Planar Diameter via Metric Compression

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CMU

Joint work with Merav Parter
Weizmann

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Tool: abstract Voronoi diagrams [Cabello 17]

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this talk

- "Simpler" algorithms: amenable to distributed comp + "standard" CONGEST tools
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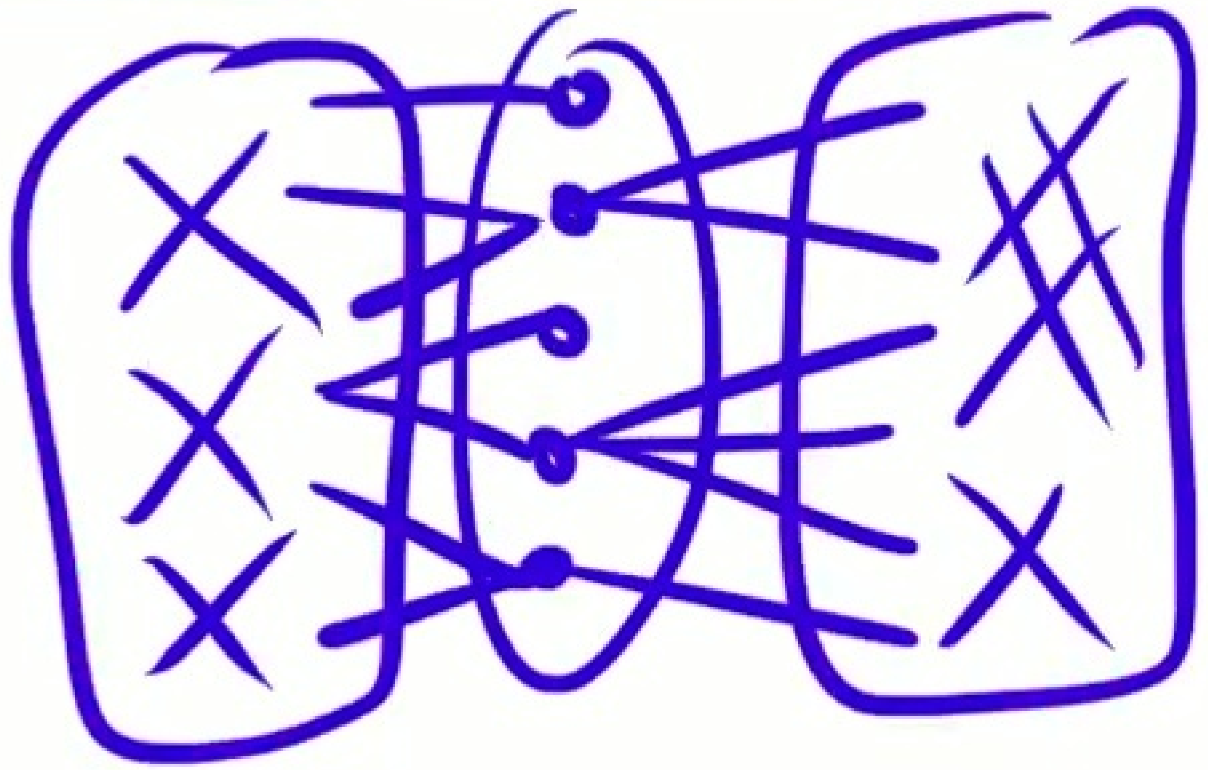
- $\tilde{\Omega}(n)$ lower bound for general graphs $(D = O(1))$ [ACK16]

Divide + Conquer

Balanced separator:



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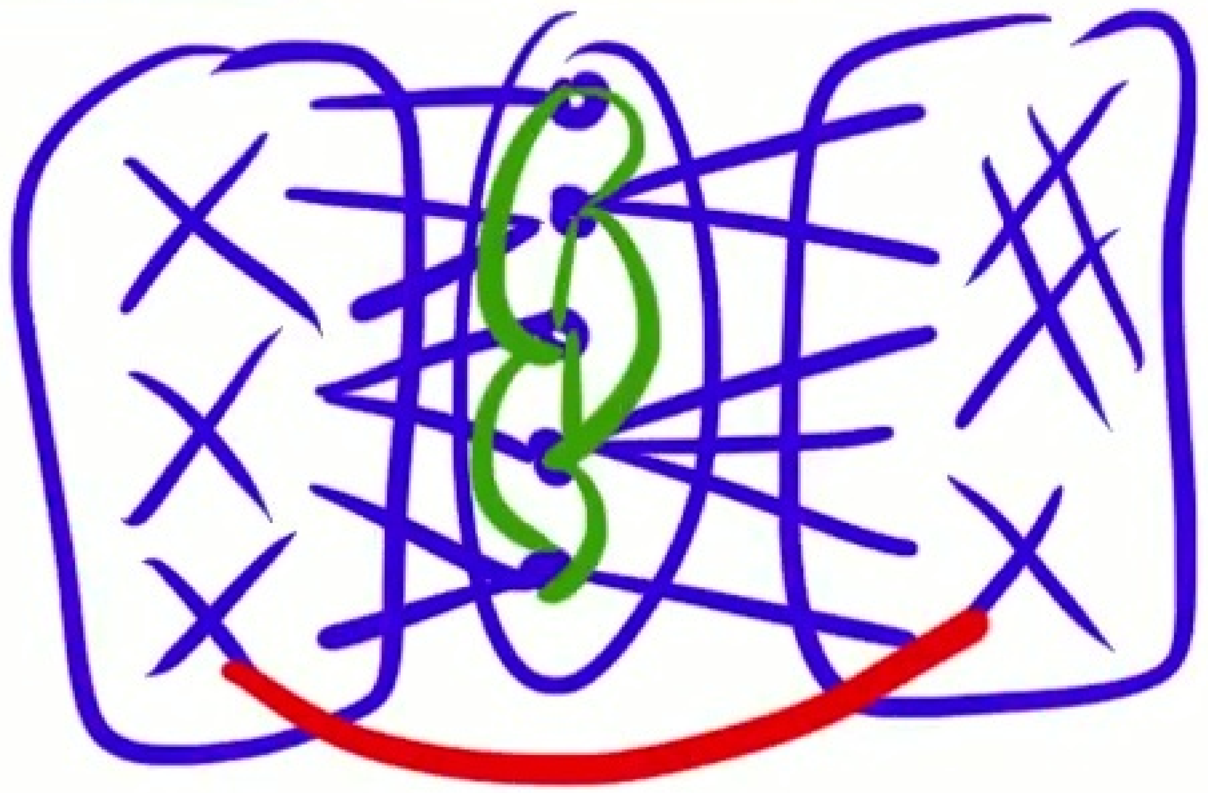

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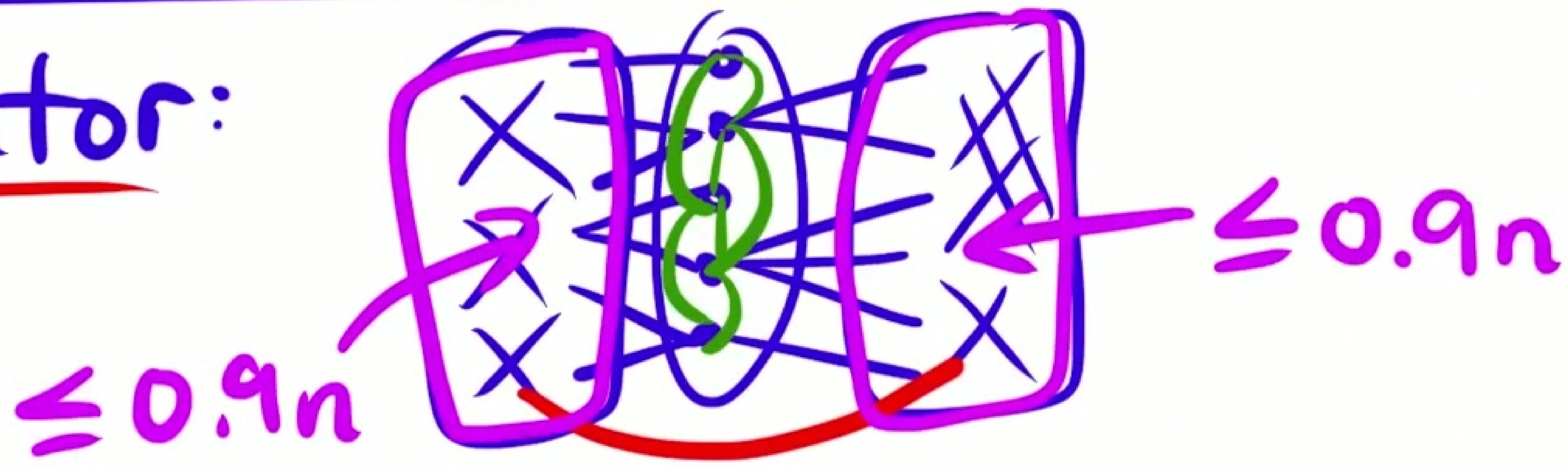

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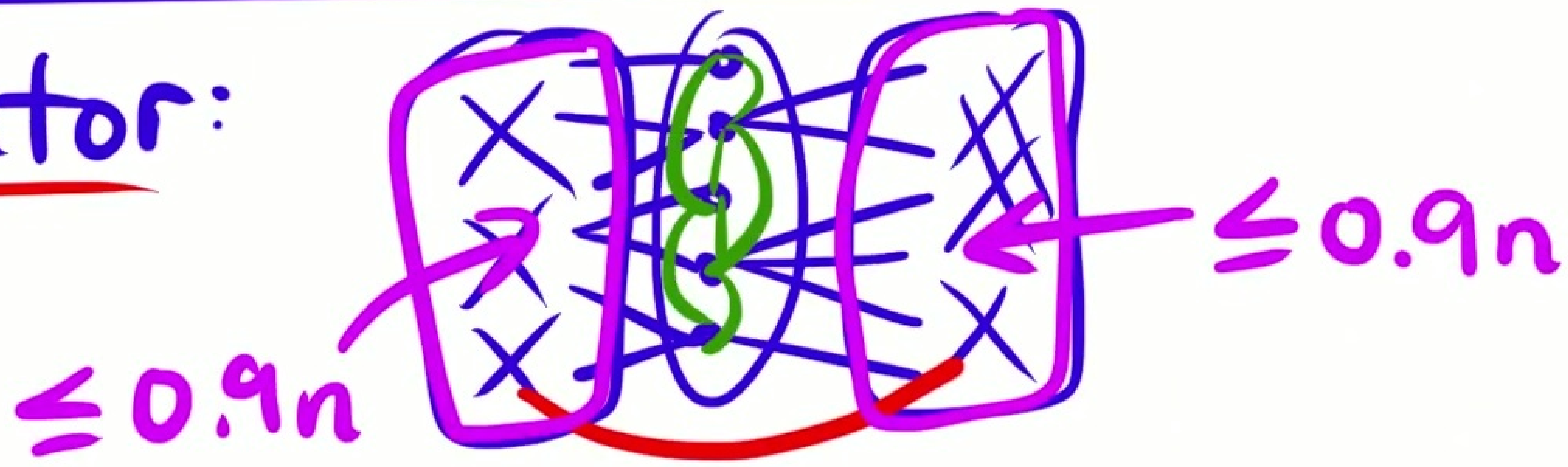
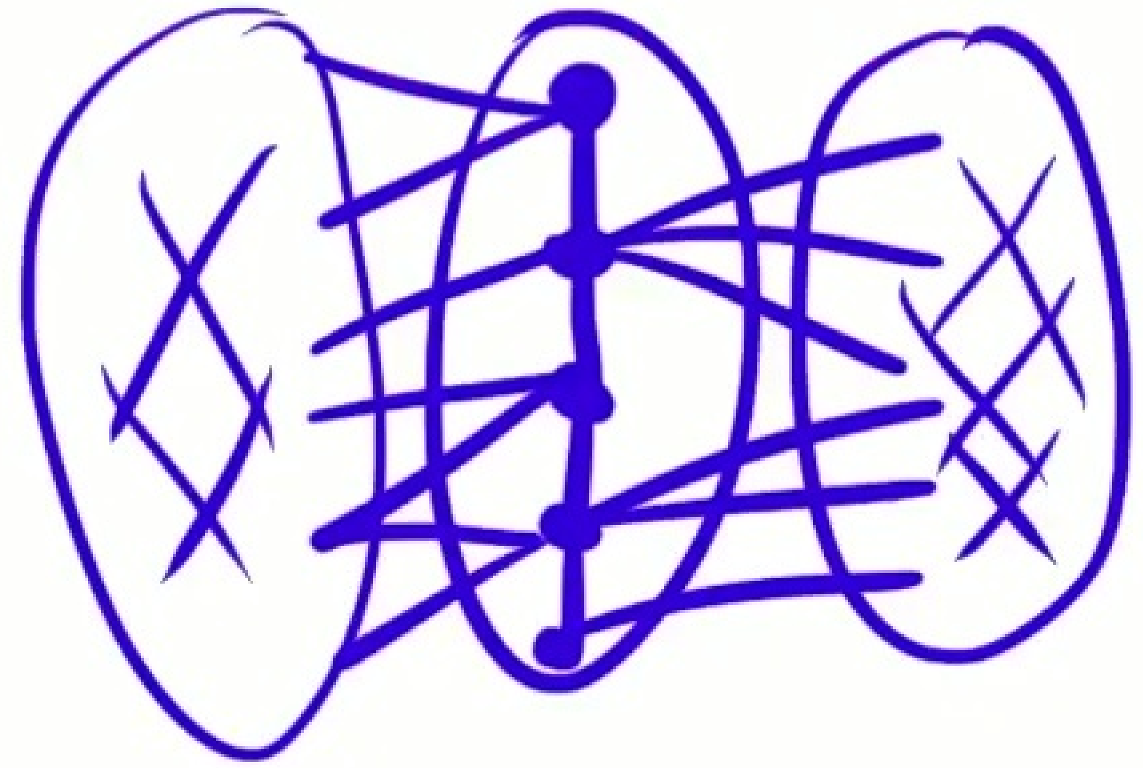


Balanced path separator:



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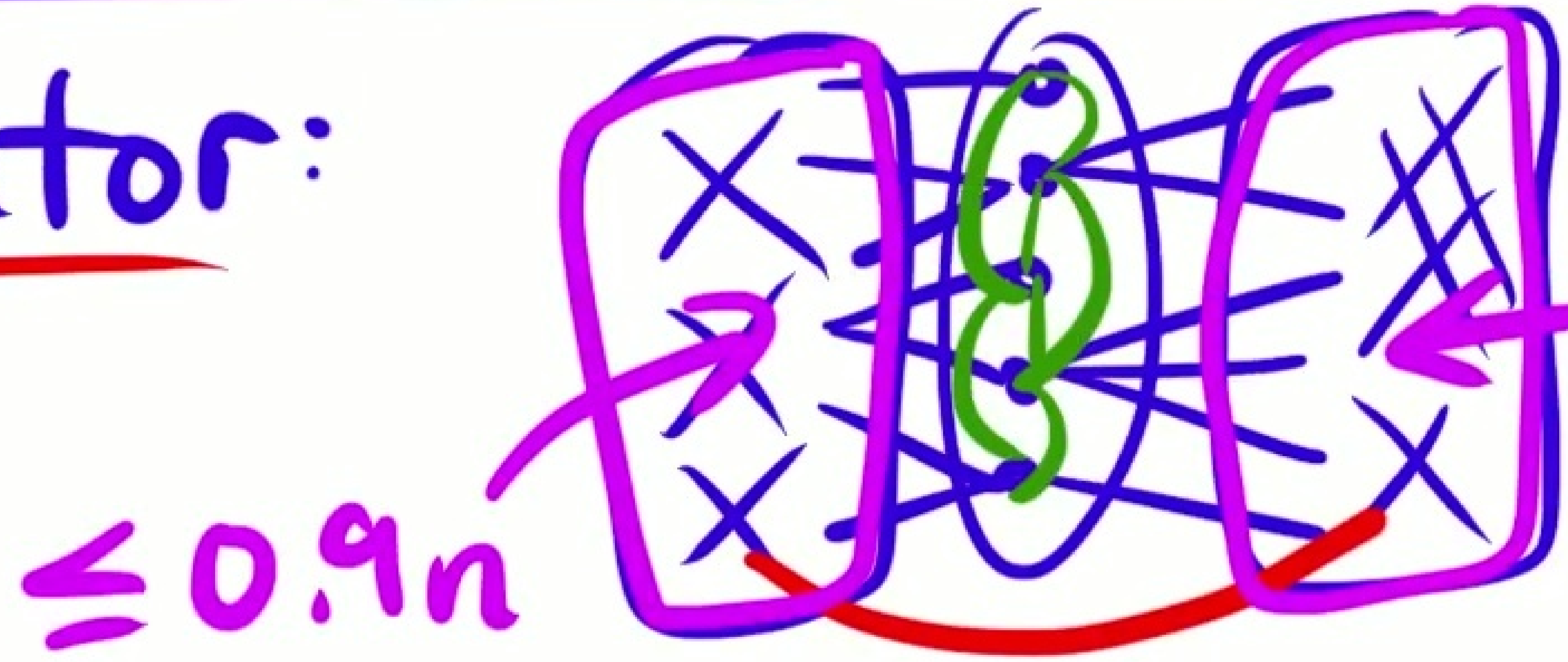

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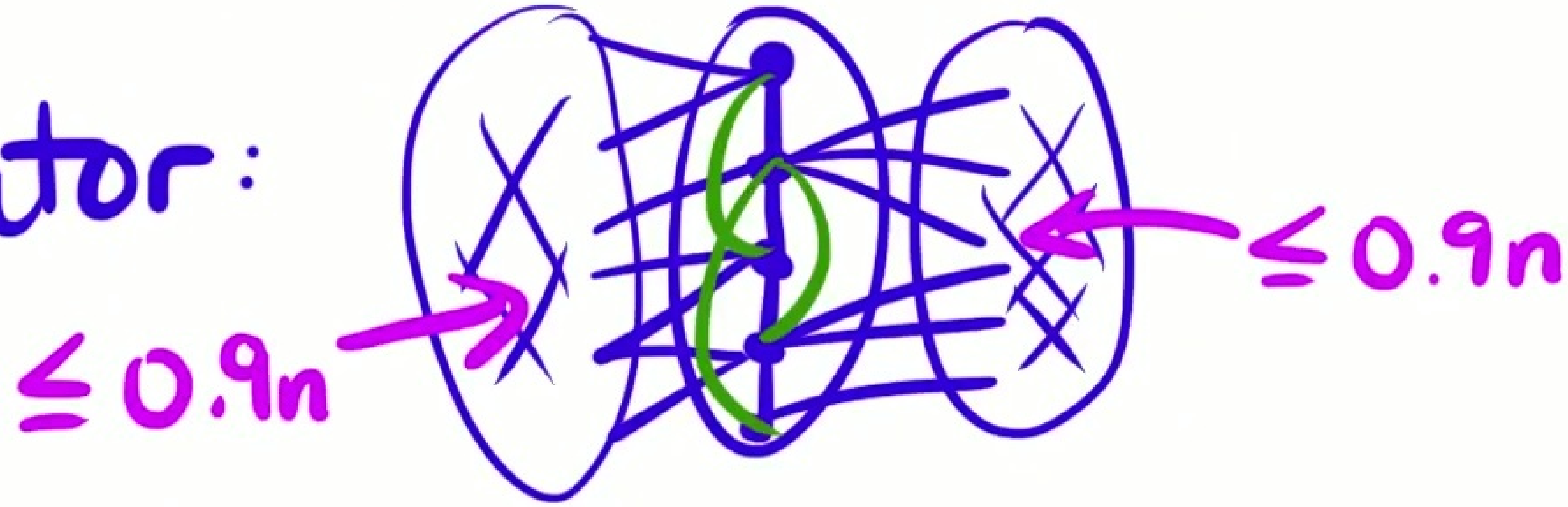

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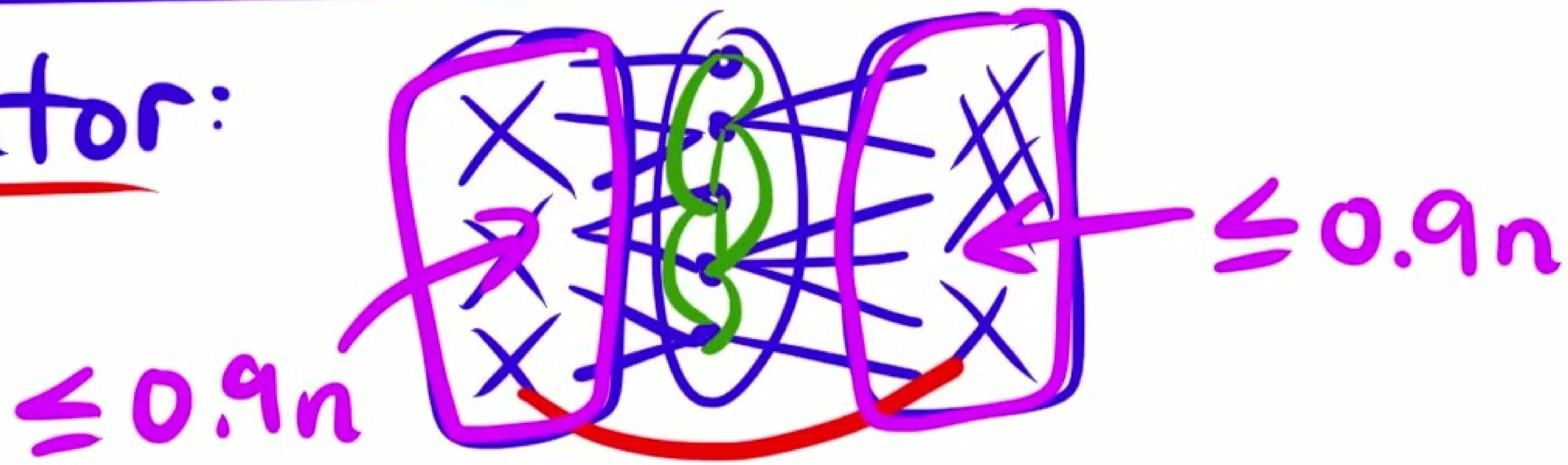

