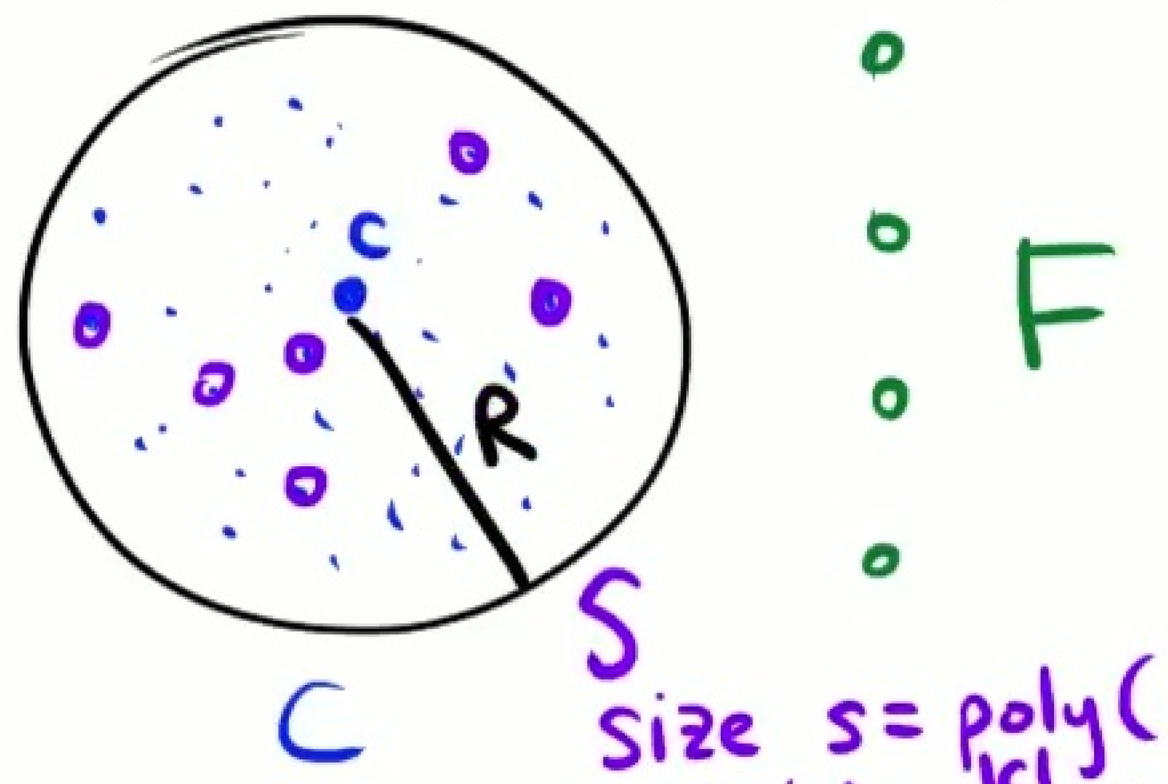
- 

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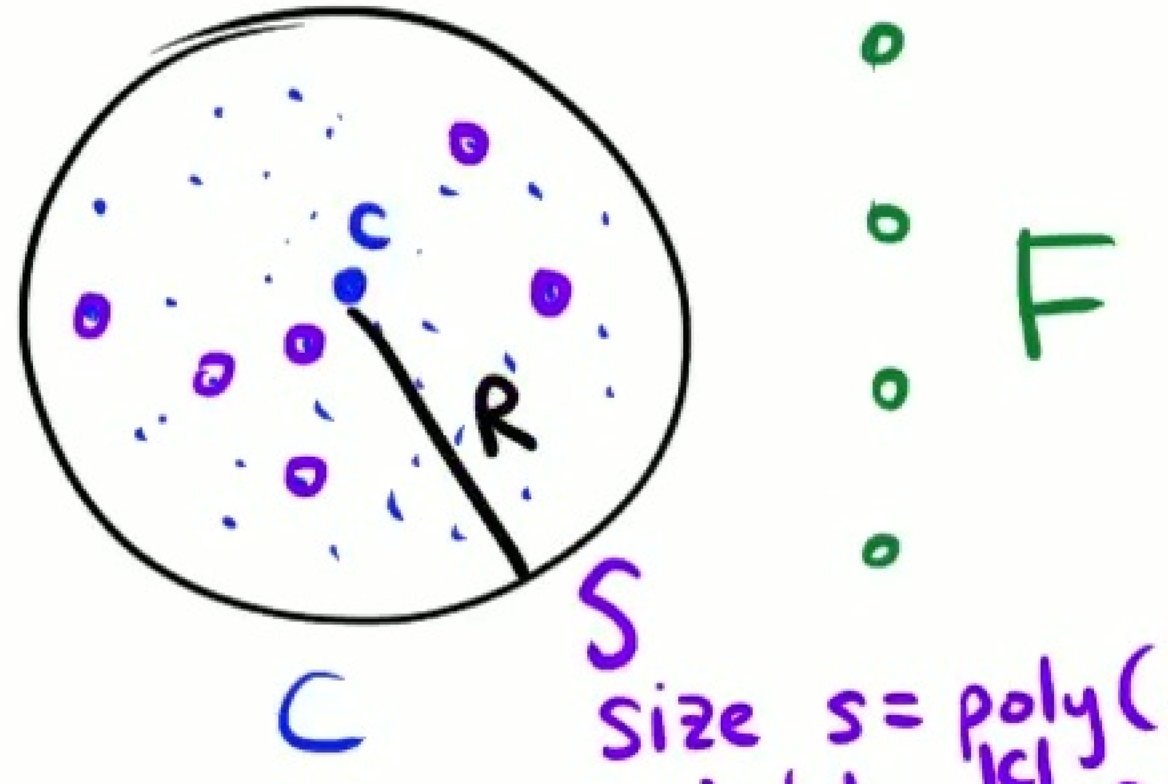
- 