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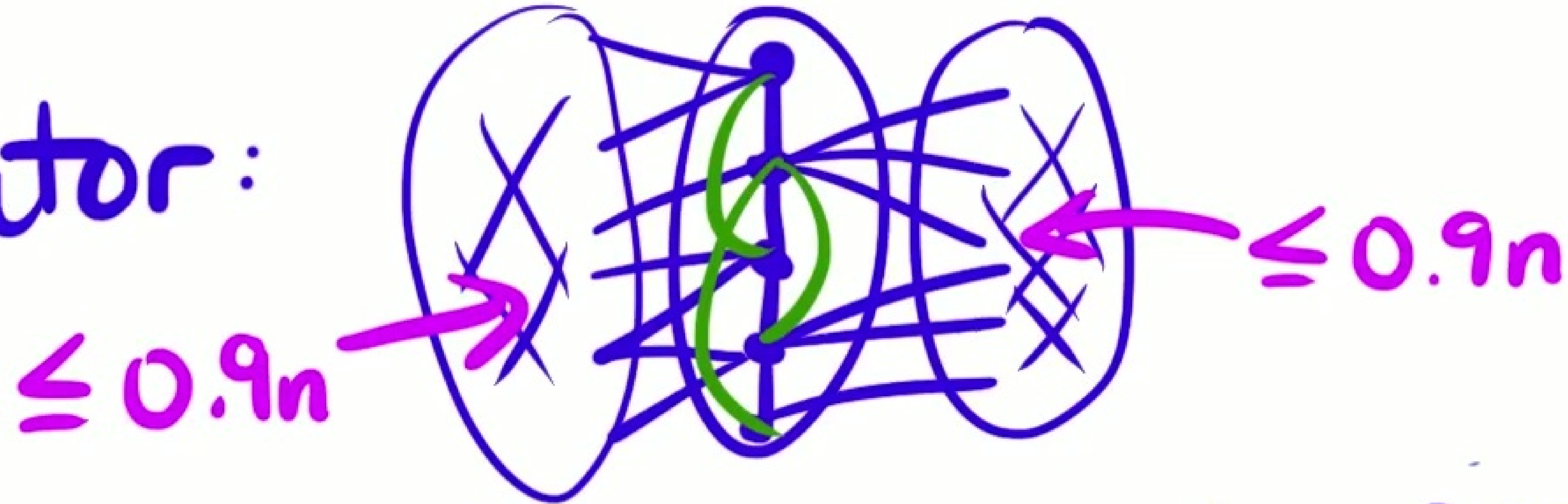
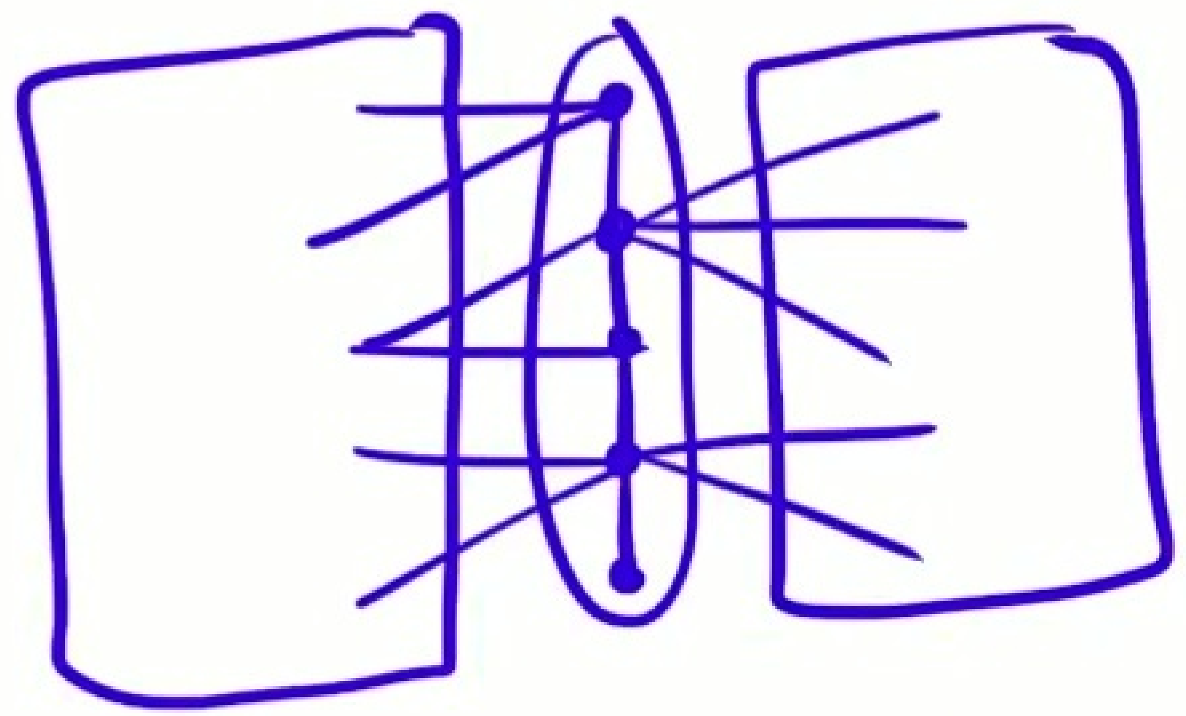


Thm: Every planar graph diameter-D has len-$O(D)$ balanced path separator.

Divide + Conquer

Diameter computation?



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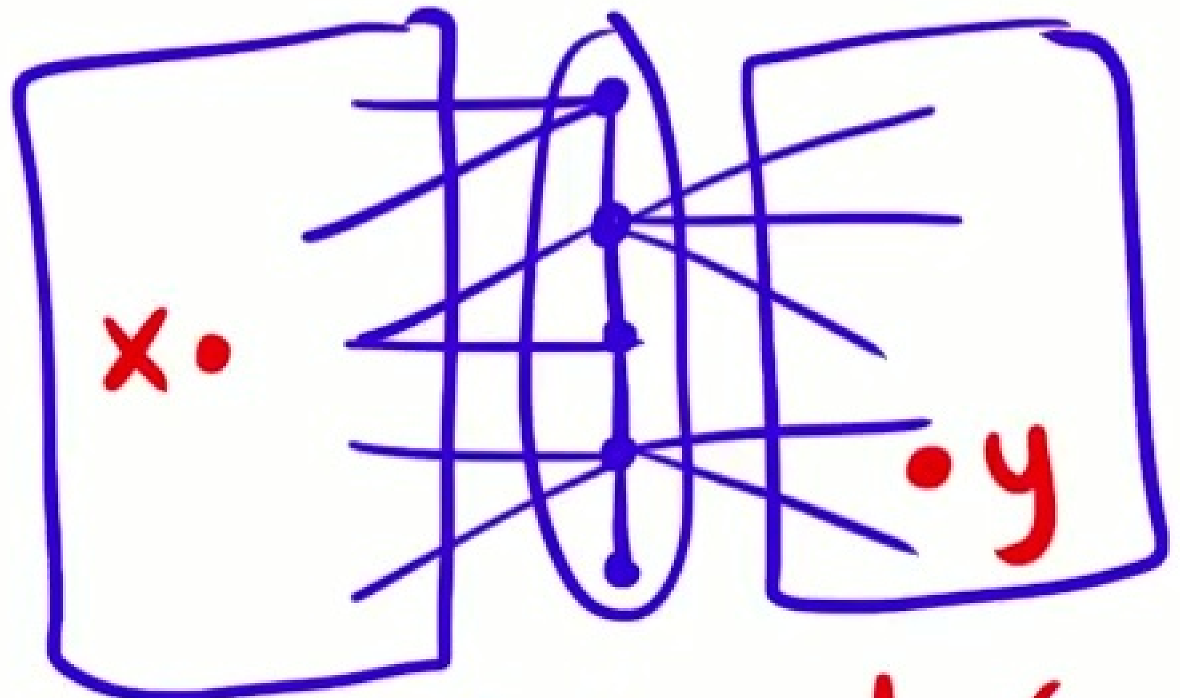

$\mathrm{diam}(G) = d_G(x,y)$

Divide + Conquer

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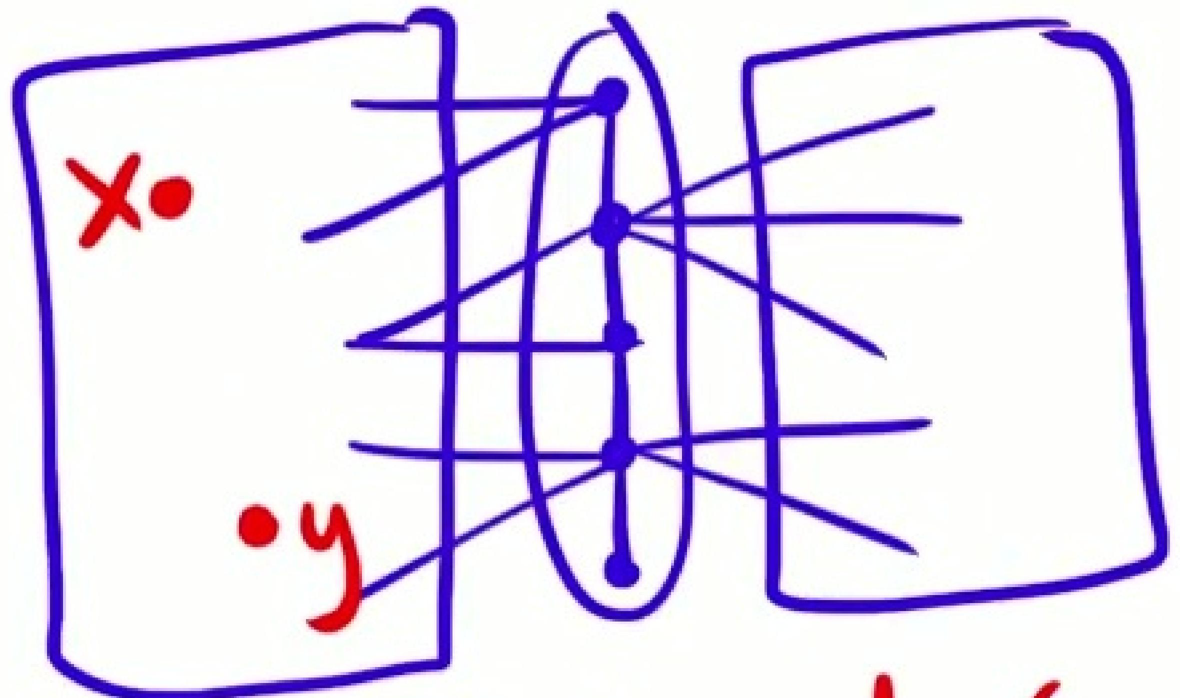

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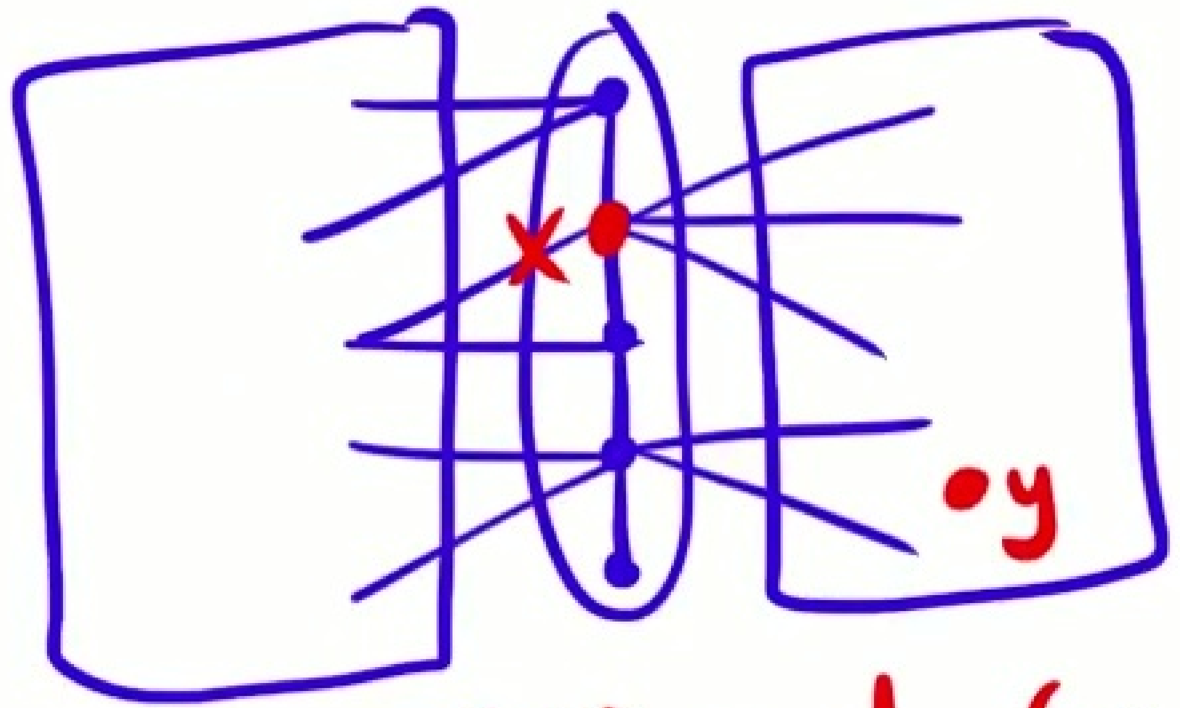

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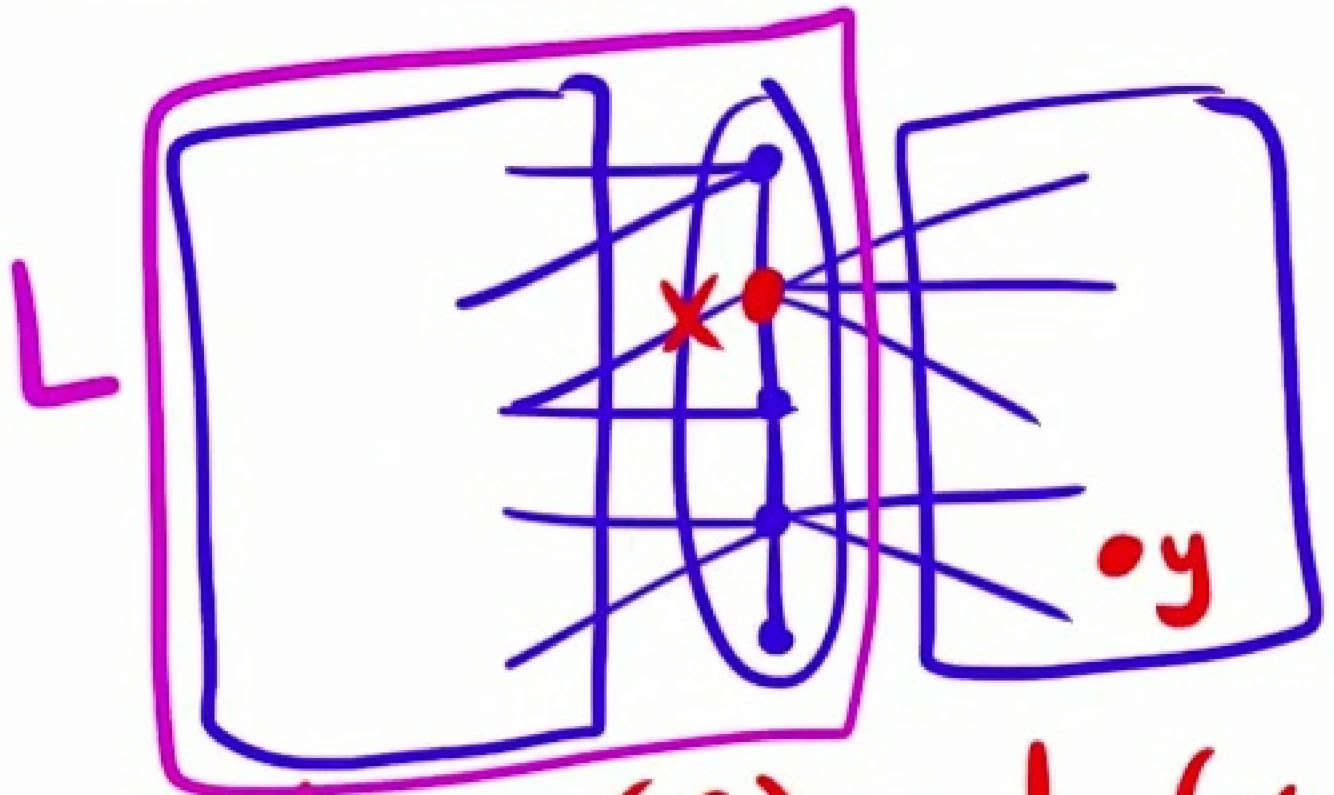

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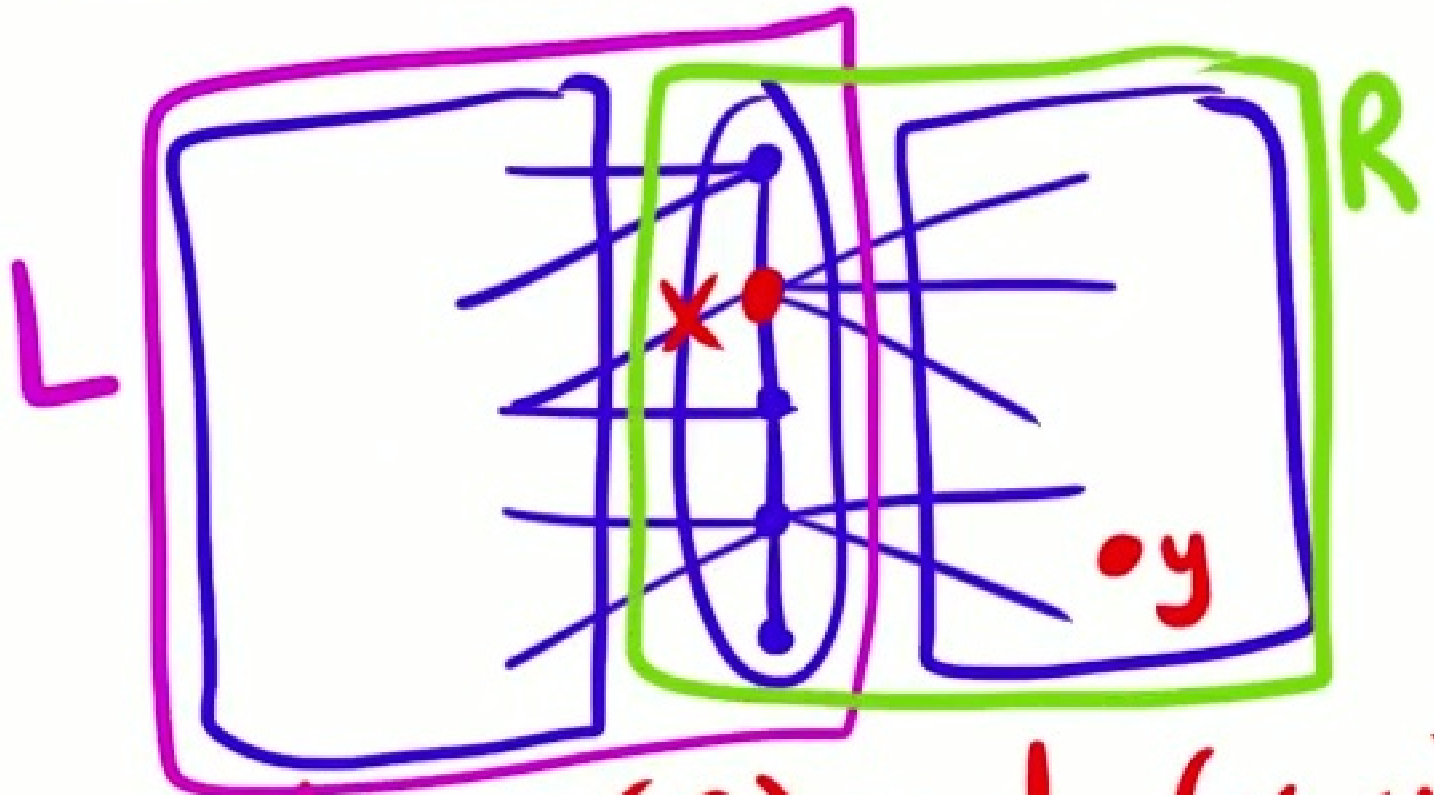