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$$\mathbb{E}[g(d)] \approx g(\mathbb{E}[d]) = g(\mathbb{1})$$

Suppose sample each $v \in C$ w.p. $\frac{s}{|C|}$ independently

Demand vector $d \in \mathbb{R}_+^C$:

$$d_v = \frac{|C|}{s} \text{ if } v \text{ sampled, else } d_v = 0$$

$g(d)$:
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total demand $|C|$

$g(d) :=$ MinCostFlow(demands, F)

$-\mathrm{cap}(f)$ demand at each $f \in F$

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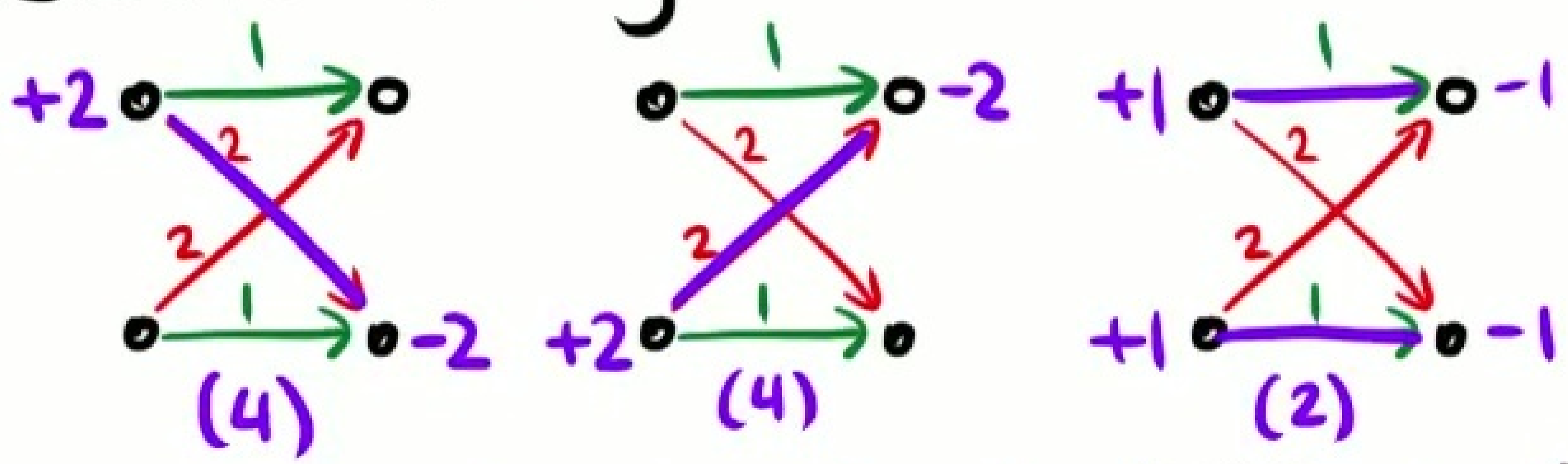
- Convexity of MinCostFlow:



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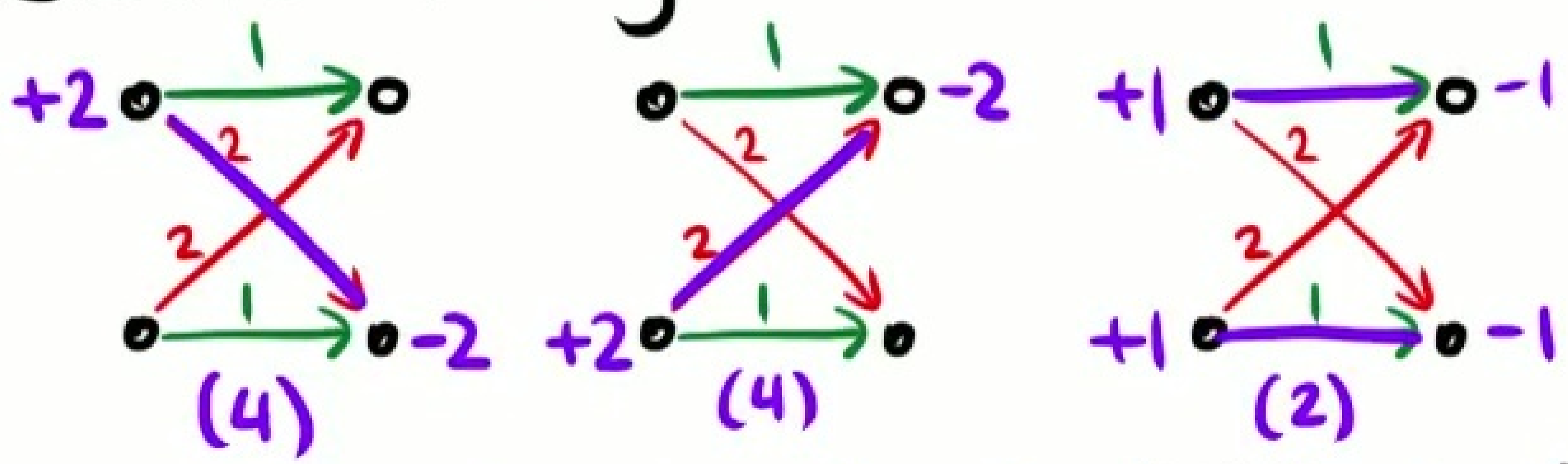

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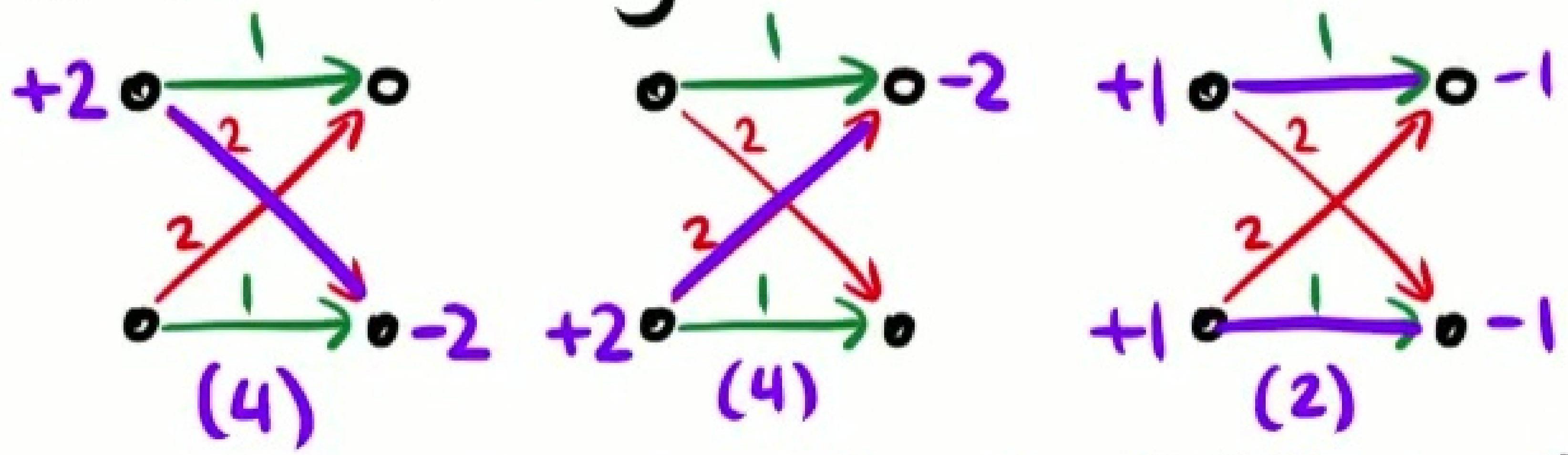

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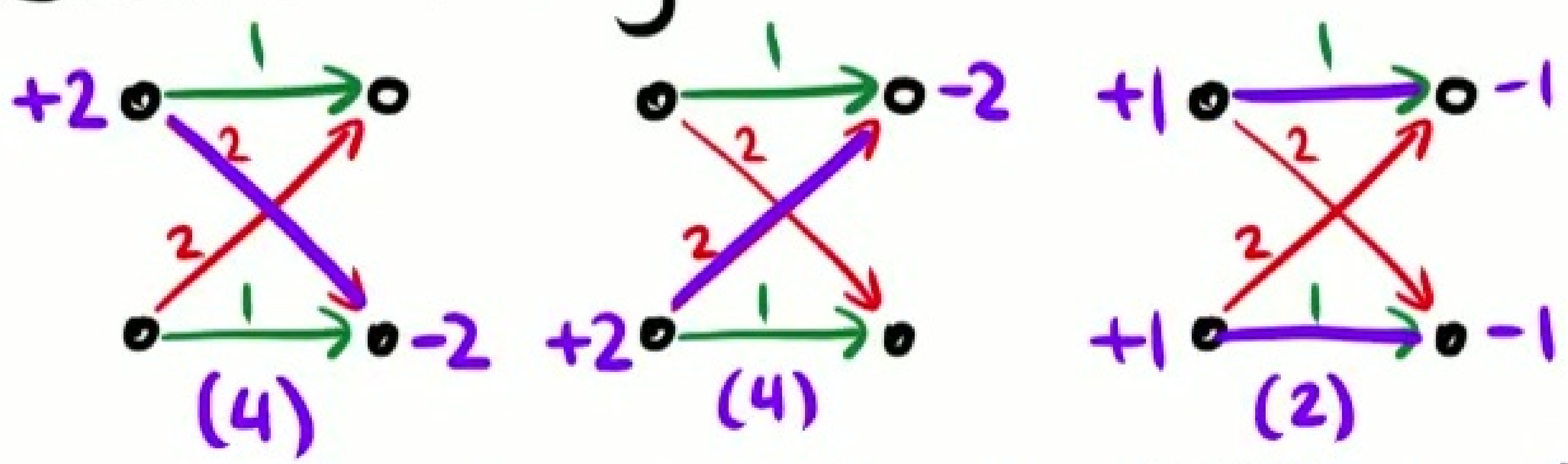

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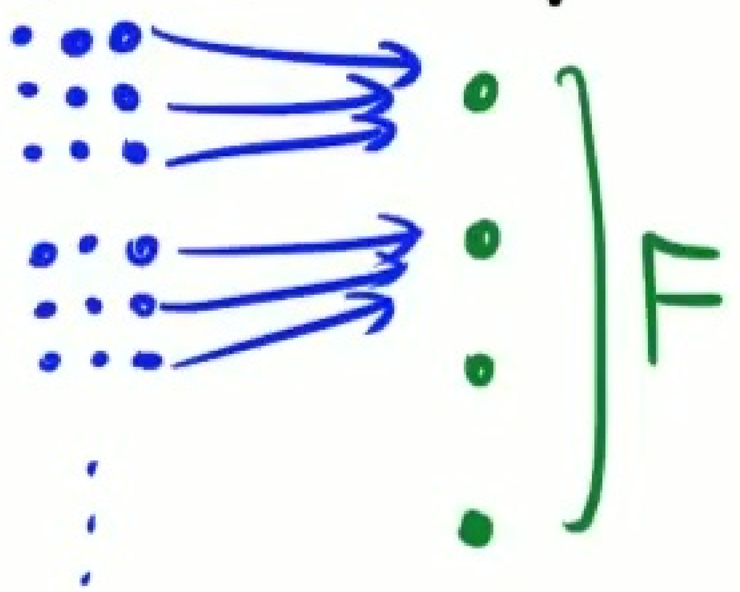

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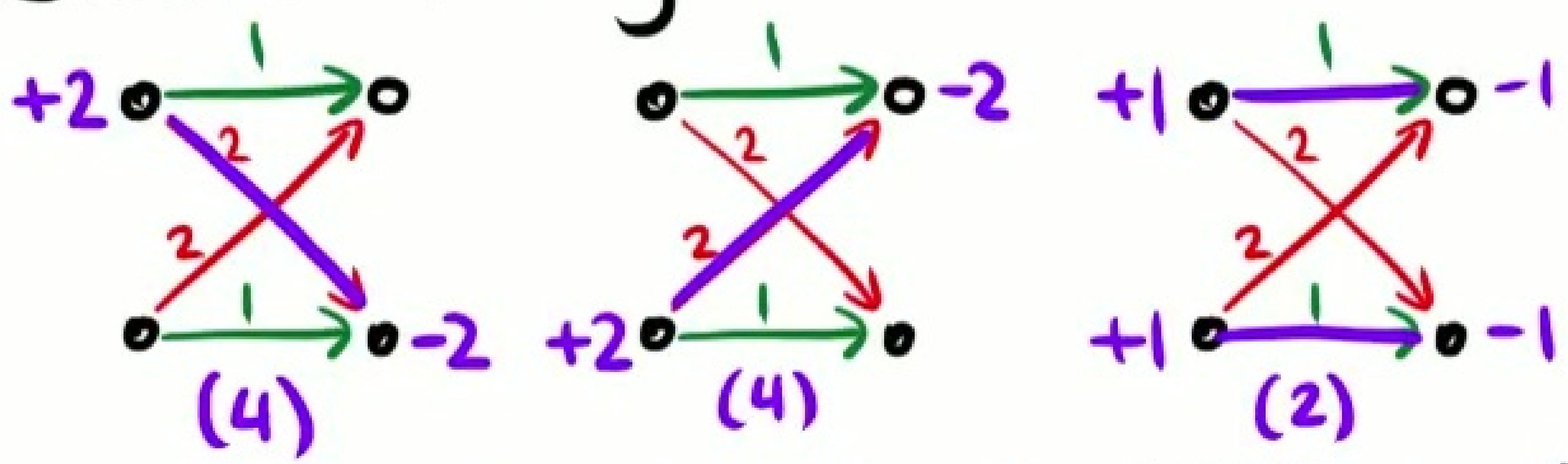