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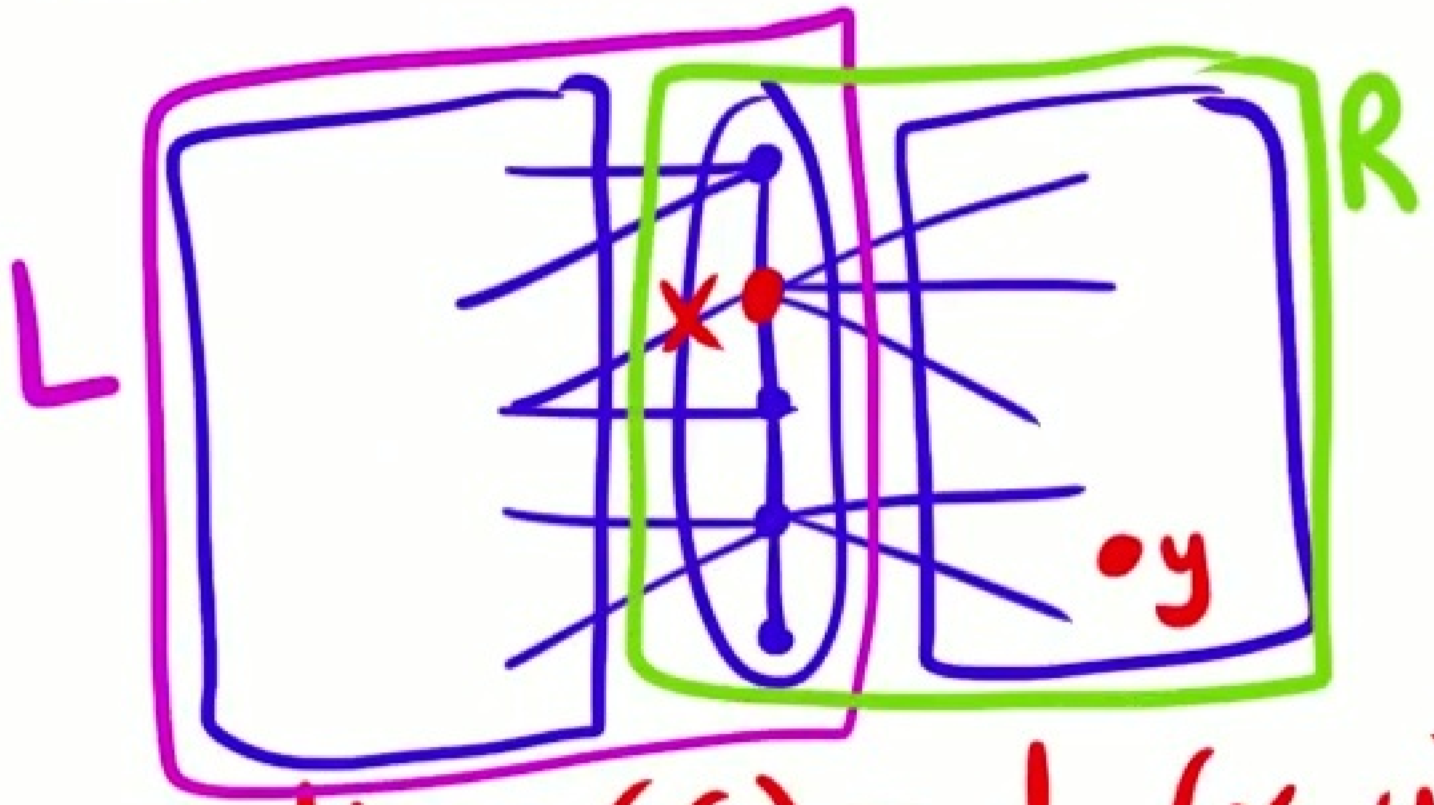
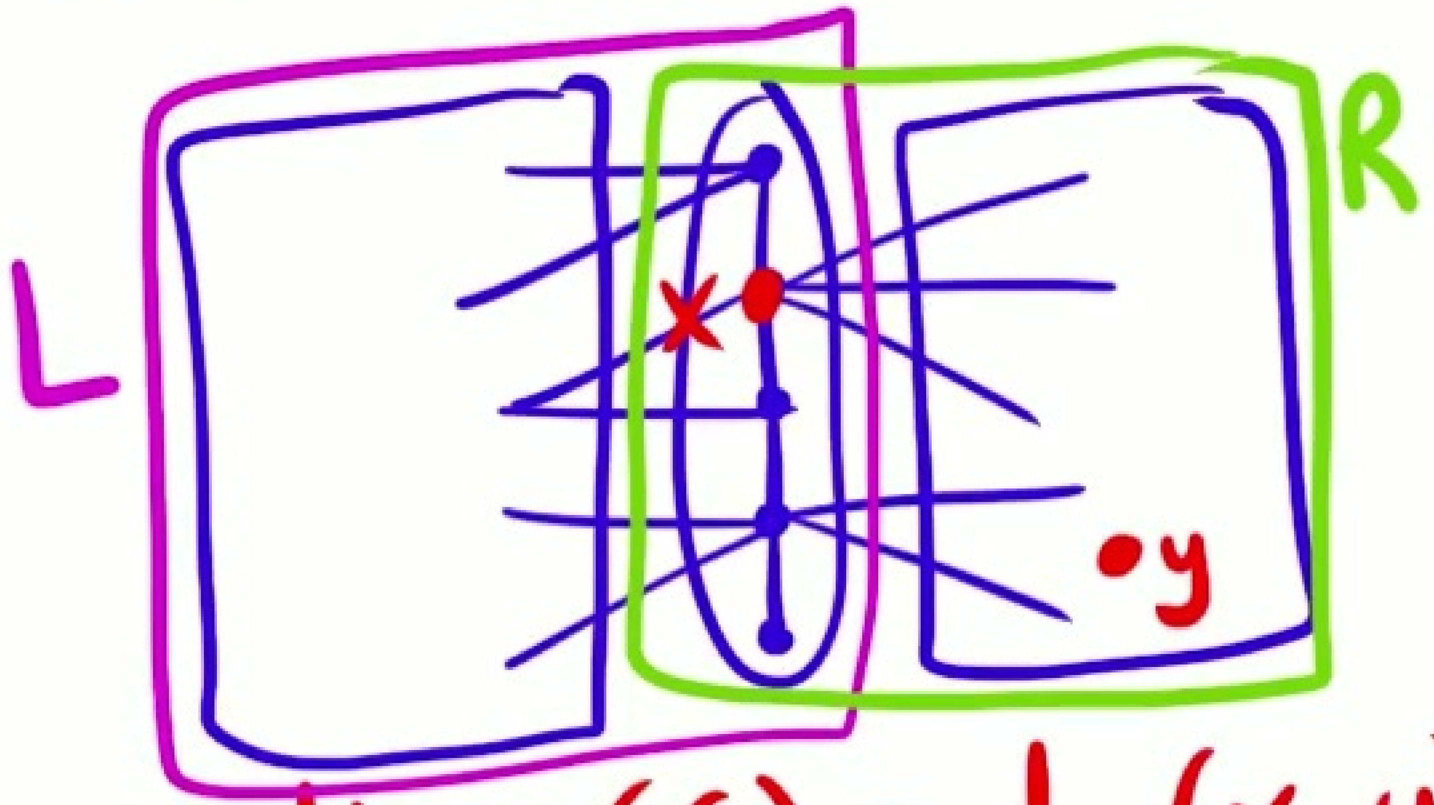


$$\mathrm{diam}(G) = d_G(x,y)$$

$$\max_{x,y} d_G(x,y) = \max \left\{ \max_{x,y \in L} d_G(x,y),\ \max_{x,y \in R} d_G(x,y),\ \max_{x \in L,\, y \in R} d_G(x,y) \right\}$$

Divide + Conquer

Diameter computation?



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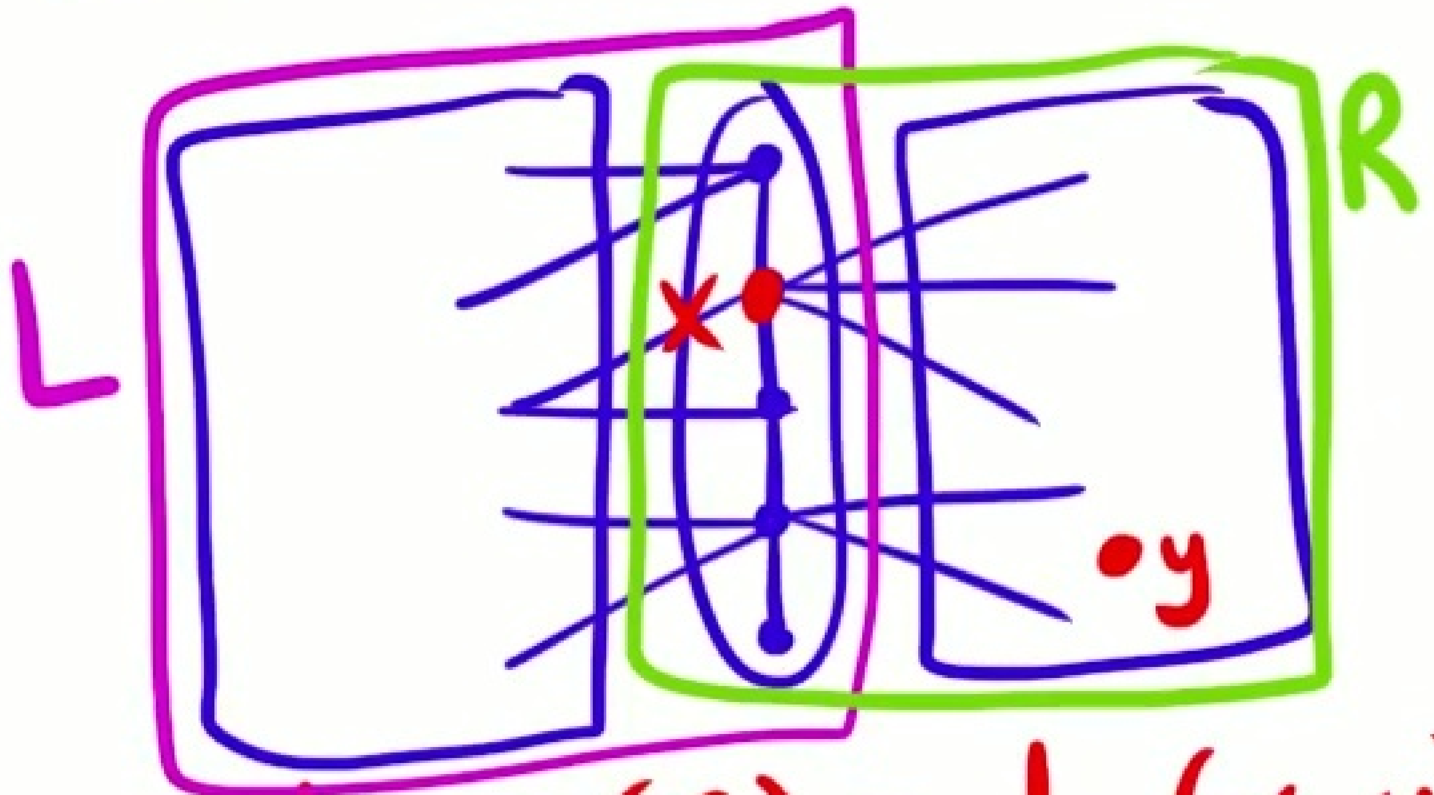
Later: $T(|L| + D^{O(1)})$

$T(|R| + D^{O(1)})$

Divide + Conquer

Diameter computation?



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$$T(n) = T(|L|+D^{O(1)}) + T(|R|+D^{O(1)}) + T_{\text{conquer}}(n)$$

Divide + Conquer

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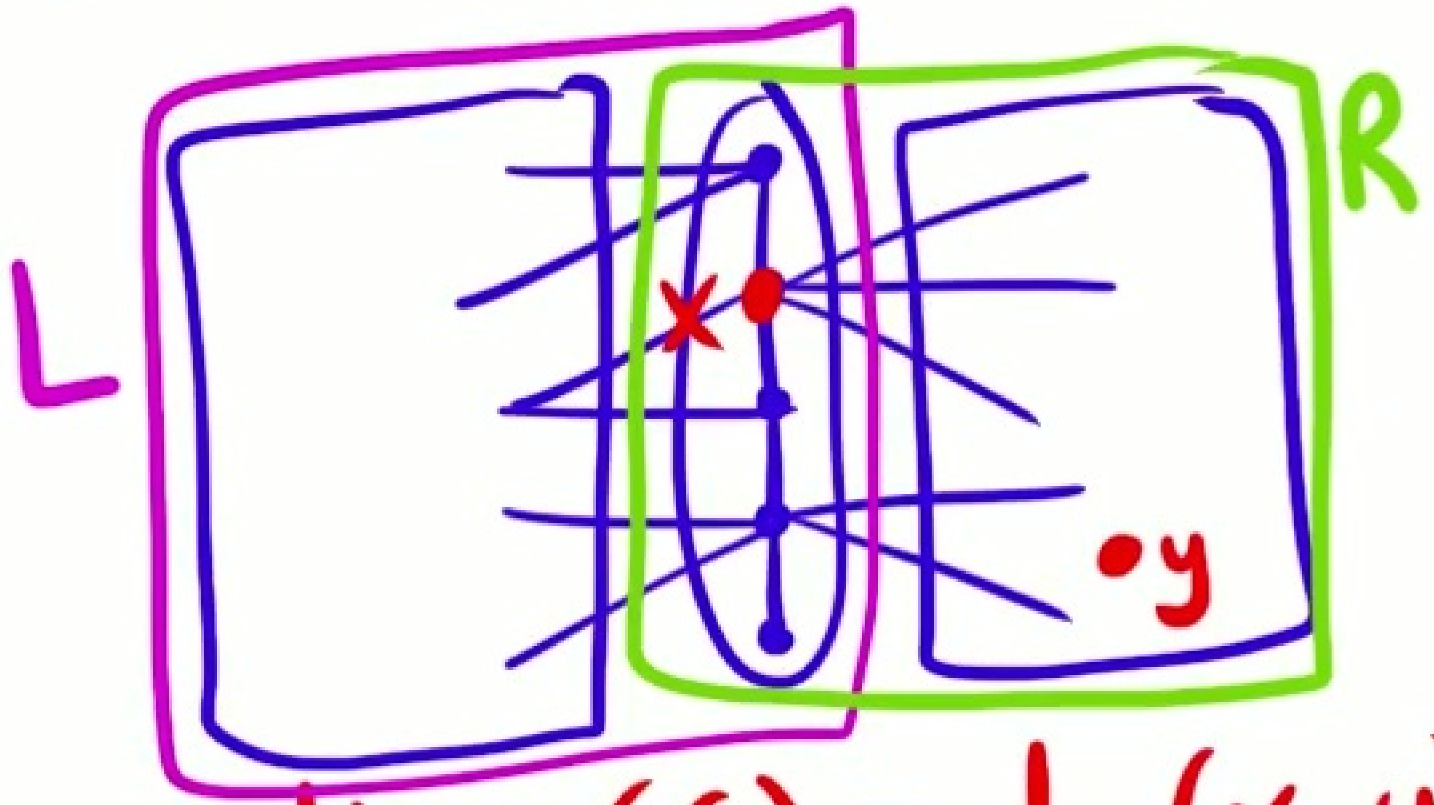

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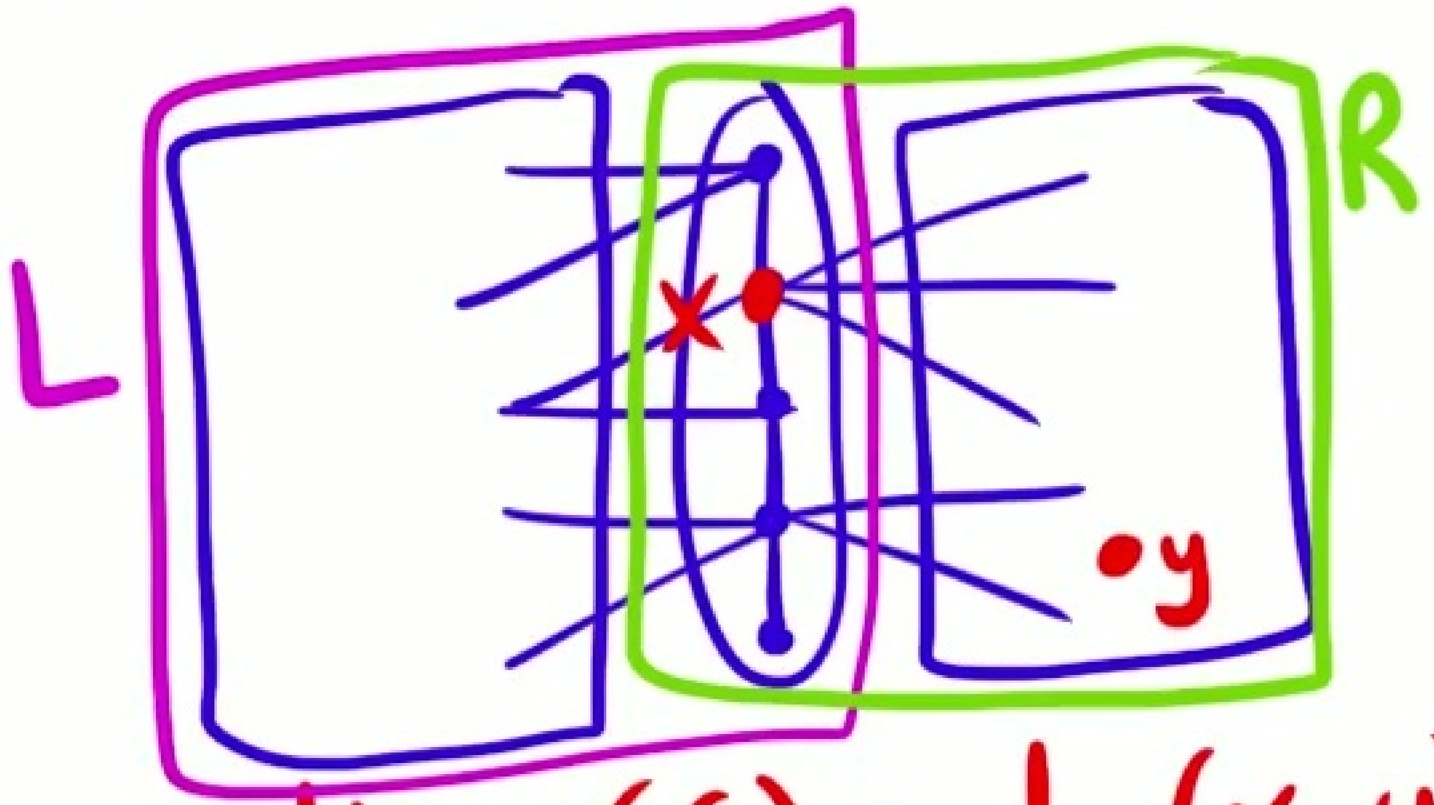
$$|L|, |R| \leq 0.9n + O(D)$$

$$|L| + |R| \leq n + O(D)$$

Divide + Conquer

Diameter computation?



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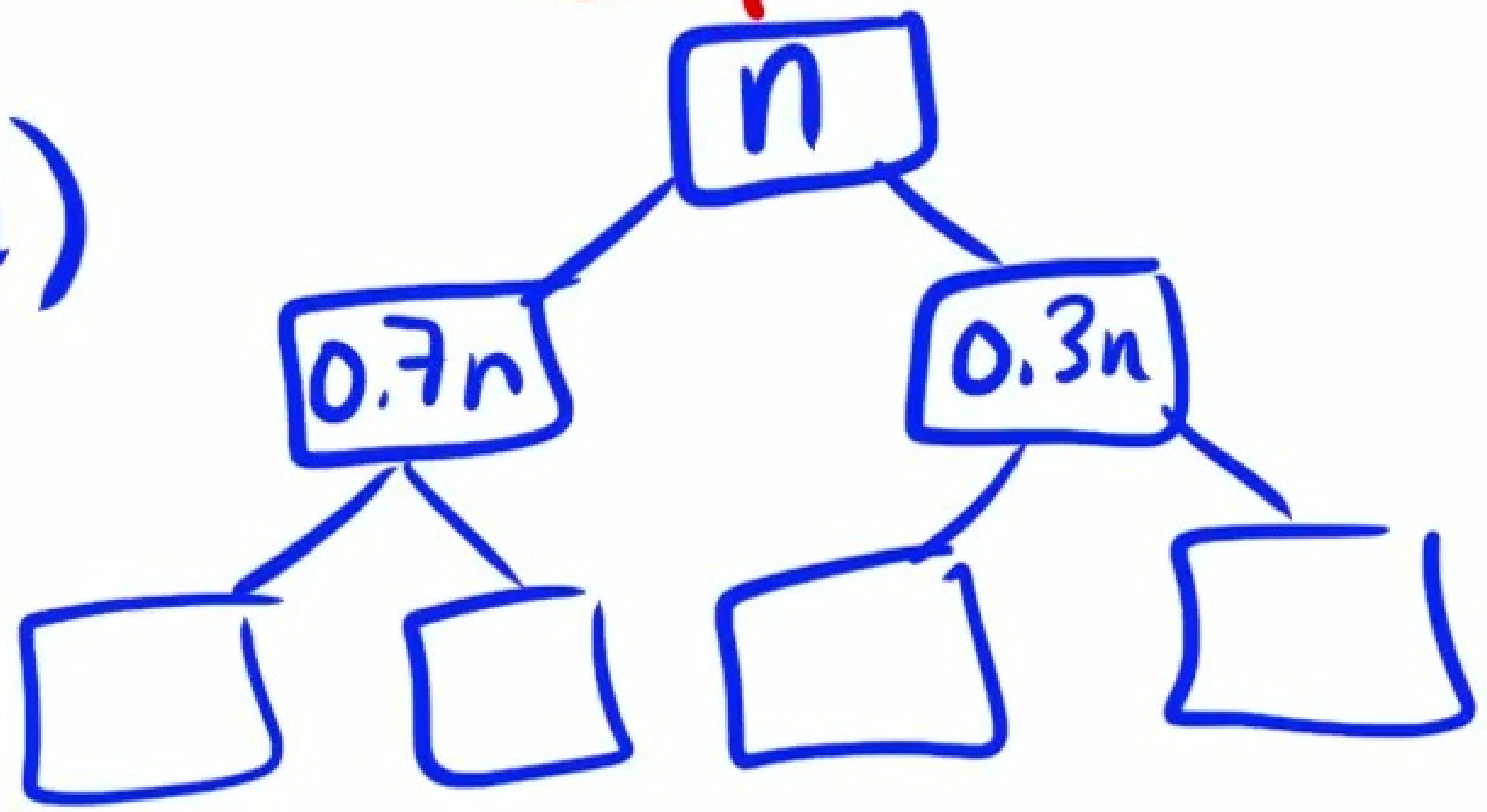

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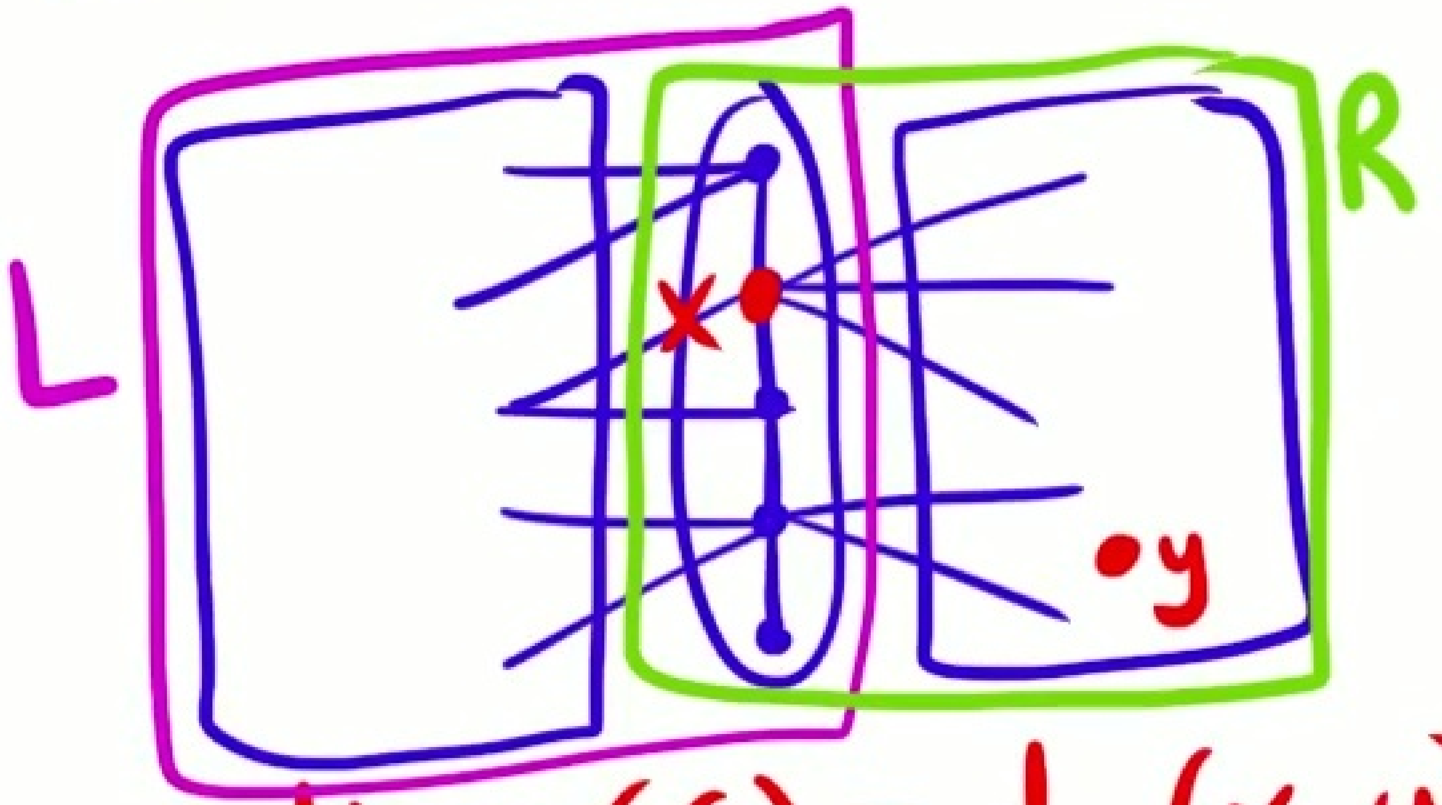

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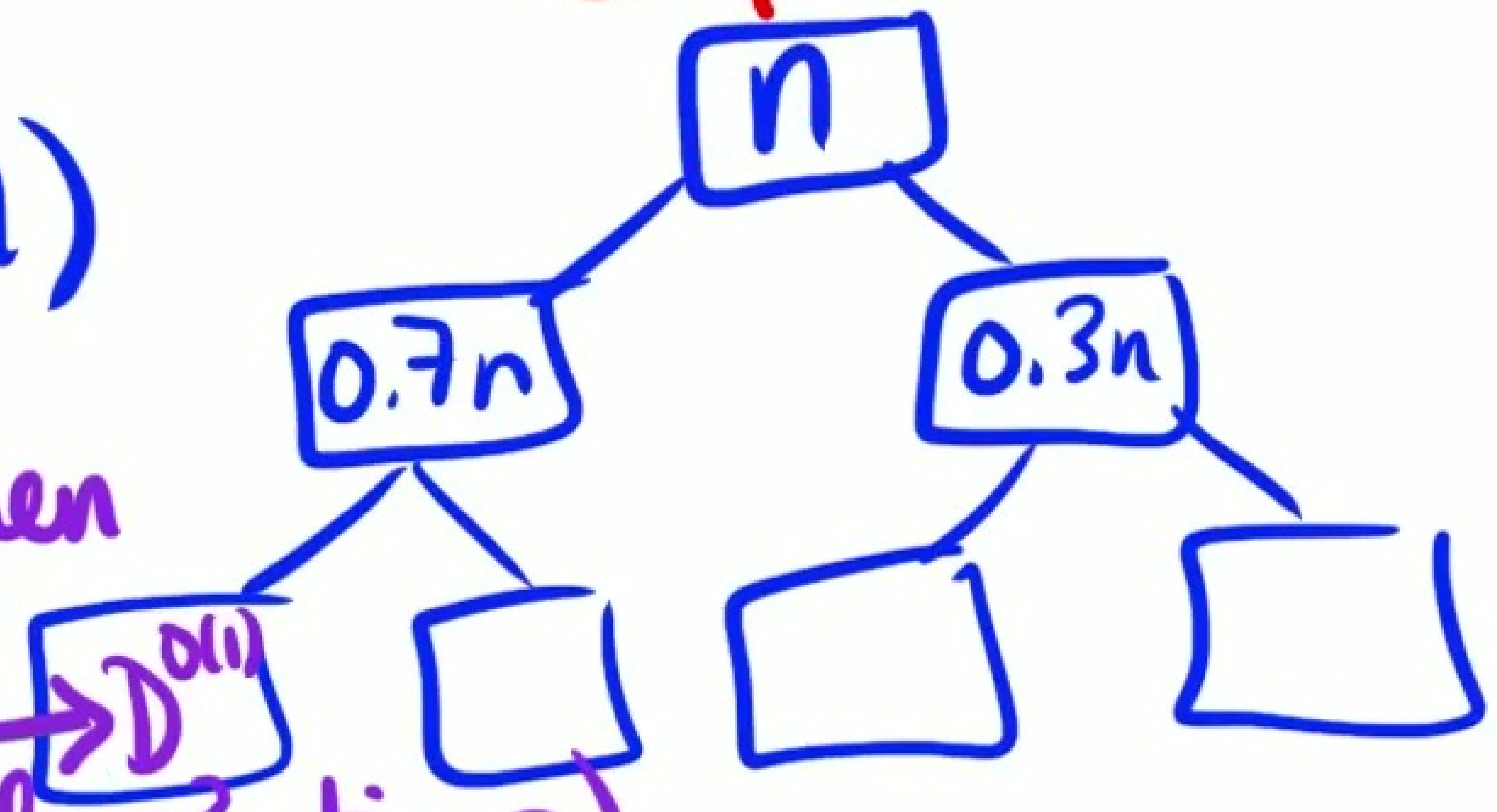
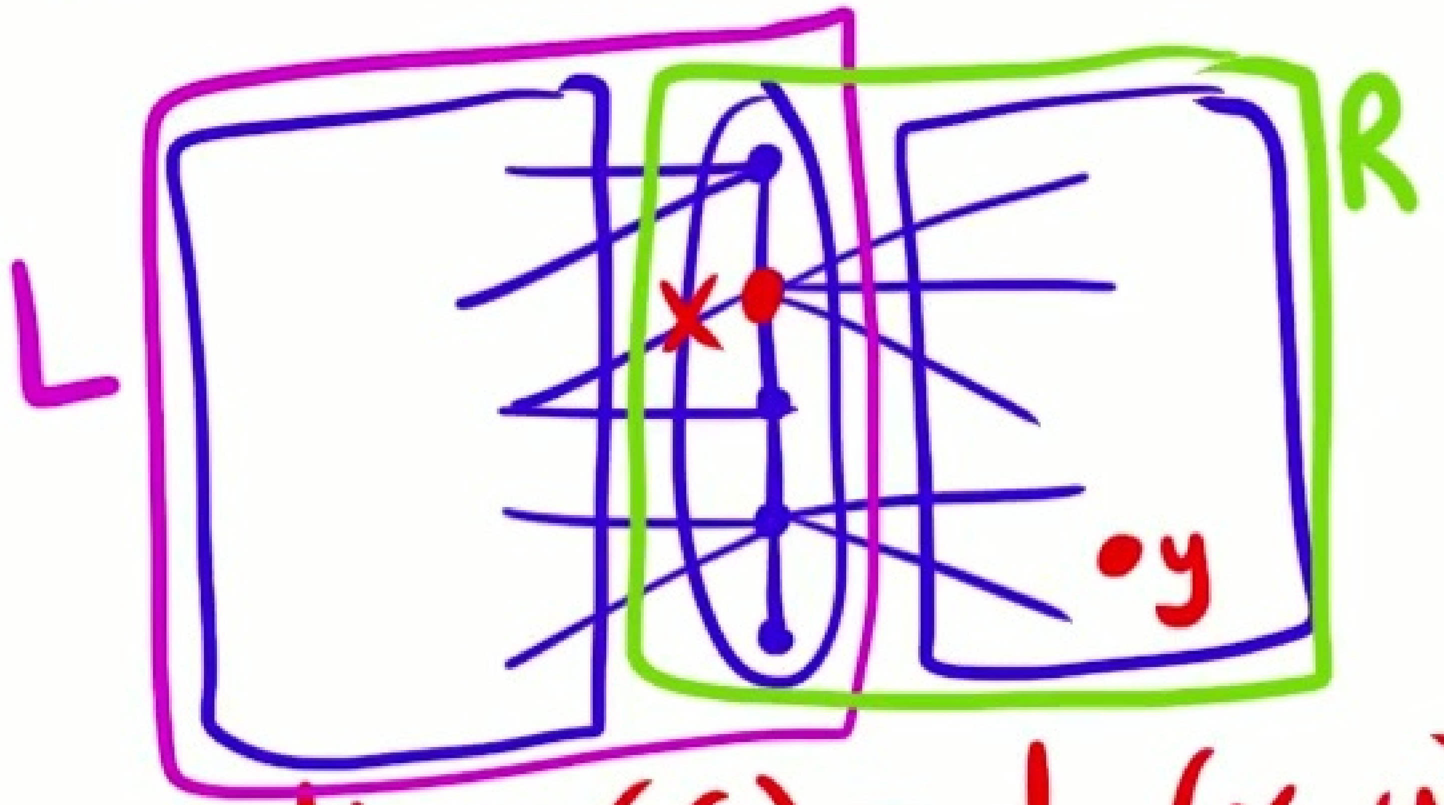


stop when size $D^{O(1)}$ (run trivial n^2 time)

Divide + Conquer

Diameter computation?



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Later: $T(|L|+D^{O(1)})$

$T(|R|+D^{O(1)})$

If $T_{conquer}(n) = nD^{O(1)}$, then $T(n) = nD^{O(1)}$.

Recursion:

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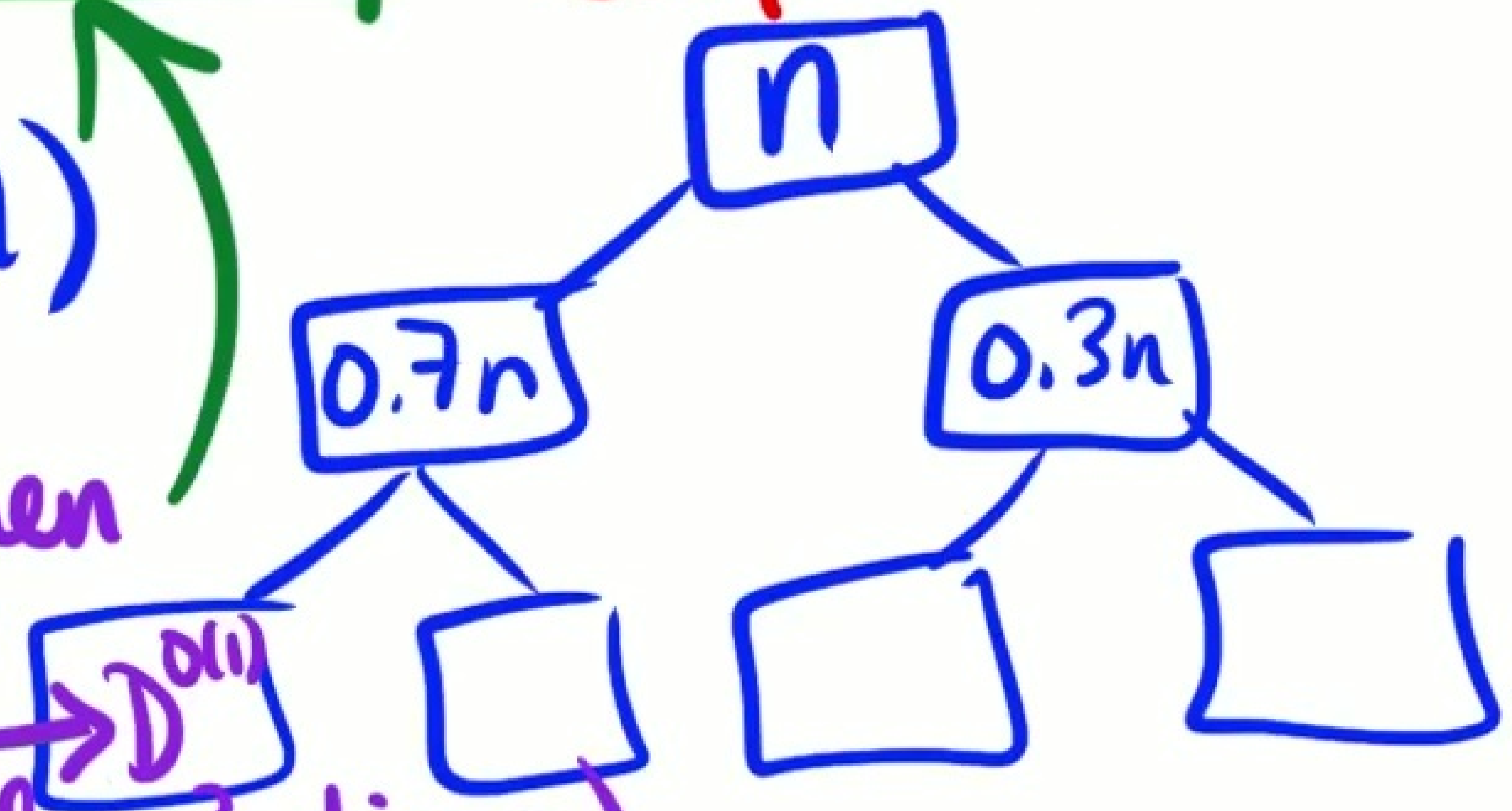

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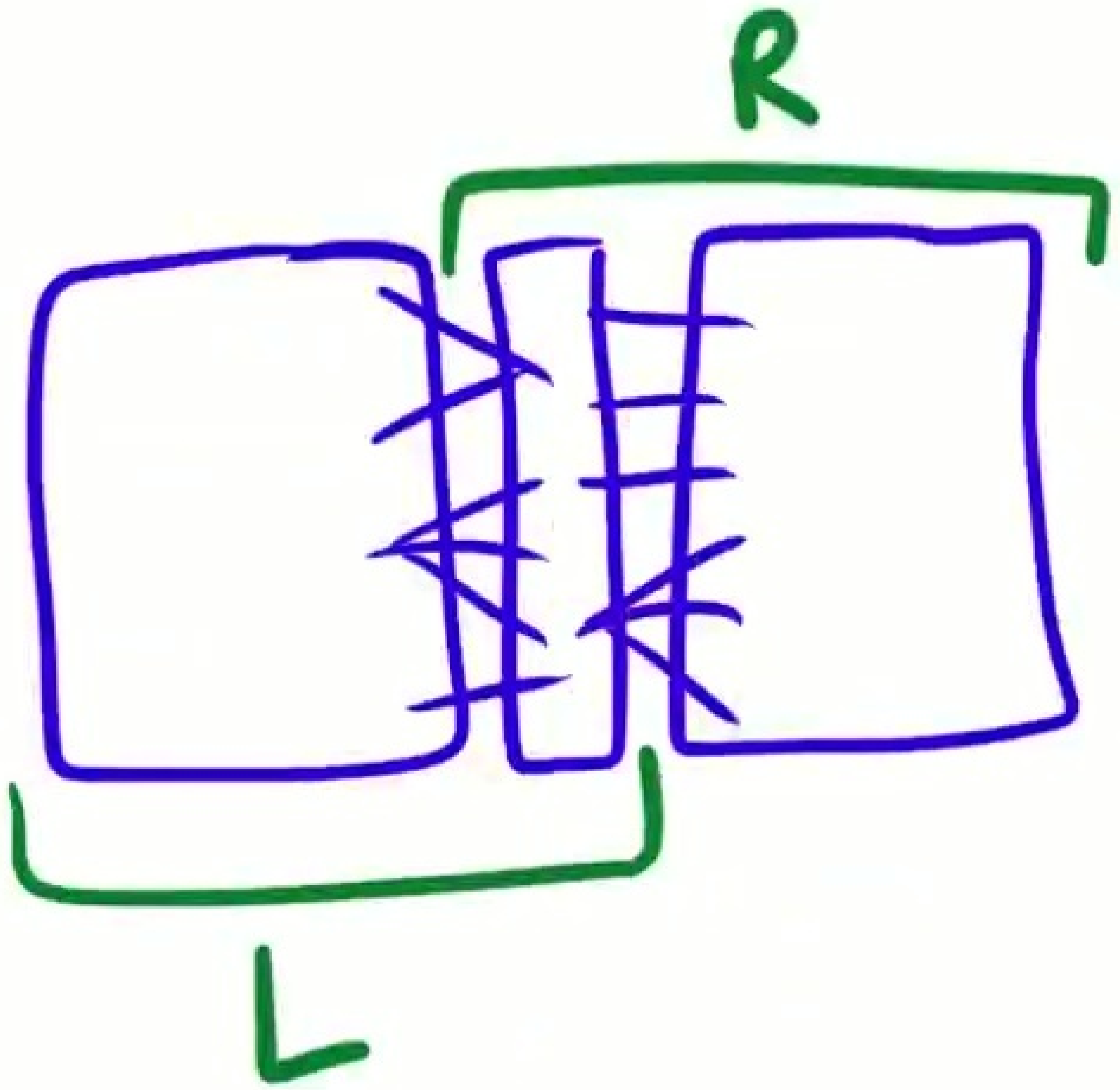
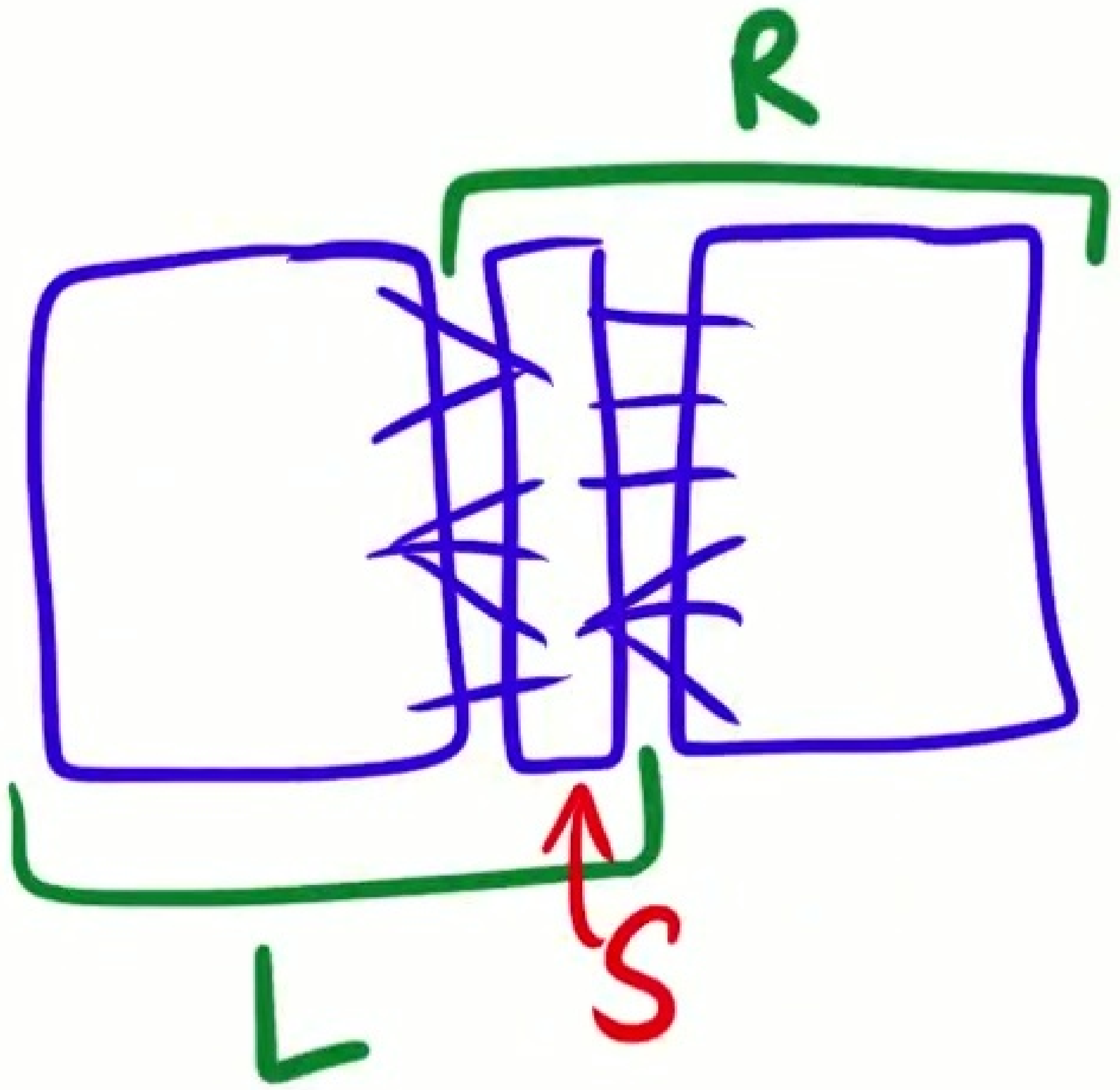
Conquer Step



$$\boxed{\text{compute } \max_{x \in L,\, y \in R} d(x,y)}$$

Conquer Step



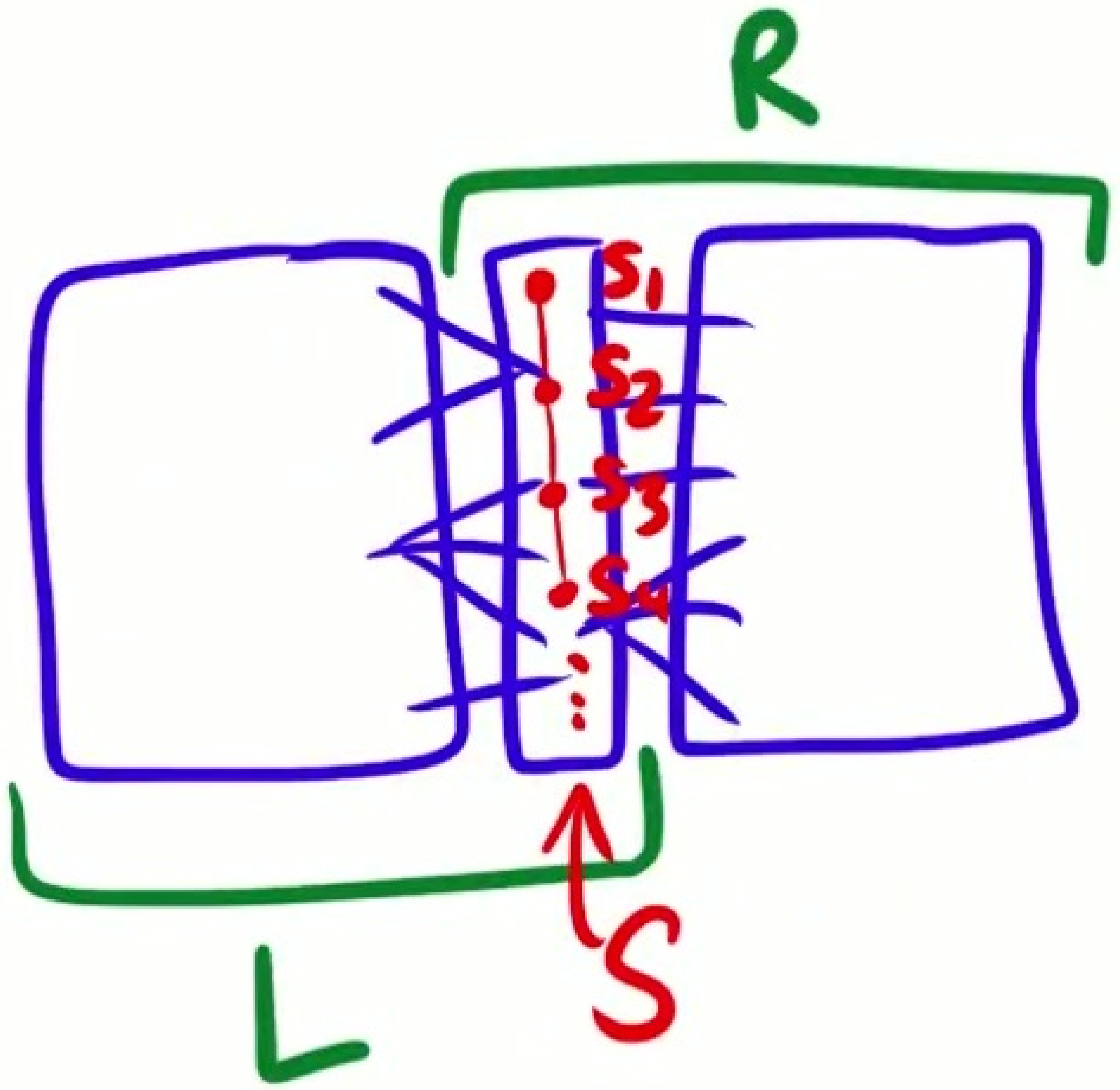
$$\boxed{\text{compute } \max_{x\in L,\, y\in R} d(x,y)}$$