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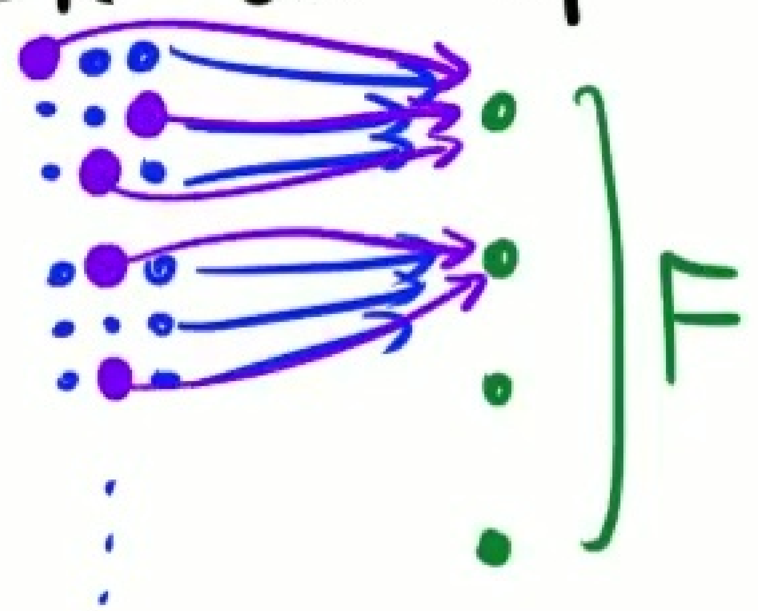

Route sampled clients to same facility in F

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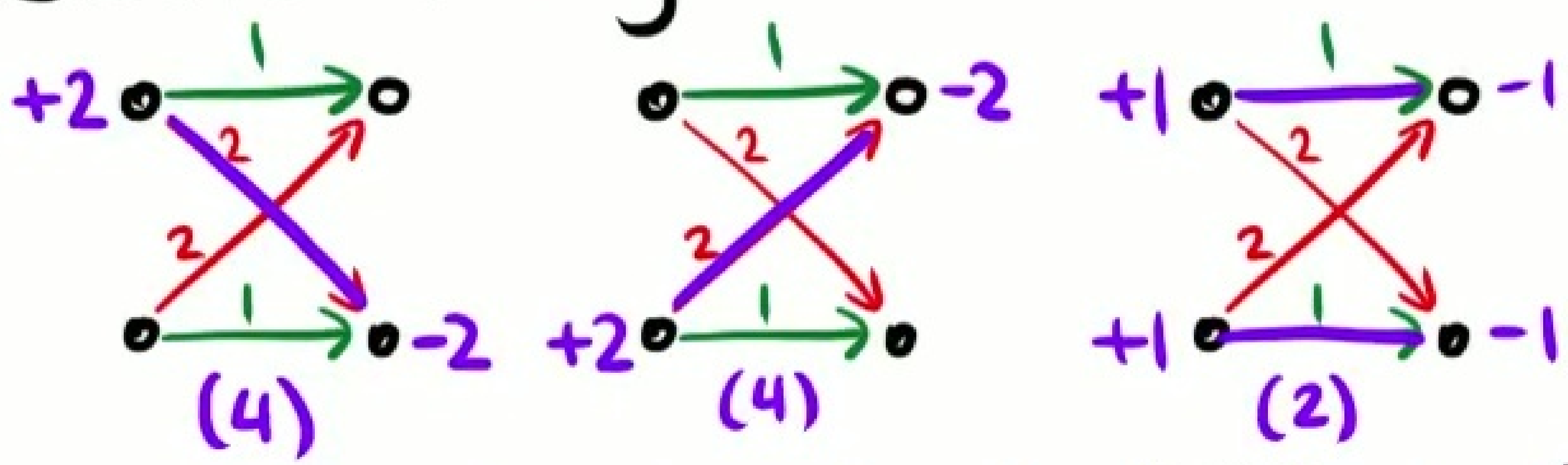

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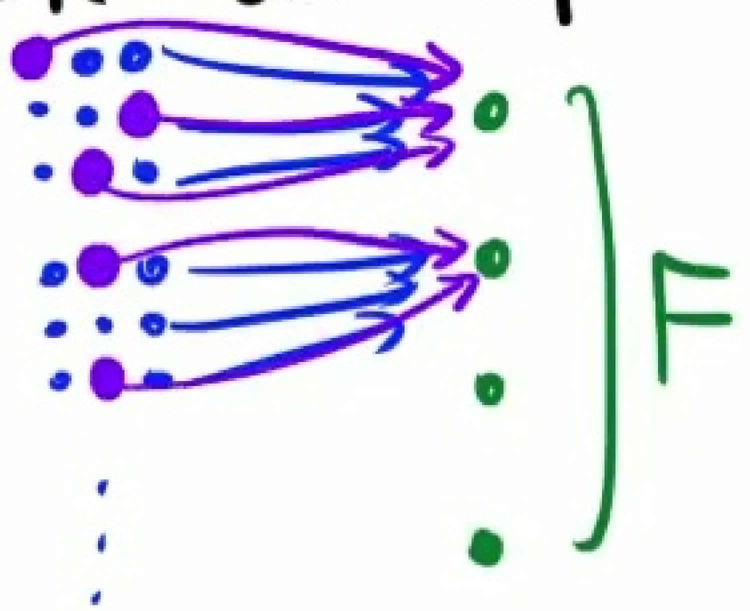

Route sampled clients to same facility in F

$(\pm\varepsilon|C|R)$

for most d (concentration)

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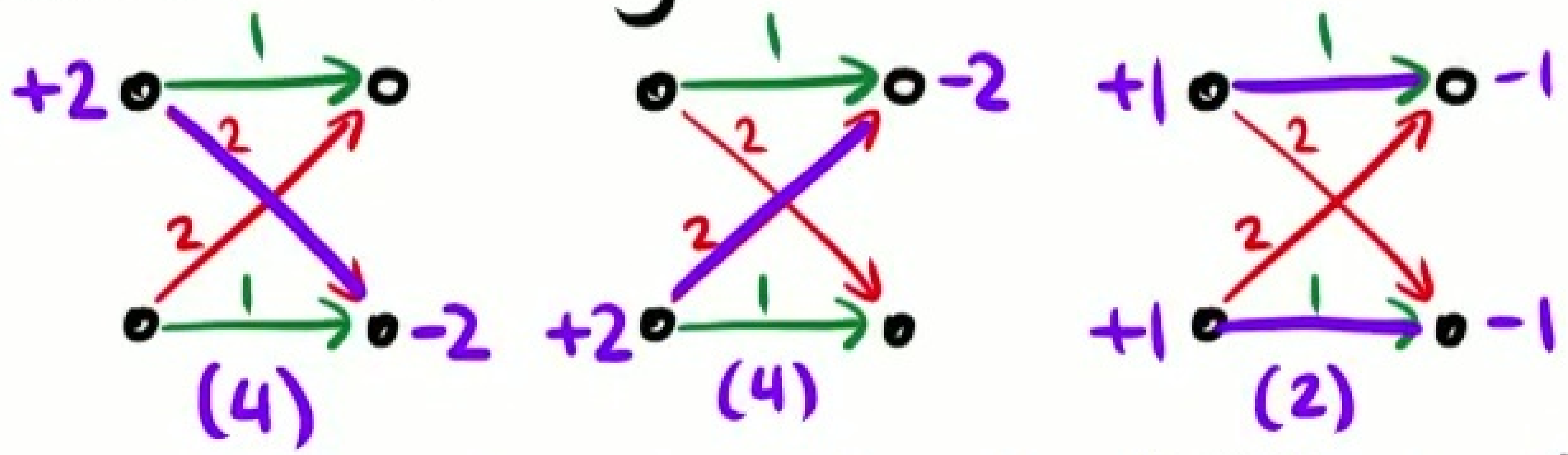

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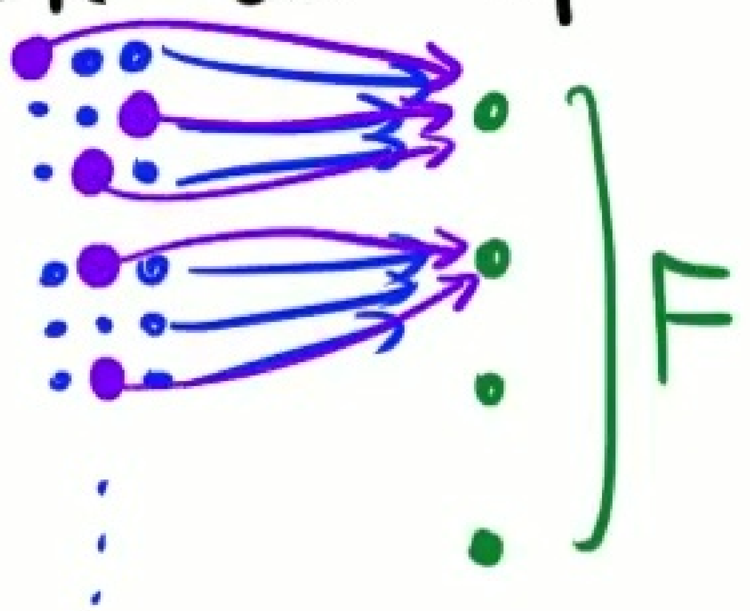