S is separator

$\Rightarrow \forall x\in L, y\in R:$

$$d(x,y) = \min_{s\in S} \left(d(x,s) + d(s,y) \right)$$

Conquer Step



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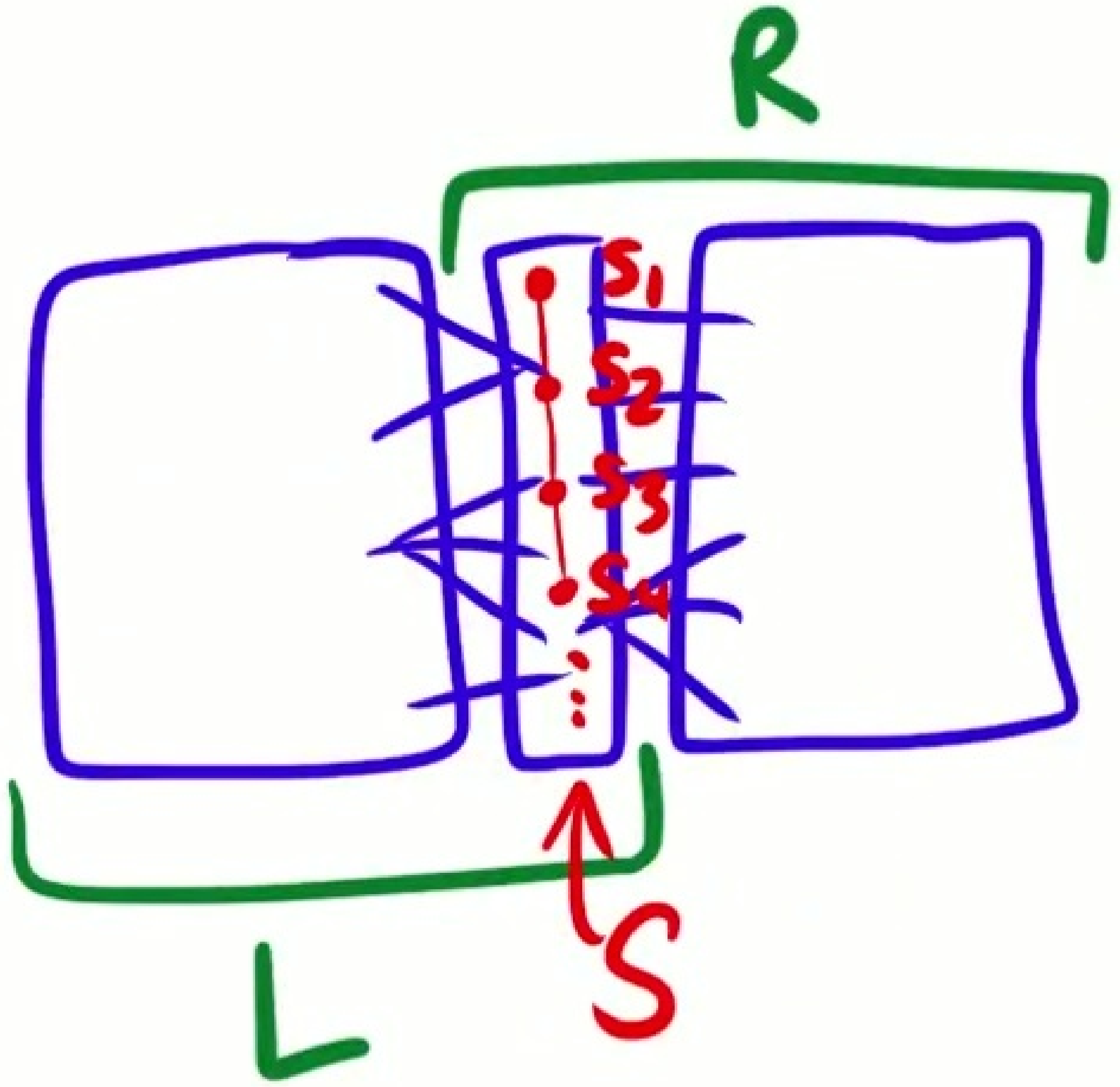
$\Rightarrow d(x,y)$ is a function of

$\langle d(x,s_1), d(x,s_2), \ldots, d(x,s_k) \rangle$ and

$\langle d(y,s_1), d(y,s_2), \ldots, d(y,s_k) \rangle$.

Conquer Step



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Let $\mathcal{L} := \{\langle d(x,s_1), \dots, d(x,s_k)\rangle : x \in L\}$

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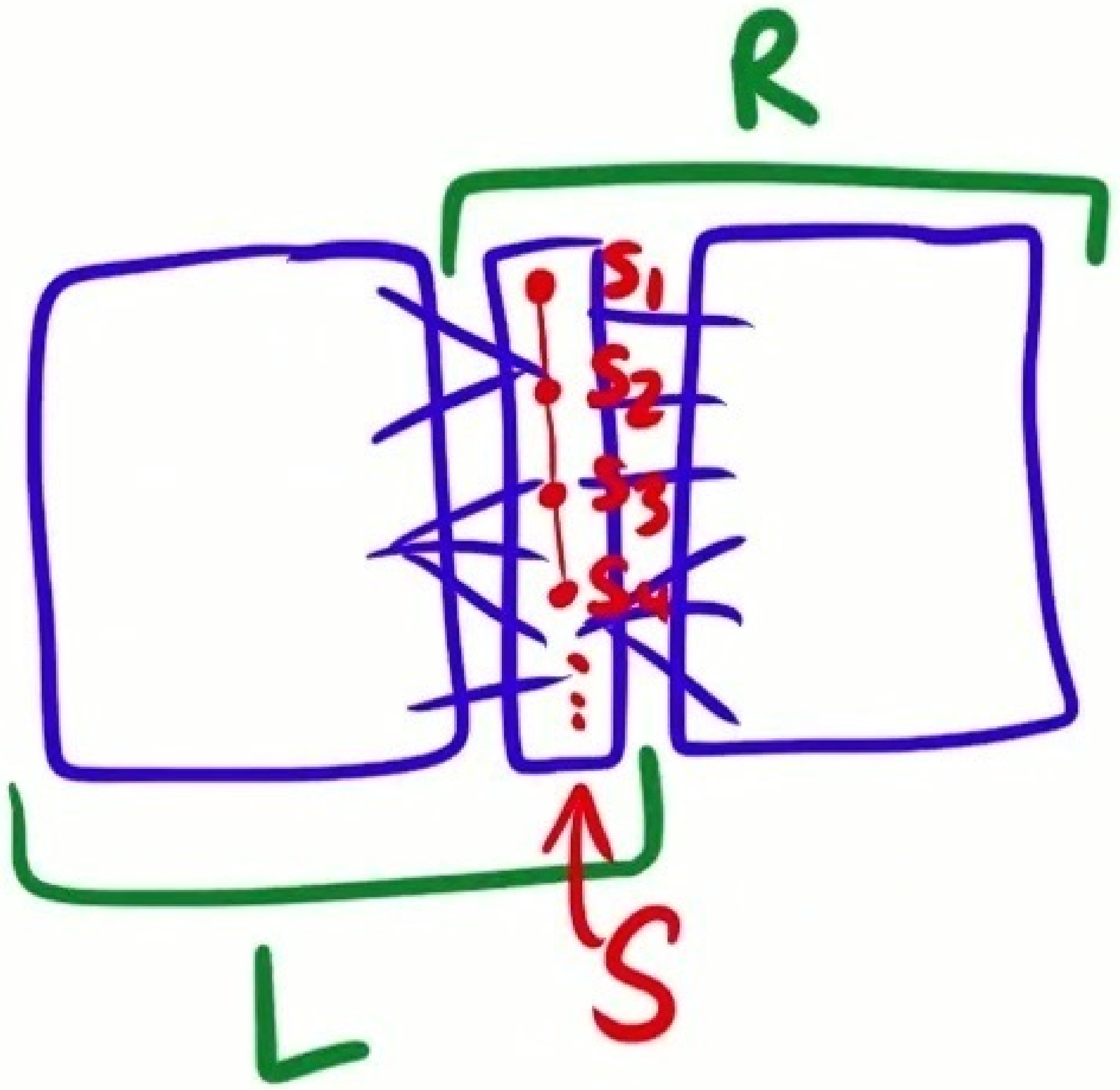
Conquer Step



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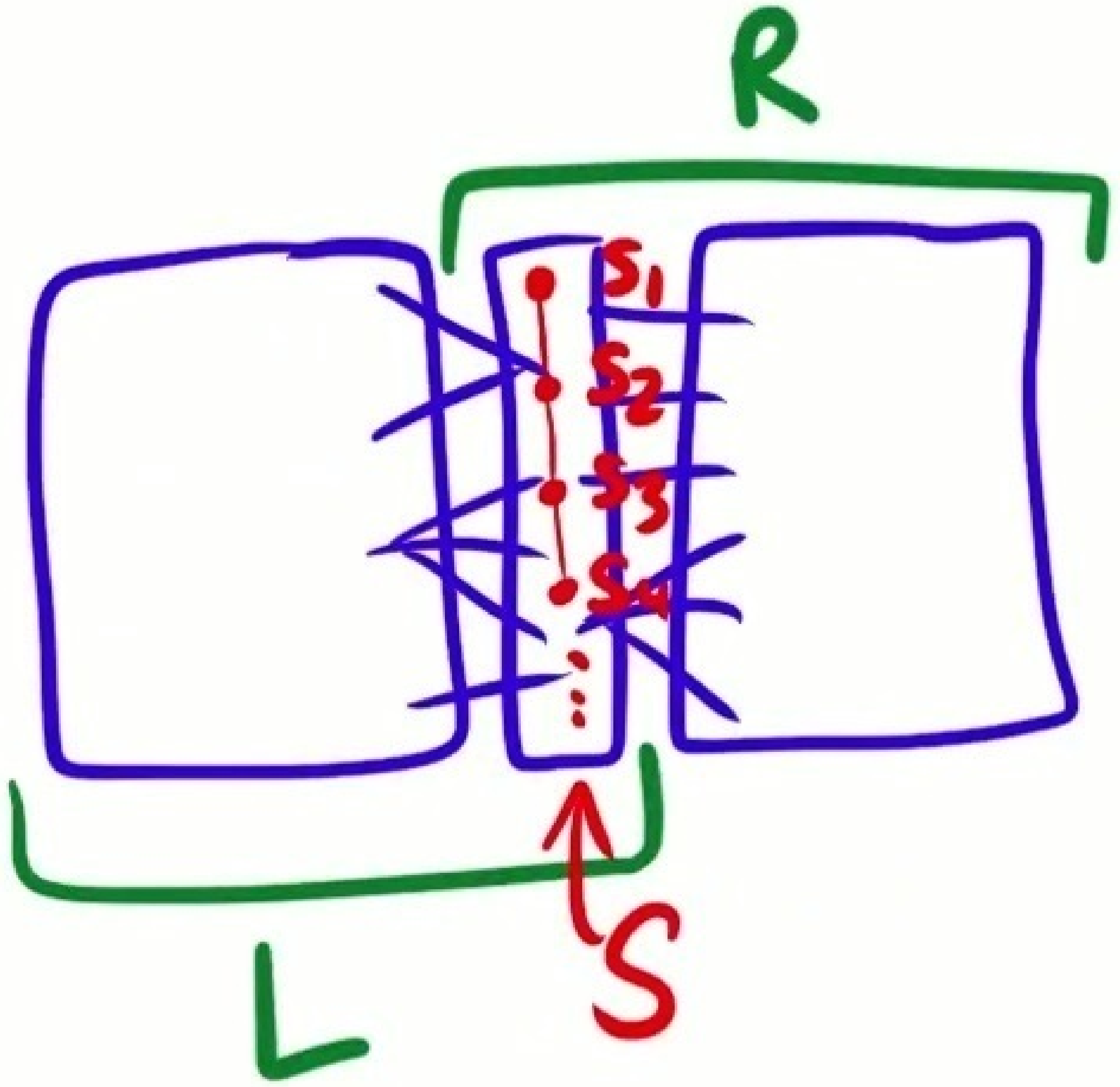
Then, want $\max_{u \in \mathcal{L},\, v \in \mathcal{R}} \text{function}(u,v)$.

Conquer Step



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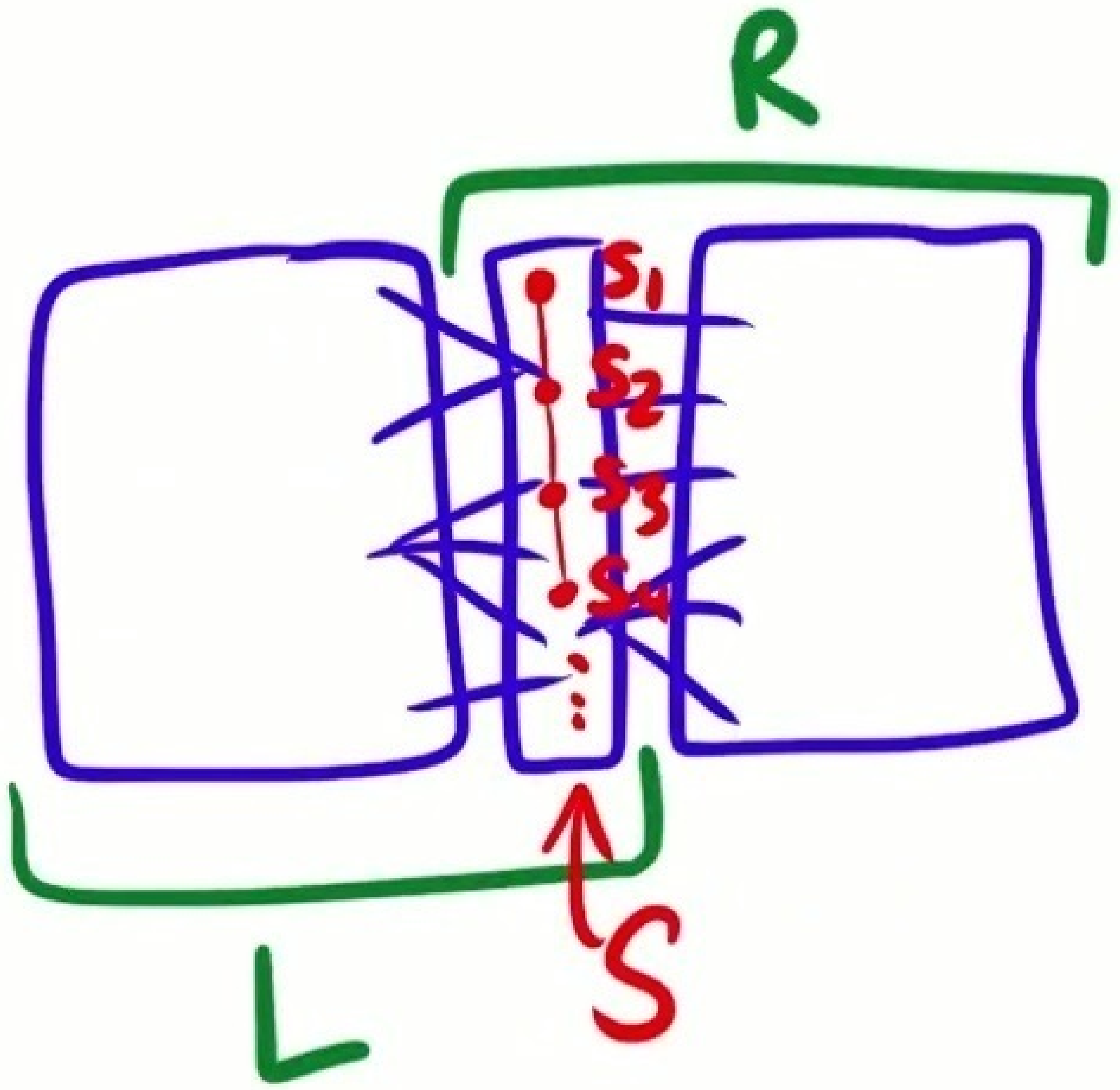
Then, want $\max_{u \in \mathcal{L}, v \in \mathcal{R}} \text{function}(u,v)$.

Runtime: $\tilde{O}(nD) + |\mathcal{L}| \cdot |\mathcal{R}| \cdot O(D)$

Conquer Step



$$\boxed{\text{compute } \max_{x \in L,\, y \in R} d(x,y)}$$

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Then, want $\max_{u \in \mathcal{L}, v \in \mathcal{R}} \text{function}(u,v)$.

Runtime: $\tilde{O}(nD) + |\mathcal{L}| \cdot |\mathcal{R}| \cdot O(D)$

Upper bound $|\mathcal{L}|, |\mathcal{R}|$?

Upper-bounding $|\mathcal{L}|$

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Easy: $|\mathcal{L}| \leq O(D \cdot 3^k)$

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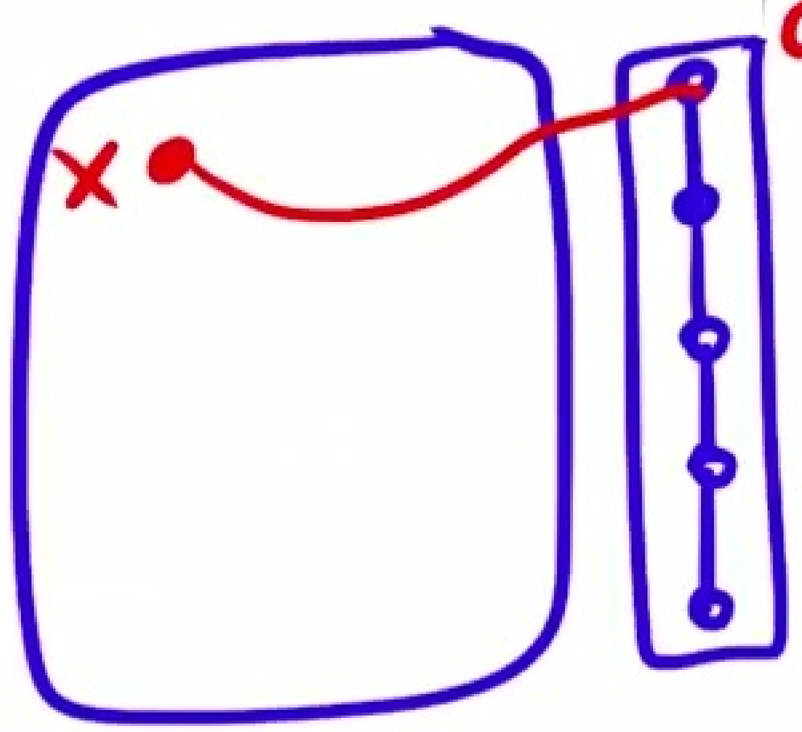
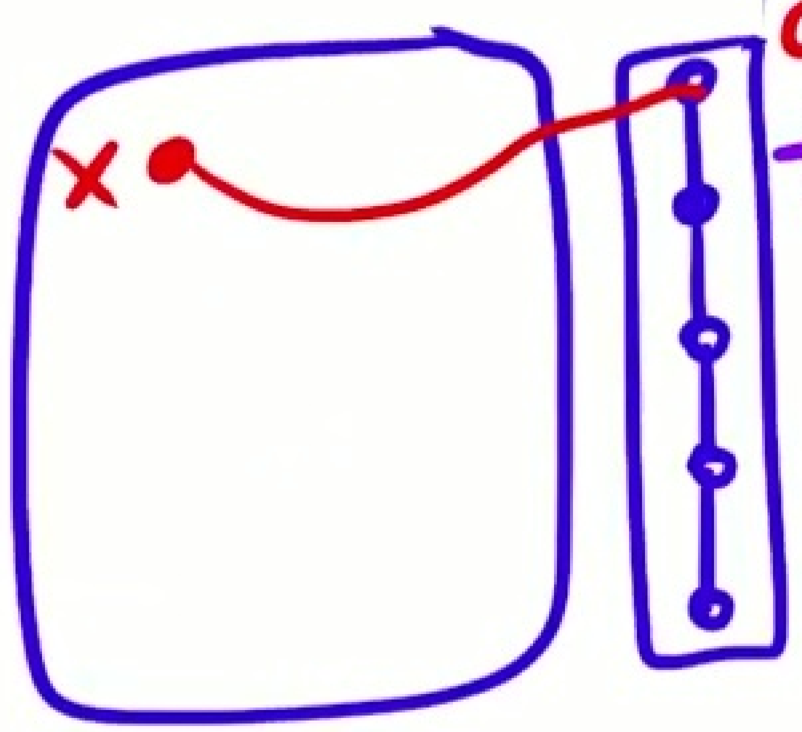