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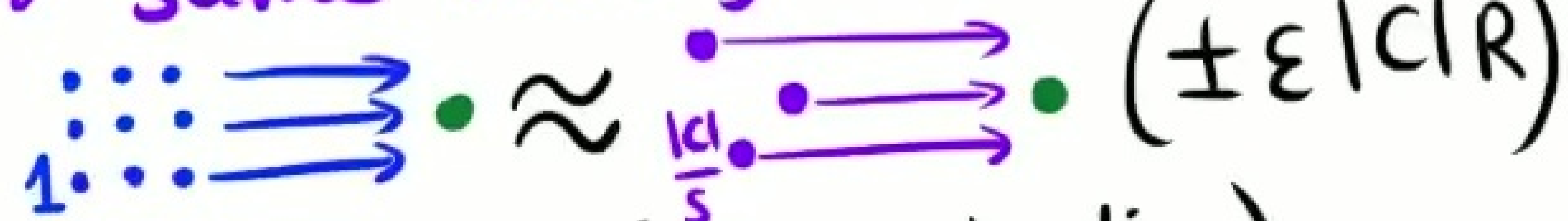

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- For most d, sampled $\approx$ size(⁘)$\cdot\frac{s}{|C|}$ many $\Rightarrow$ demands almost preserved

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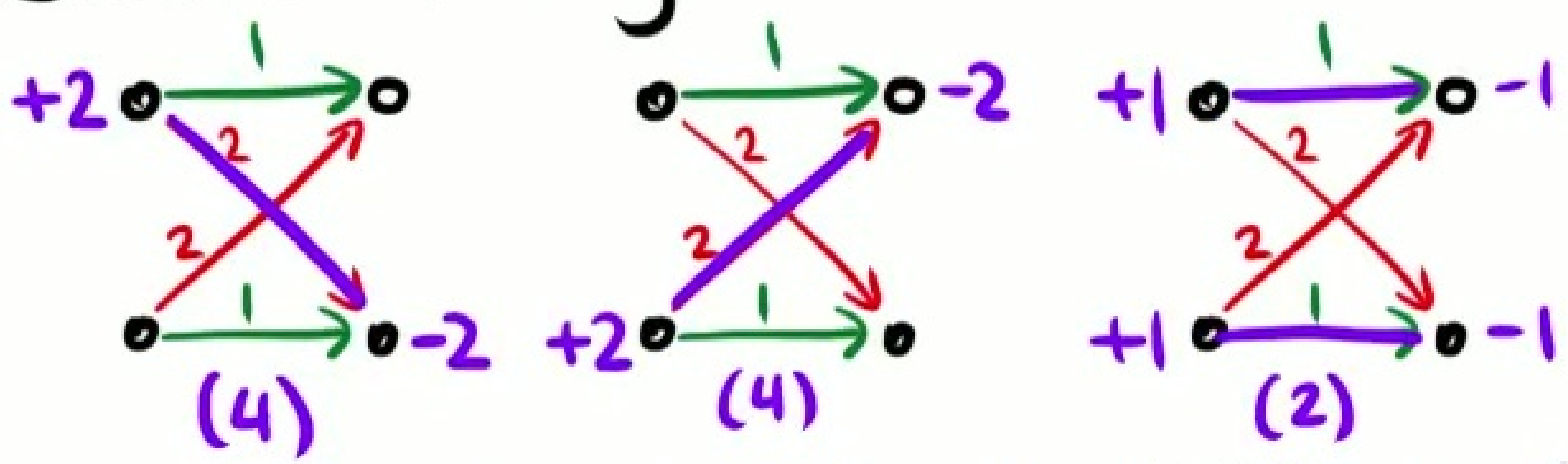
- Convexity of MinCostFlow:



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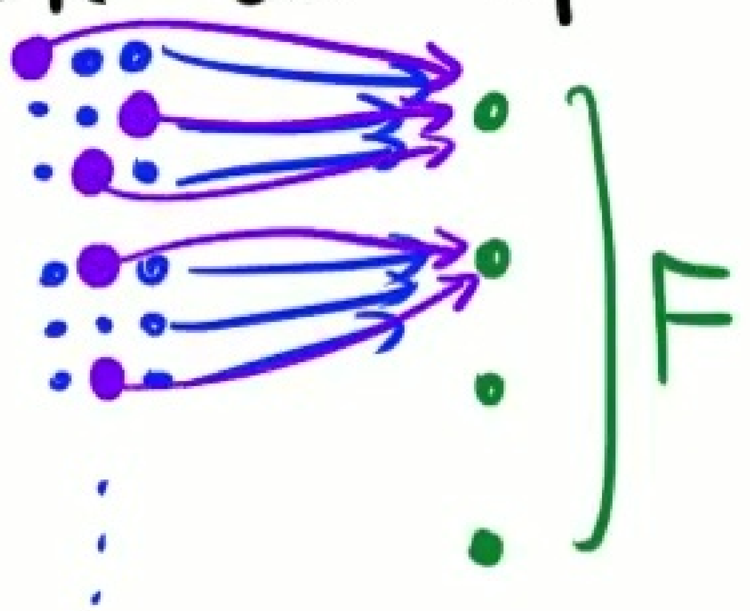

Route sampled clients to same facility in F

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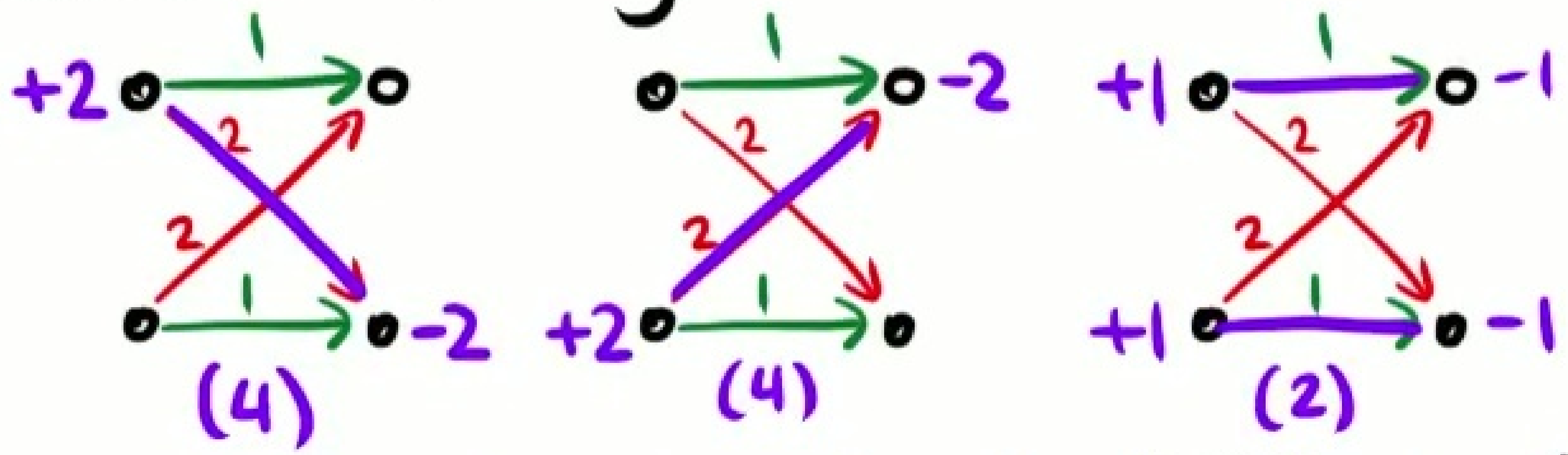

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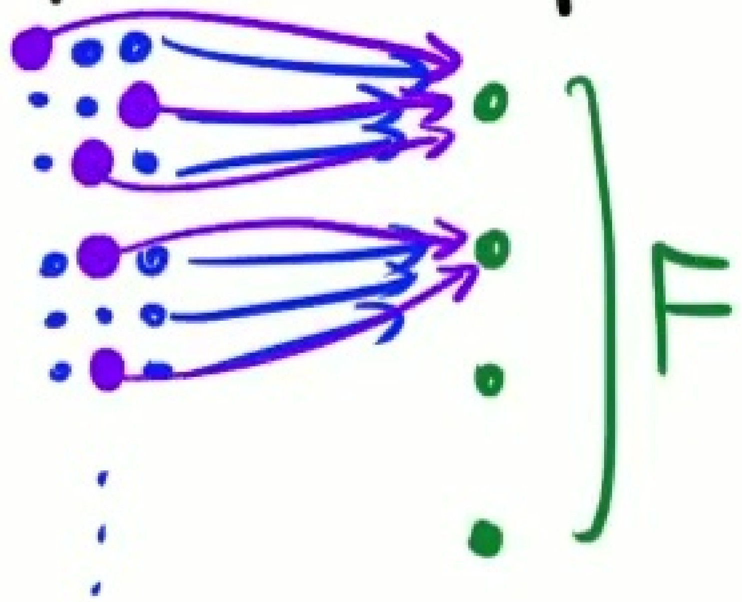

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- Total error $\leq O(\varepsilon|C|R) \cdot k$ over all •

Reset $\varepsilon \leftarrow \Theta(\varepsilon/k)$

Open problems

- Improve (3+eps)-approx in FPT?
  (Lower bound 1+2/e even for
  k-median in FPT)
- Hardness > (1+2/e), even for polytime?
- More problems where FPT improves
  approximation?