$d(x, s_1) \in \{0, 1, \ldots, D\}$

Upper-bounding $|\mathcal{L}|$

Easy: $|\mathcal{L}| \le O(D \cdot 3^k)$



$d(x, s_1) \in \{0, 1, \dots, D\}$

triangle ineq:

$|d(x, s_1) - d(x, s_2)| \le 1$

$\Rightarrow d(x, s_2) = d(x, s_1) + \begin{cases} -1 \\ 0 \\ +1 \end{cases}$

Upper-bounding $|\mathcal{L}|$

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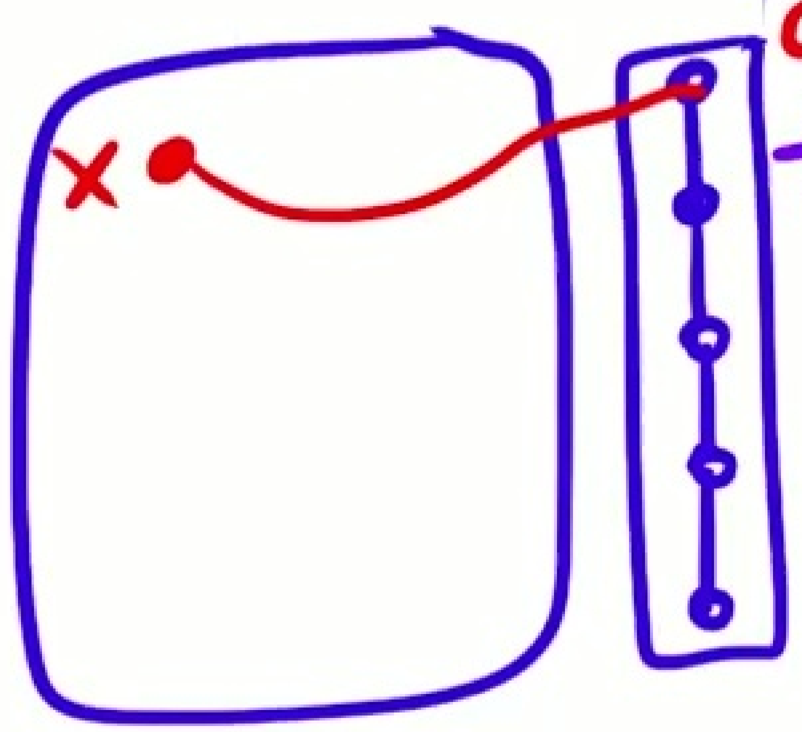

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$\Rightarrow$ 3 choices of $d(x, s_{i+1})$ given $d(x, s_i)$

$\Rightarrow (D+1) \cdot 3^{\ell - 1}$ choices total

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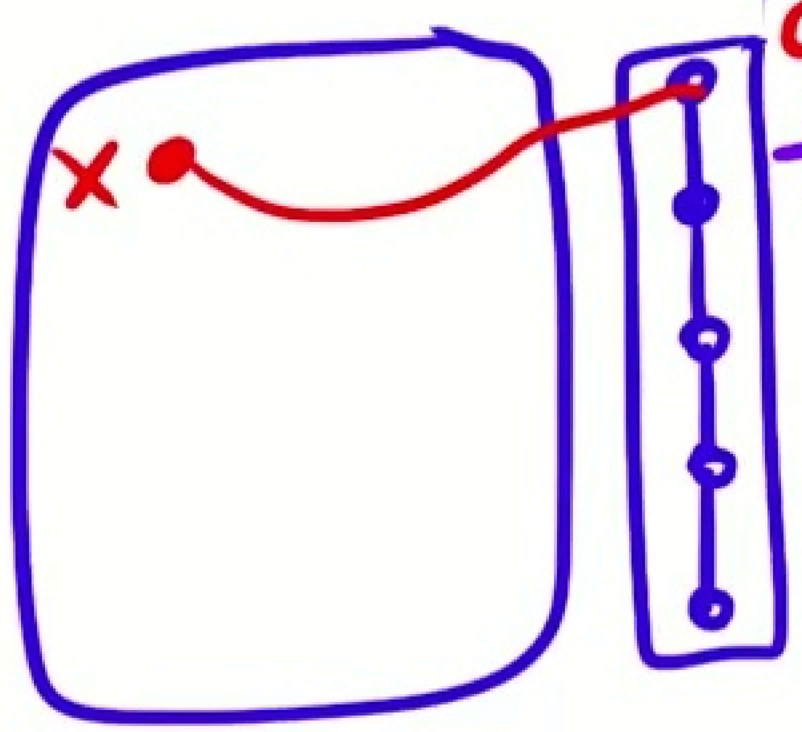

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Main Theorem: $|\mathcal{L}| \leq O(D \cdot k^3)$.

$$|\mathcal{L}| = O(Dk^3)$$

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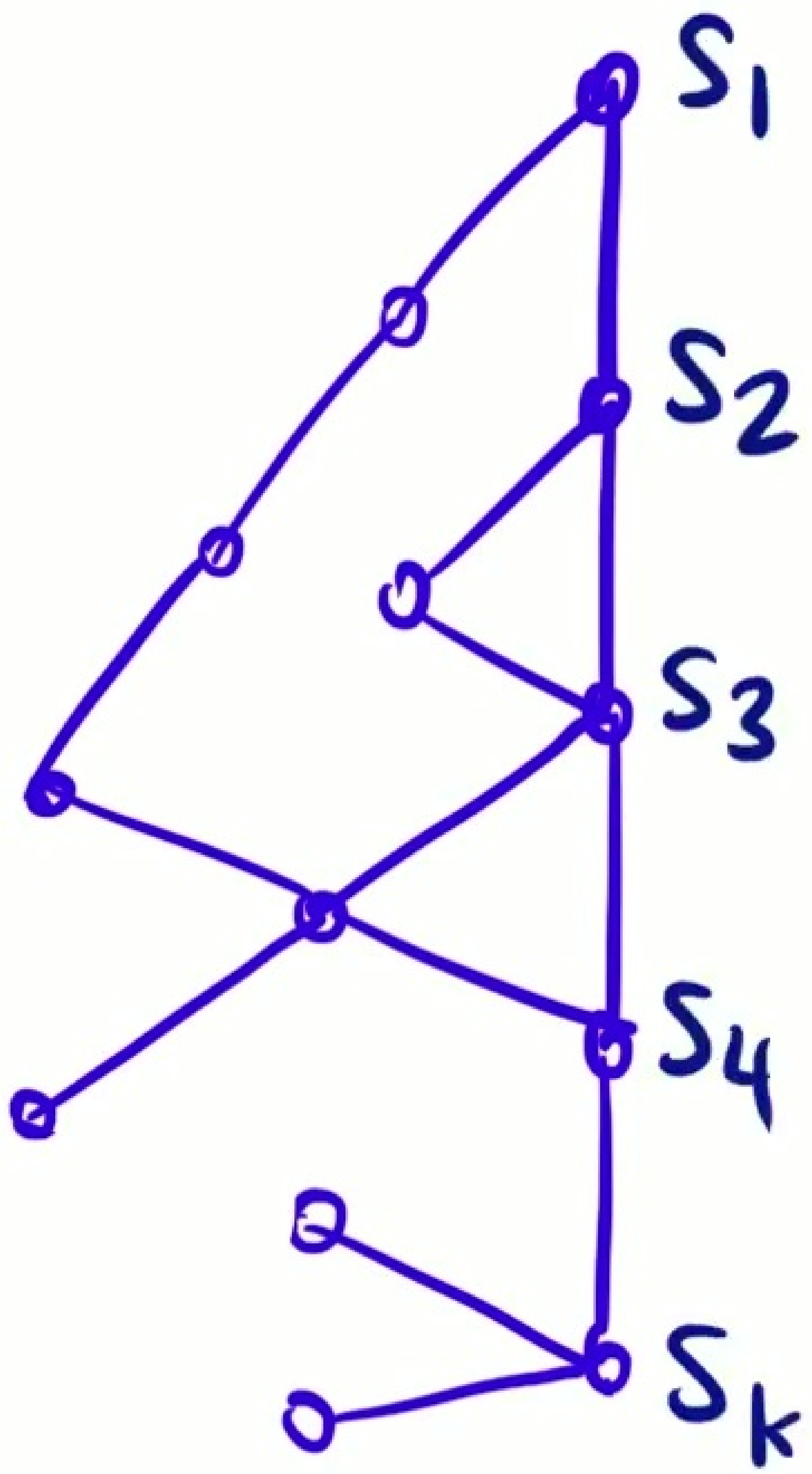

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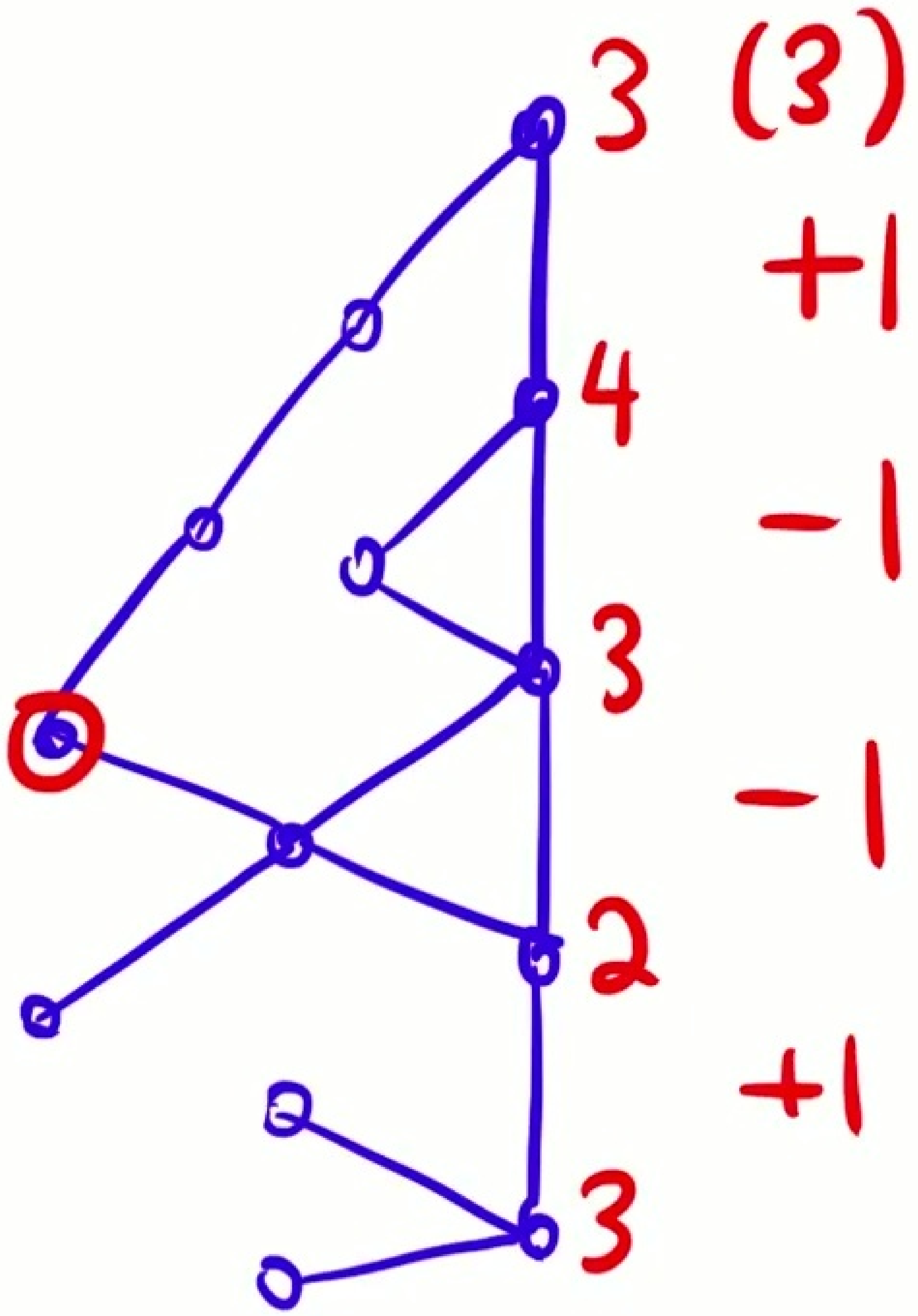

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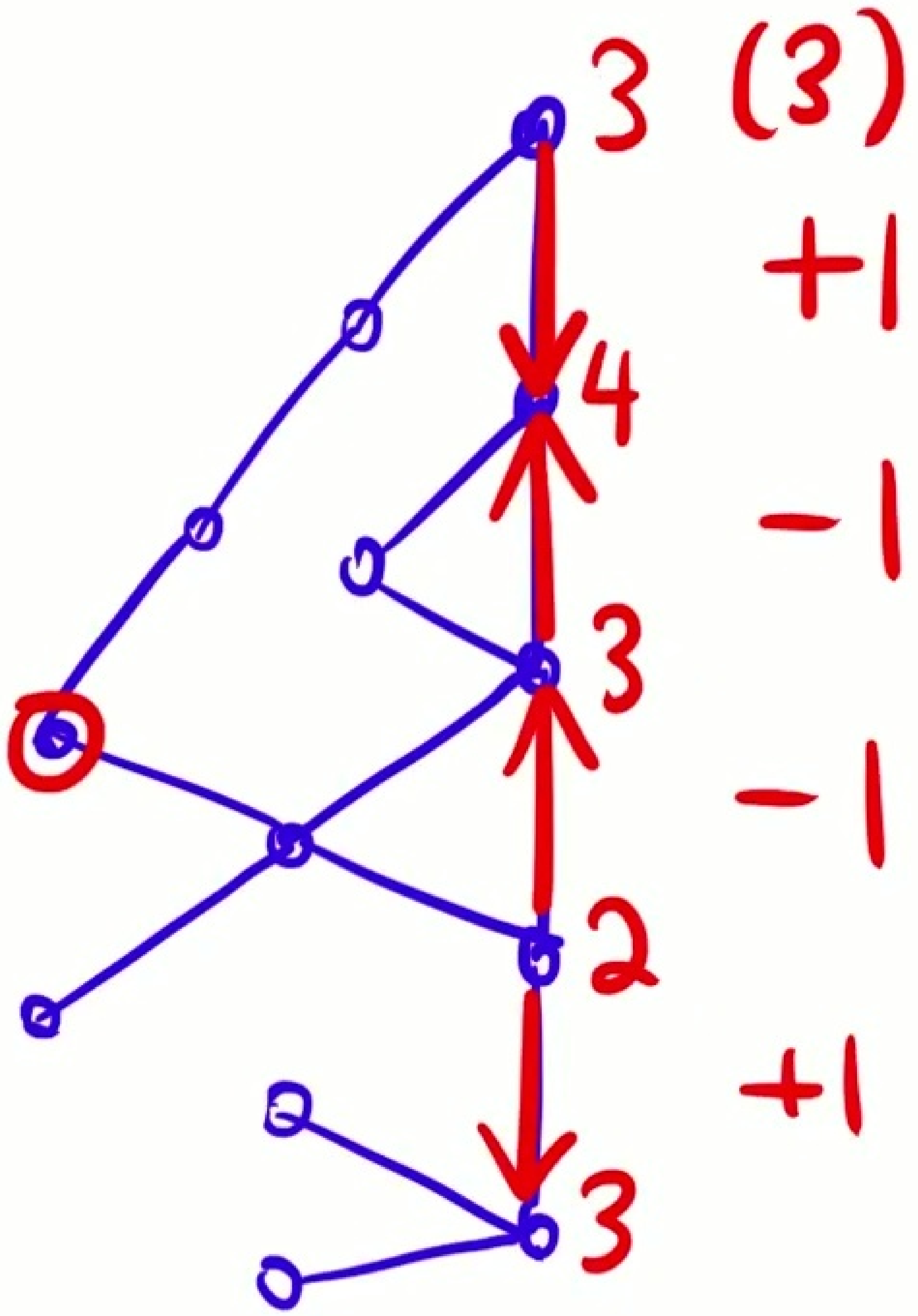

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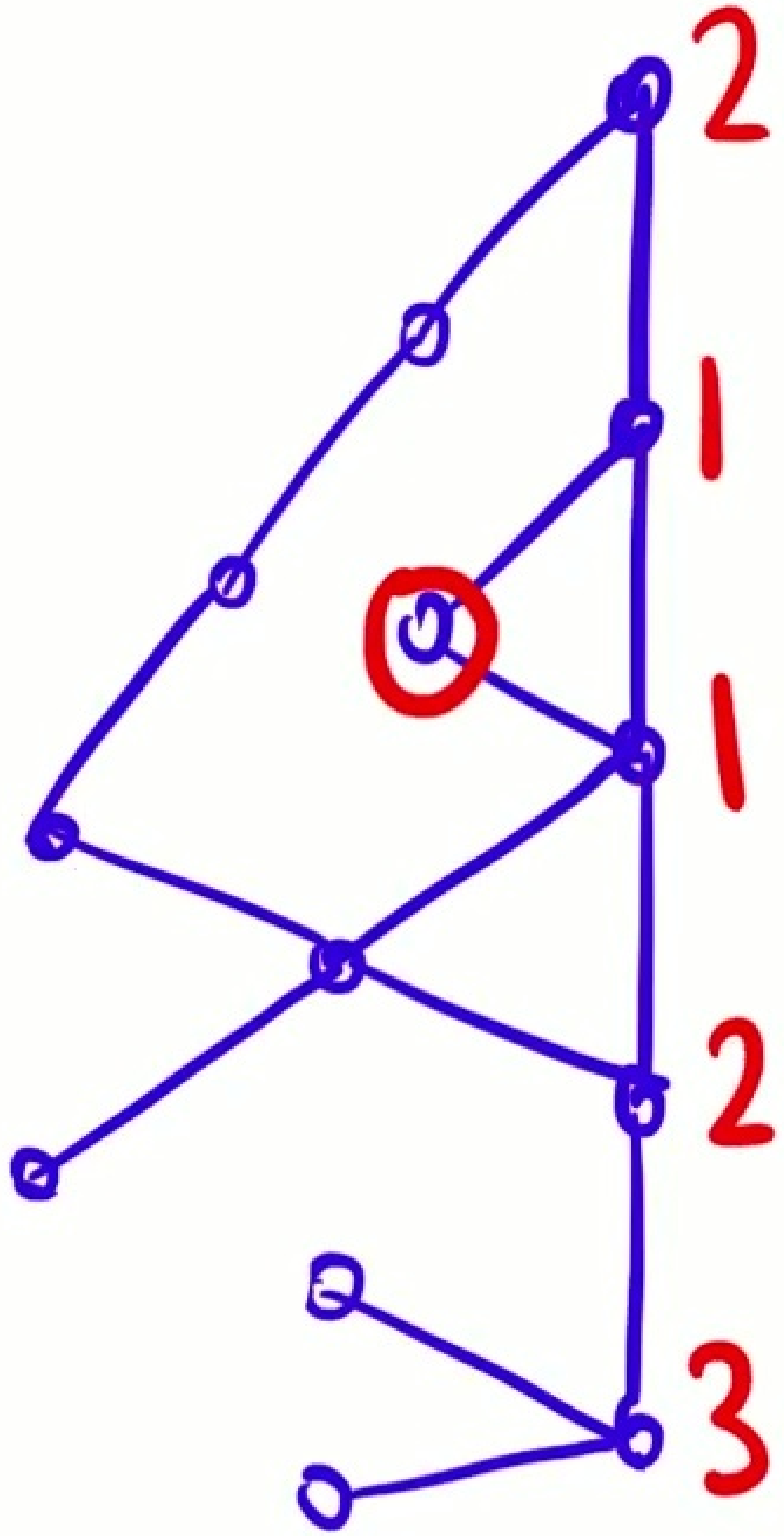

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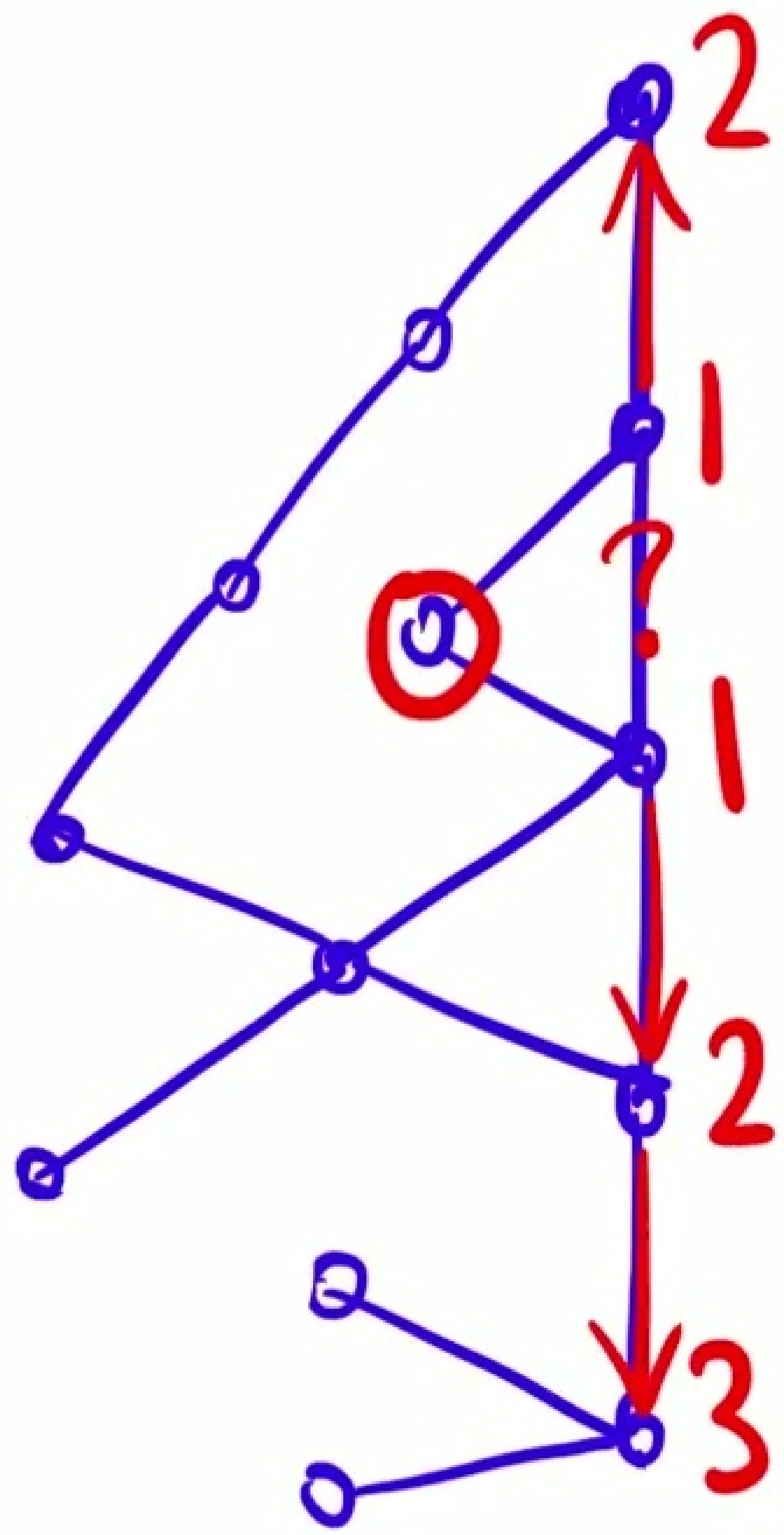

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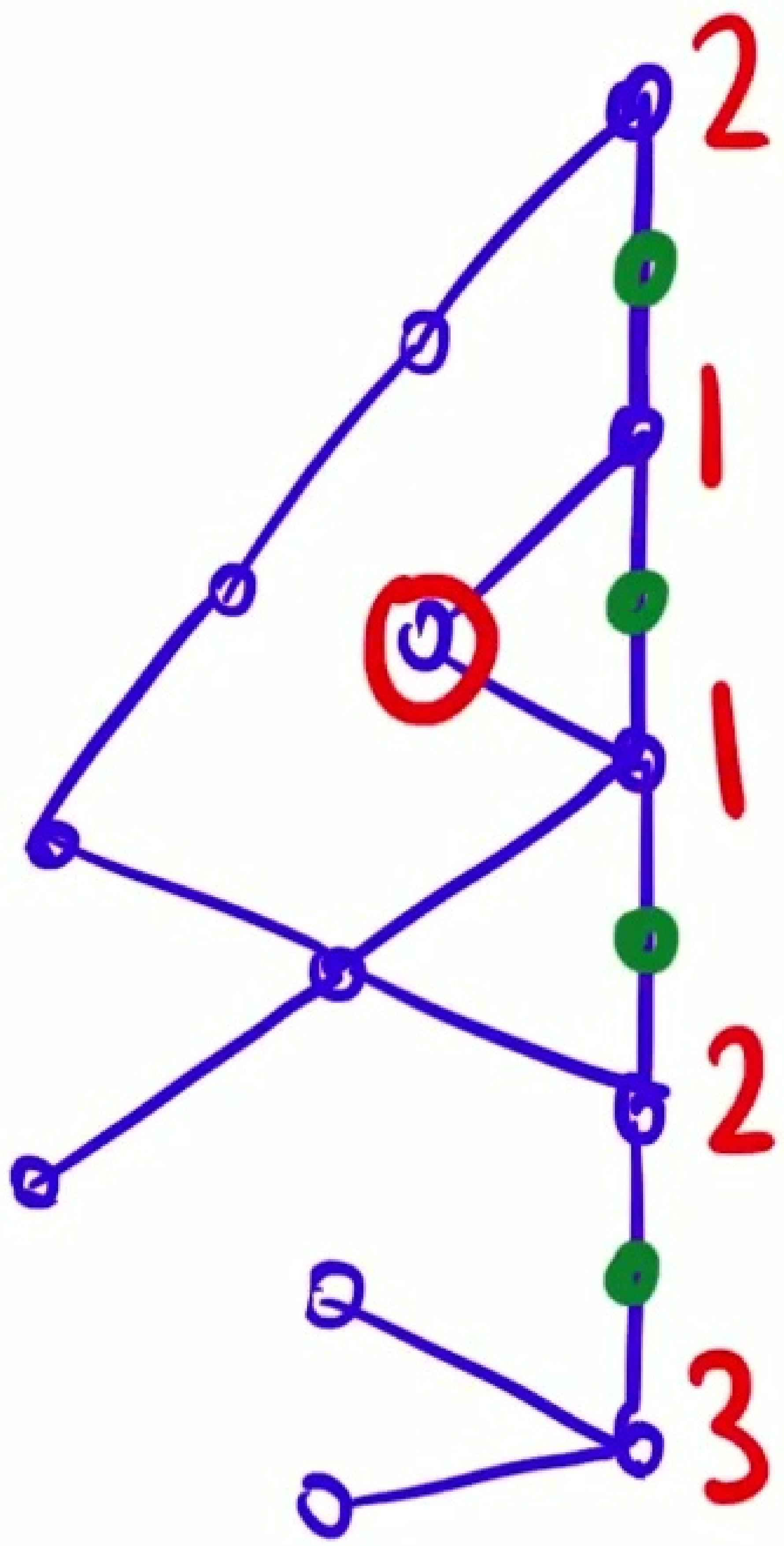

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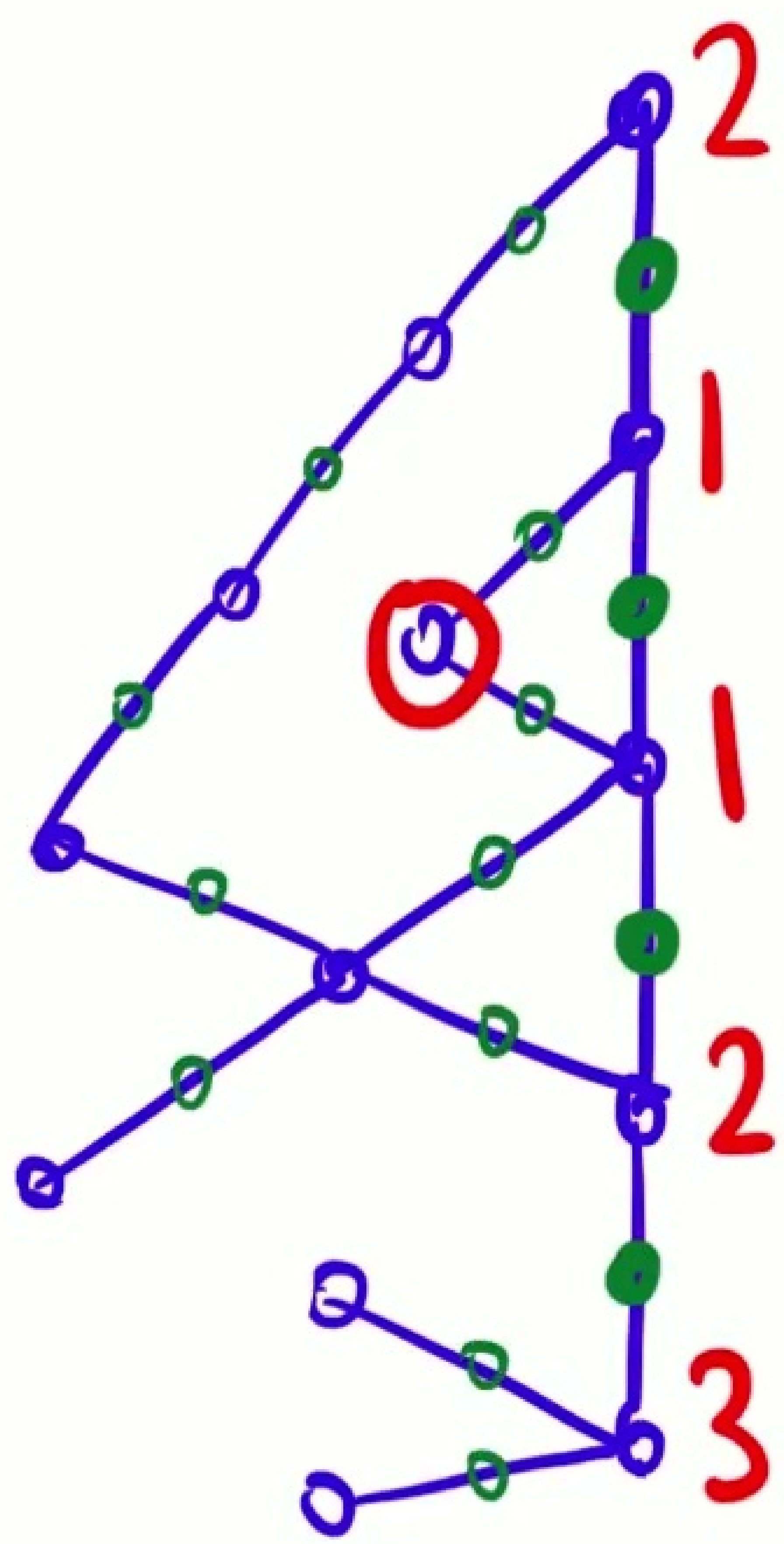
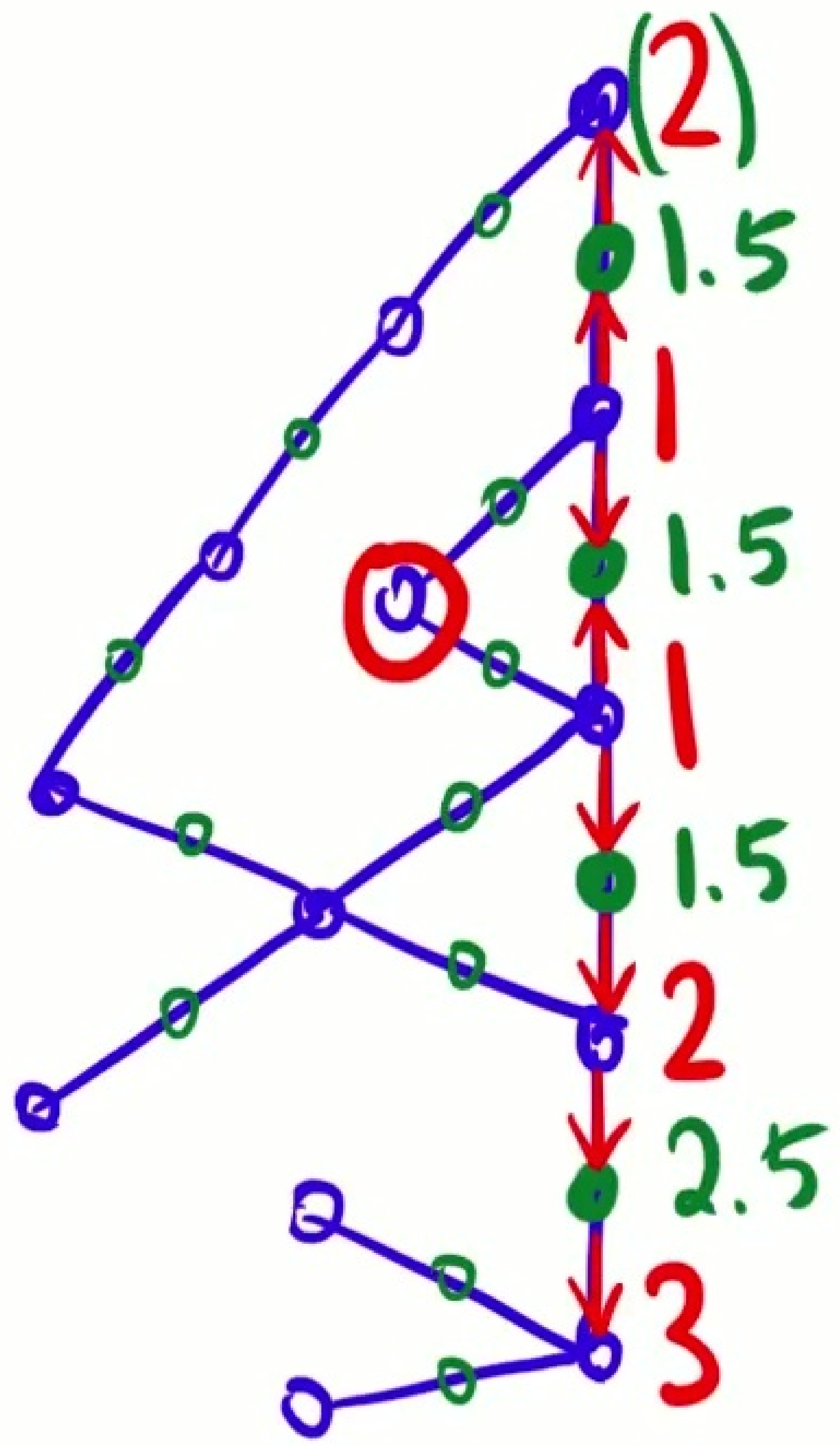
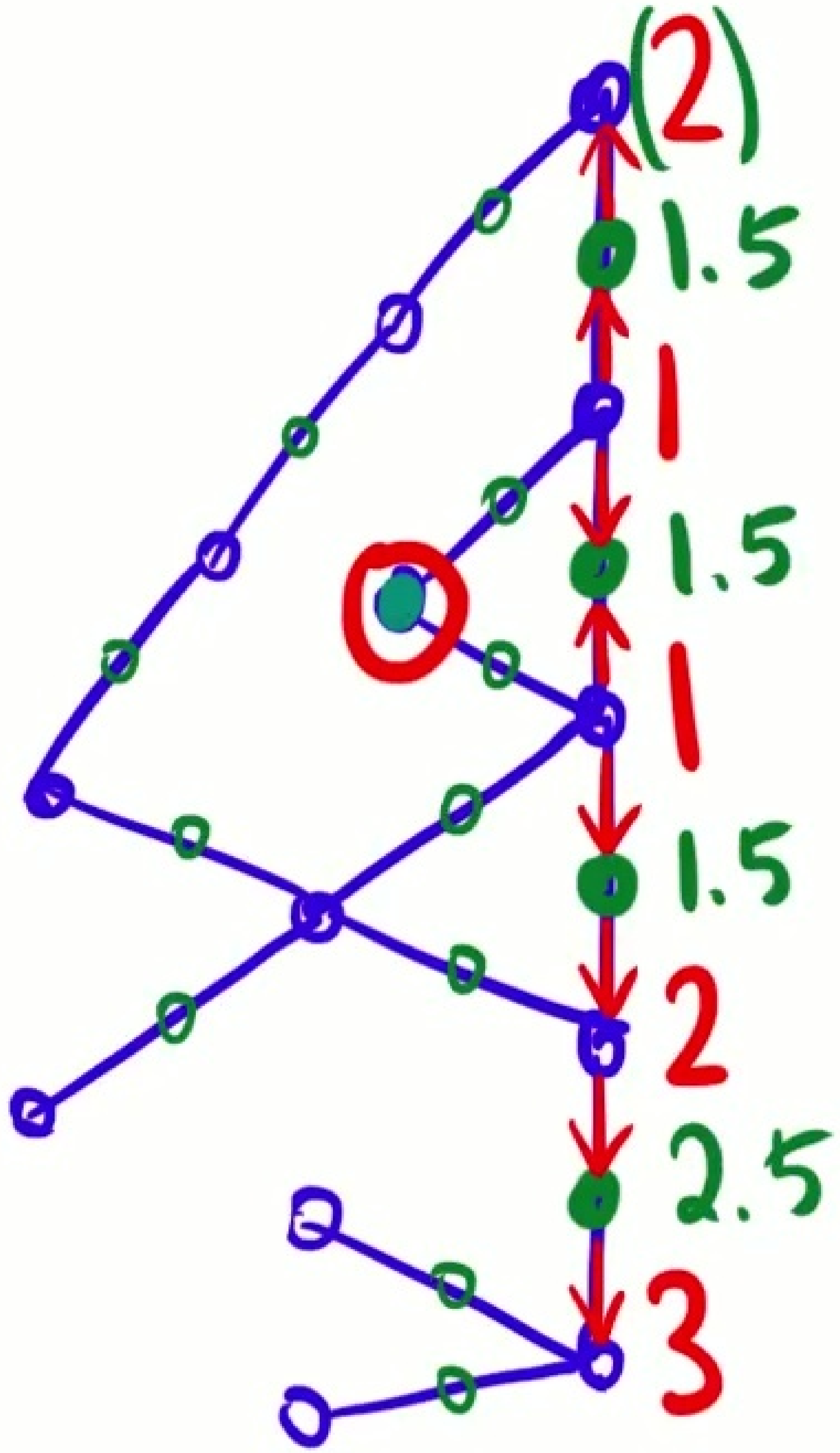


$$|\mathcal{L}| = O(Dk^3)$$

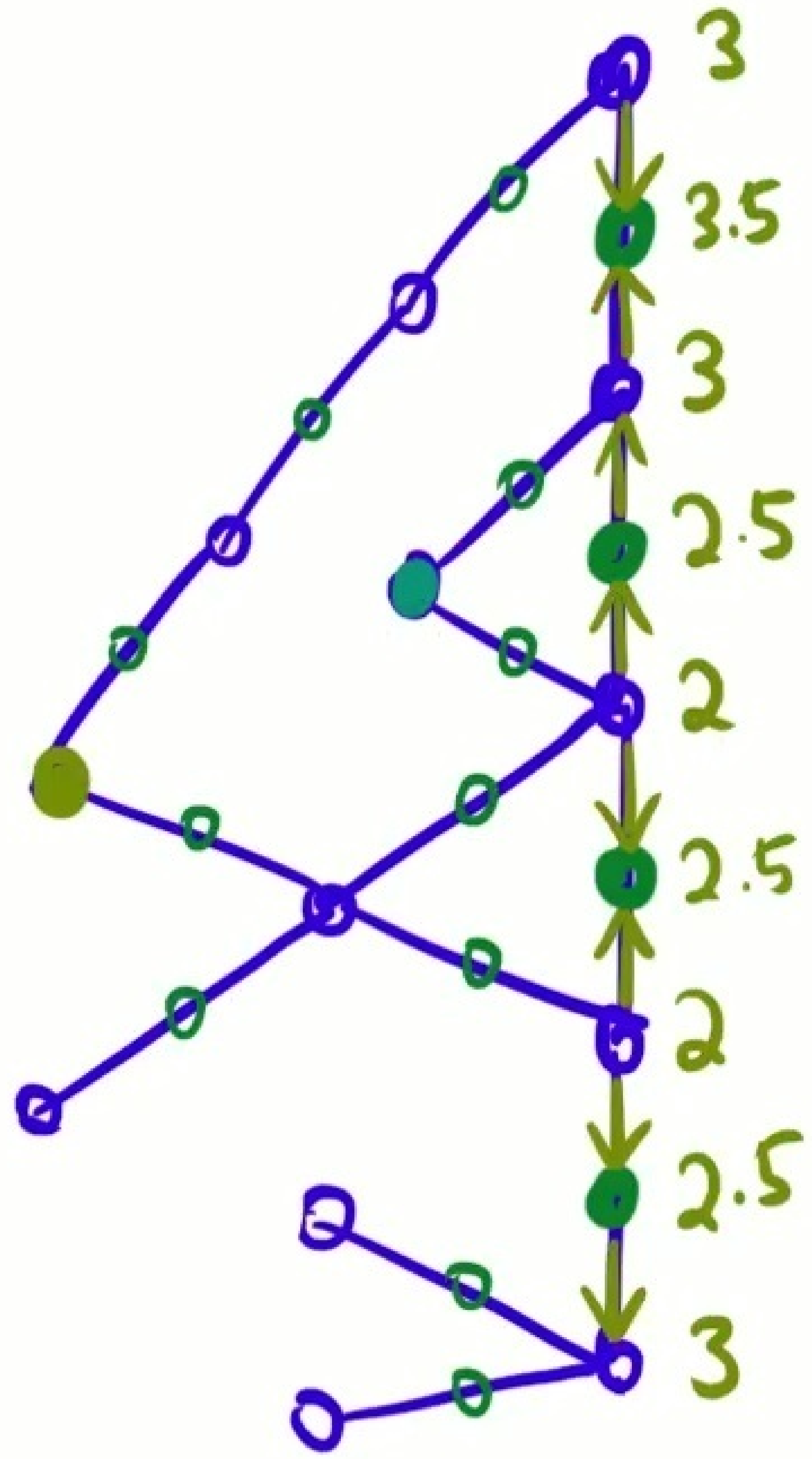


$$|\mathcal{L}| = O(Dk^3)$$



(2) ↑ ↑ ↓ ↑ ↓ ↓ ↓ ↓

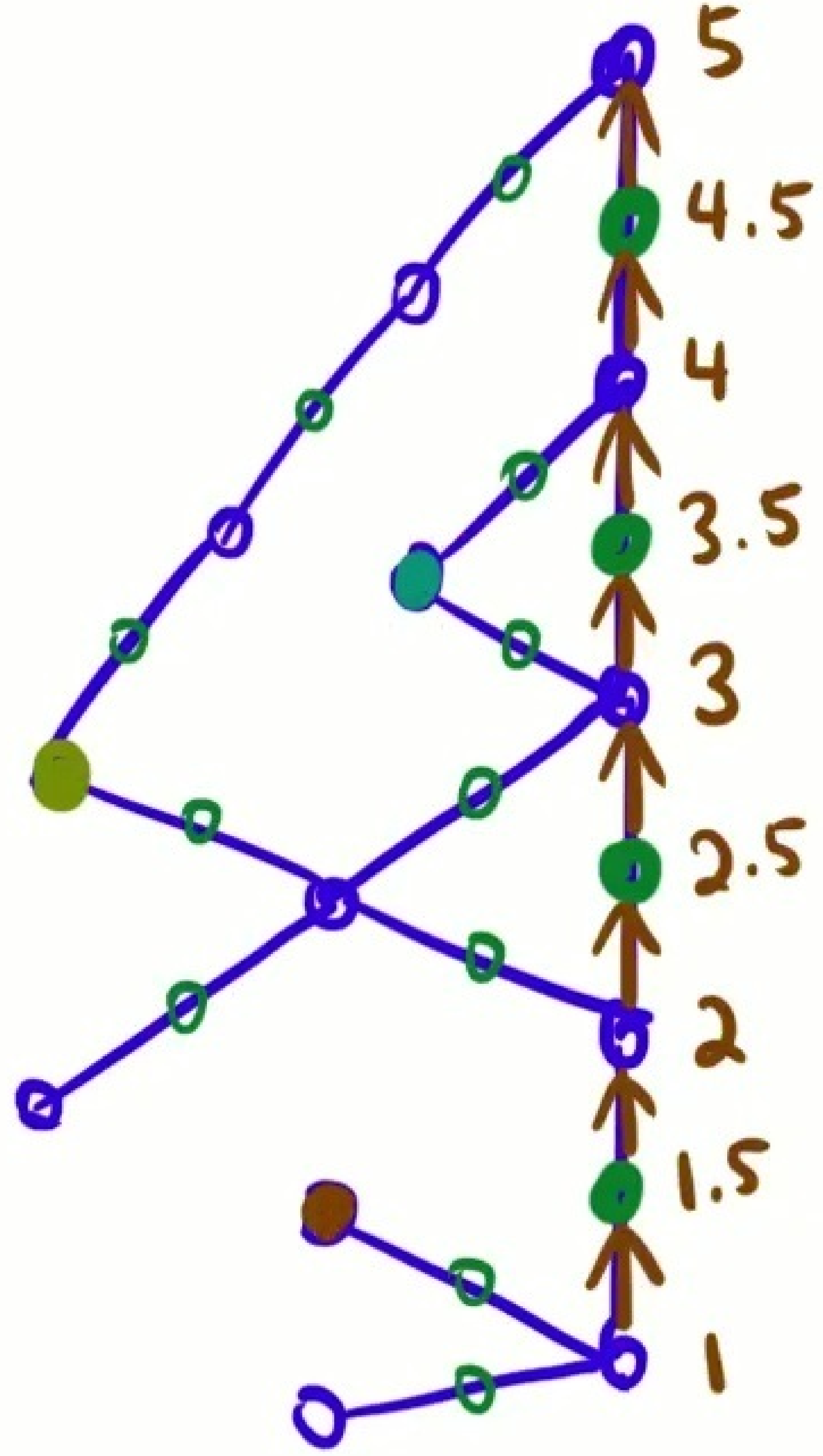
$|\mathcal{L}| = O(Dk^3)$



(2) $\uparrow \uparrow \downarrow \uparrow \downarrow \downarrow \downarrow \downarrow$

(3) $\downarrow \uparrow \uparrow \uparrow \downarrow \uparrow \downarrow \downarrow$

$$|\mathcal{L}| = O(Dk^3)$$

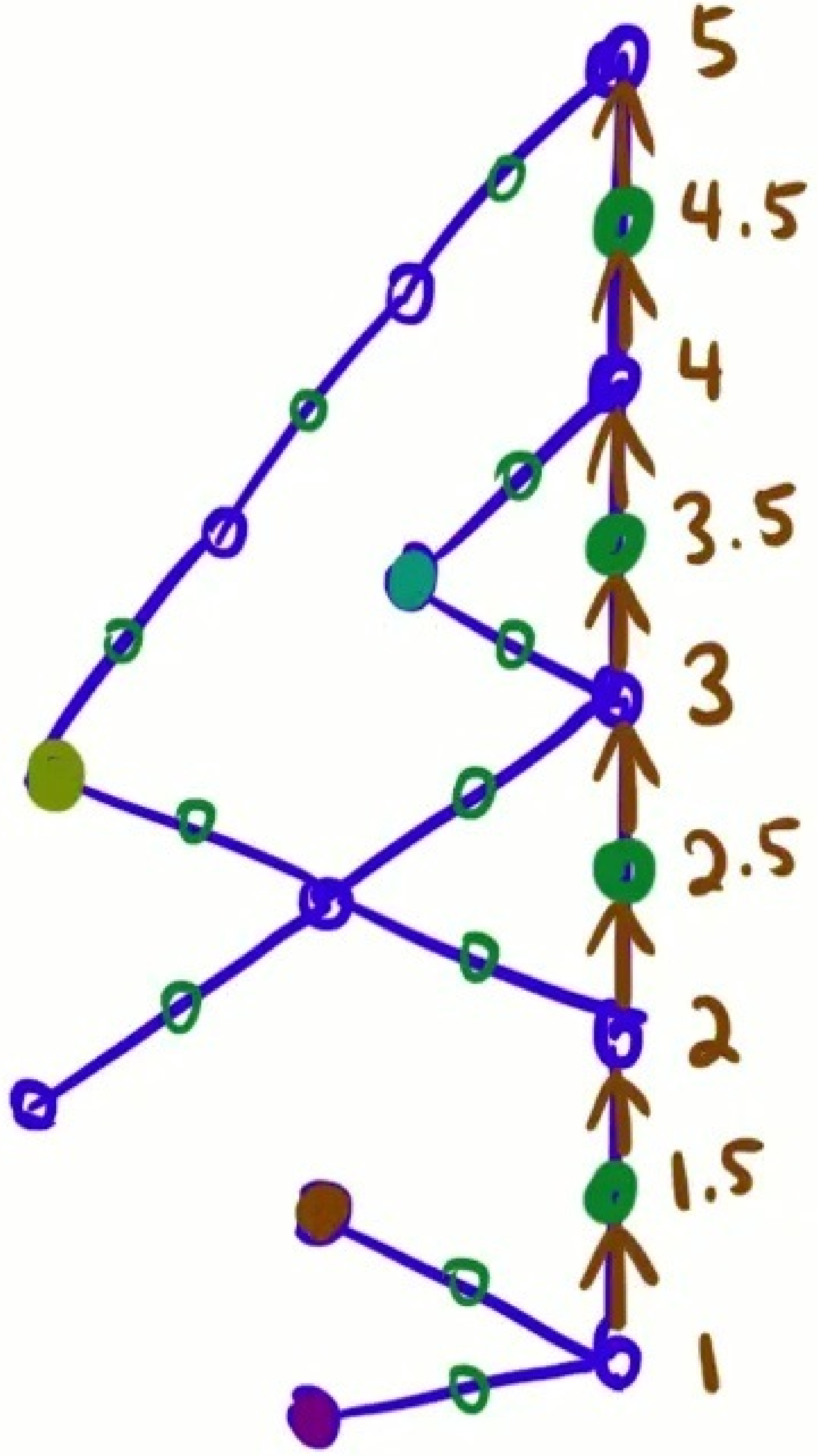


(2) ↑ ↑ ↓ ↑ ↓ ↓ ↓ ↓

(3) ↓ ↑ ↑ ↑ ↓ ↑ ↓ ↓

(5) ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

$|\mathcal{L}| = O(Dk^3)$



(2) ↑ ↑ ↓ ↑ ↓ ↓ ↓ ↓

(3) ↓ ↑ ↑ ↑ ↓ ↑ ↓ ↓

(5) ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

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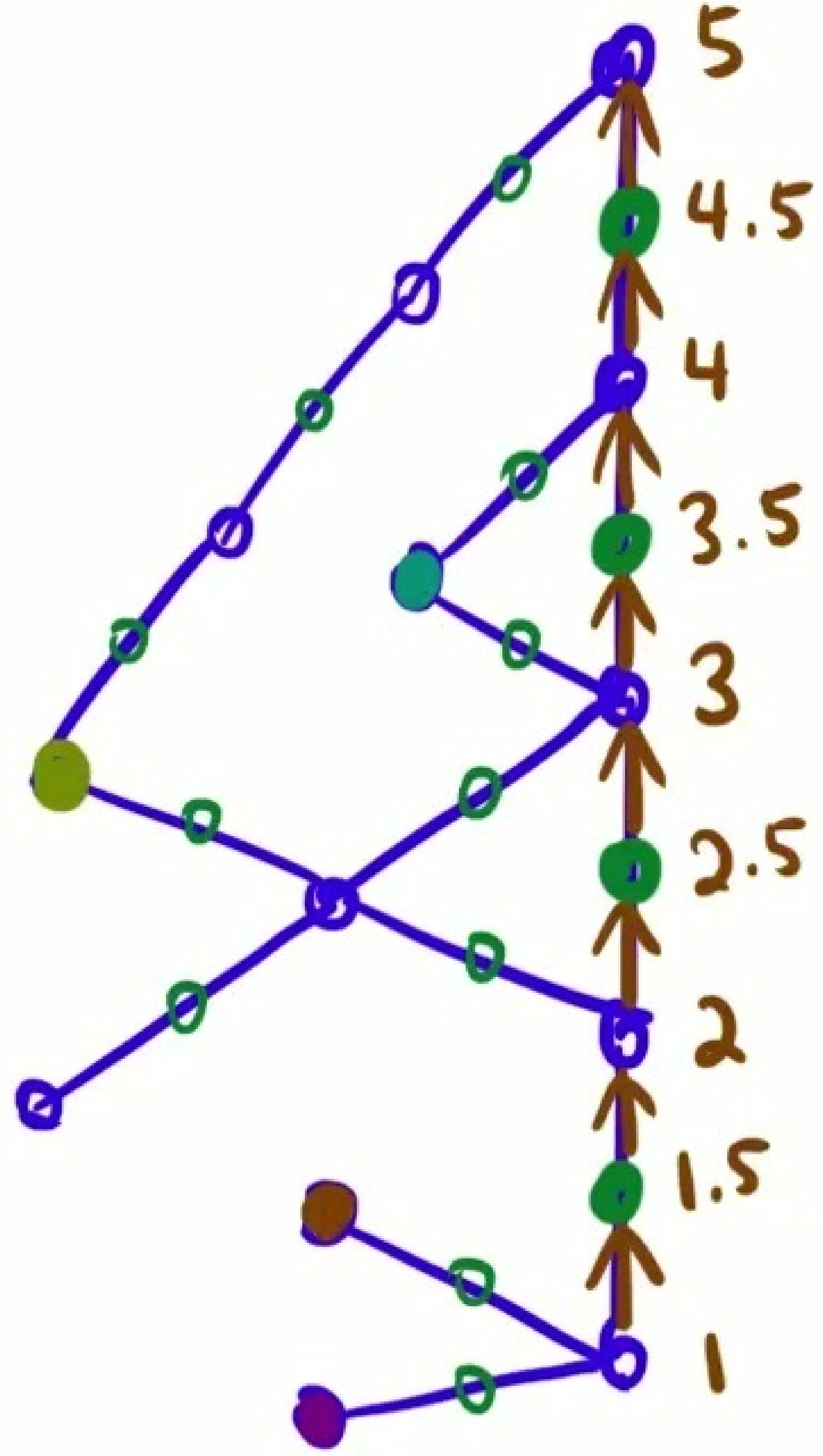
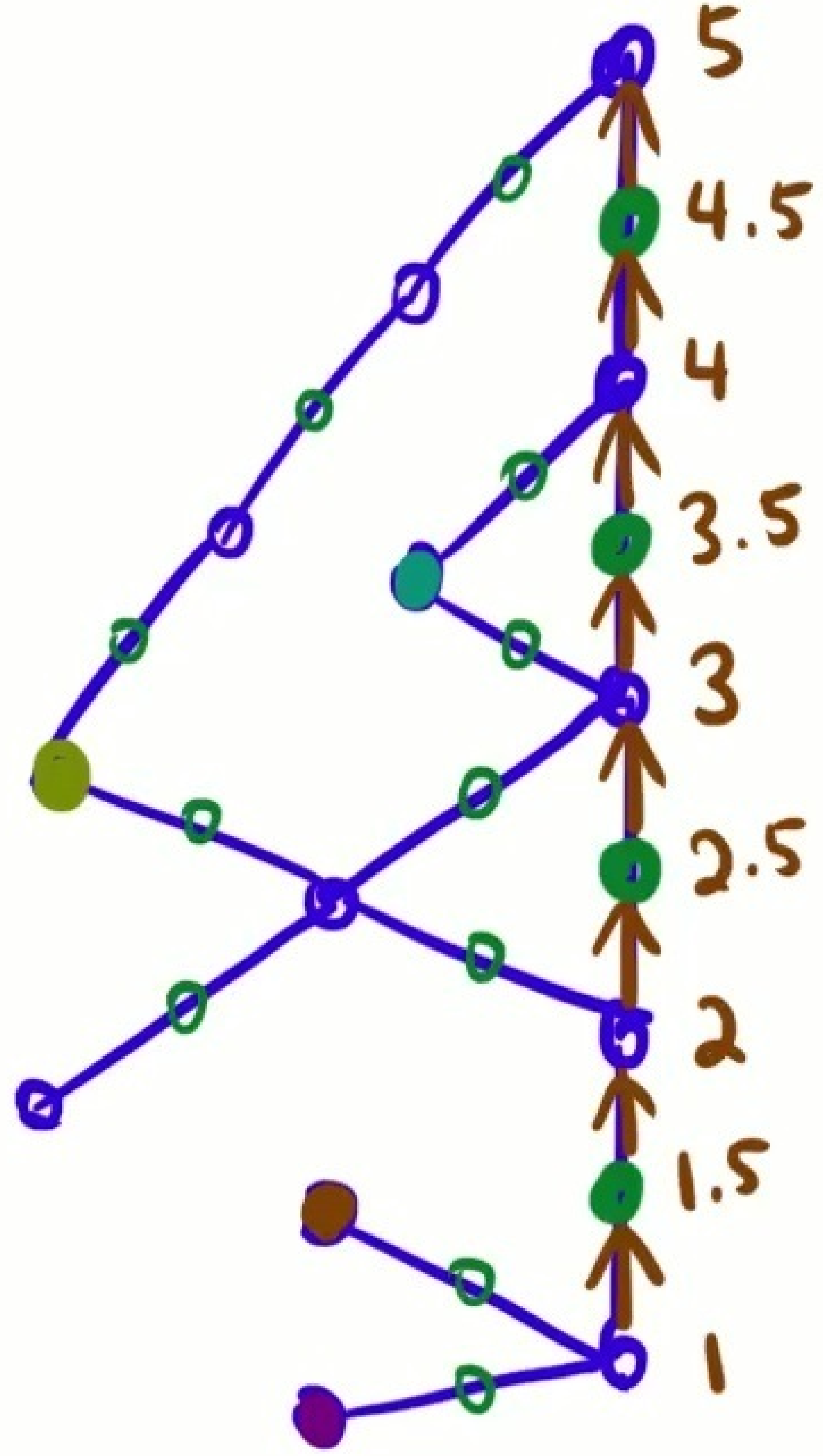


(2) ↑ ↑ ↓ ↑ ↓ ↓ ↓ ↓
(3) ↓ ↑ ↑ ↑ ↓ ↑ ↓ ↓
(5) ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑
(5) ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

$\leq O(D)$
choices

$|\mathcal{L}| = O(Dk^3)$

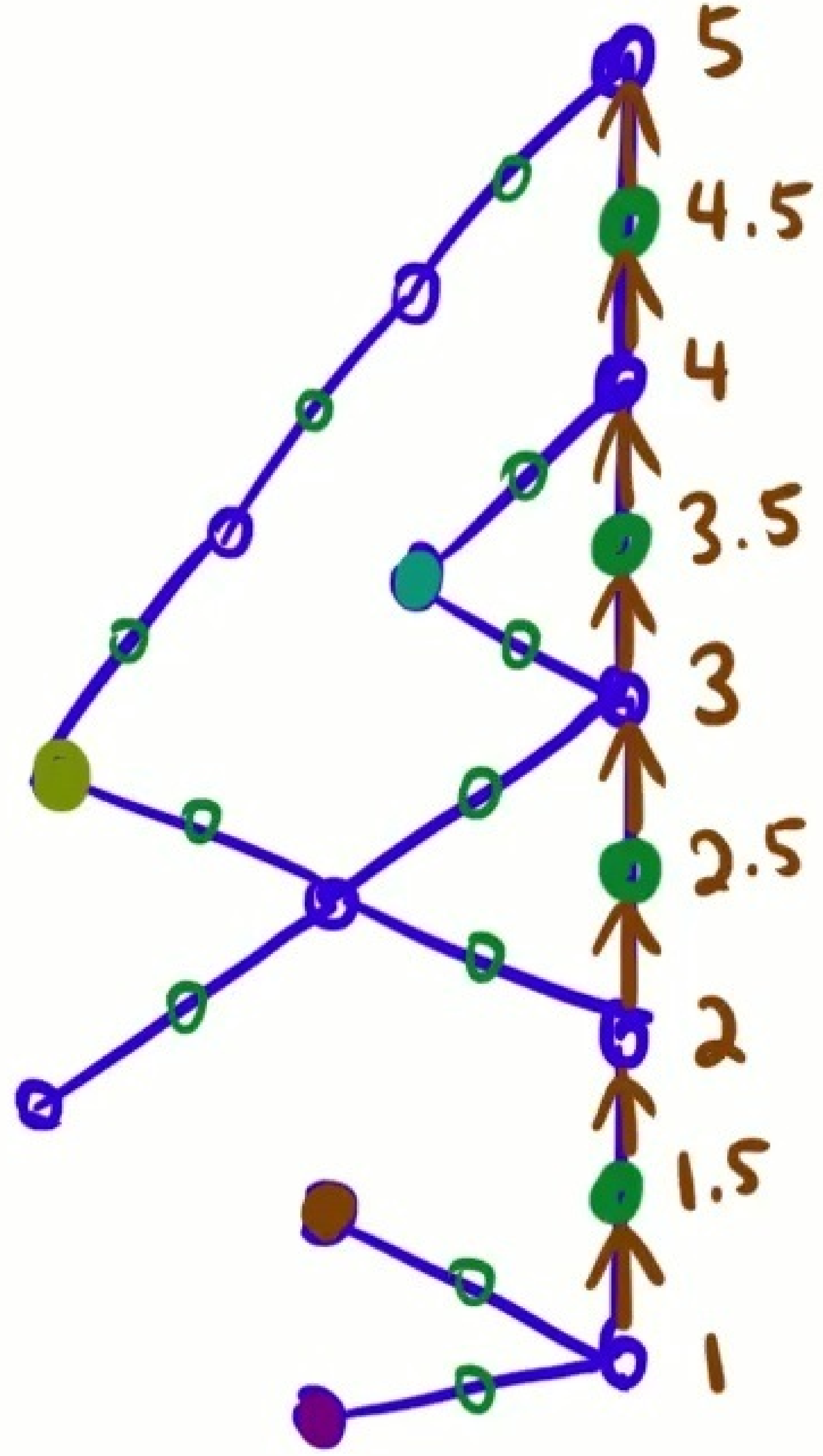


(2) ↑ ↑ ↓ ↑ ↓ ↓ ↓ ↓
(3) ↓ ↑ ↑ ↑ ↓ ↑ ↓ ↓
(5) ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑
(5) ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

$\leq O(D)$ choices

How many distinct strings?

$|\mathcal{L}| = O(Dk^3)$



$$\begin{array}{c|cccccccc}
(2) & \uparrow & \uparrow & \downarrow & \uparrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(3) & \downarrow & \uparrow & \uparrow & \uparrow & \downarrow & \uparrow & \downarrow & \downarrow \\
(5) & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
(5) & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
\end{array}$$

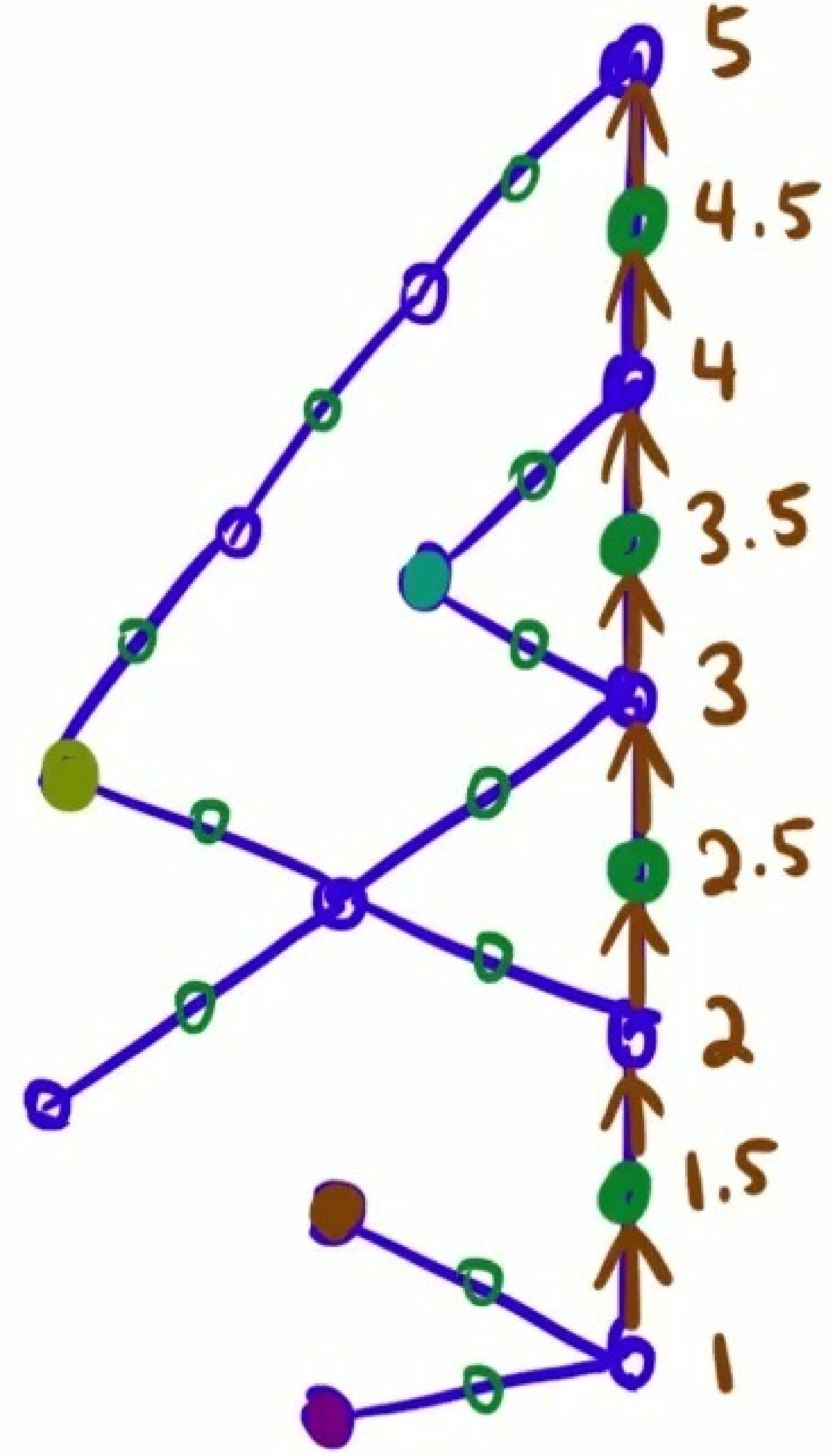
$\leq O(D)$ choices

How many <u>distinct</u> strings?

Def [VC dimension]

A matrix of $\uparrow/\downarrow$'s has <u>VC dim</u> $< d$ if it contains no 2^d-by-d submatrix whose rows span all length-k strings.

$|\mathcal{L}| = O(Dk^3)$



(2)	↑	↑	↓	↑	↓	↓	↓	↓
(3)	↓	↑	↑	↑	↓	↑	↓	↓
(5)	↑	↑	↑	↑	↑	↑	↑	↑
(5)	↑	↑	↑	↑	↑	↑	↑	↑

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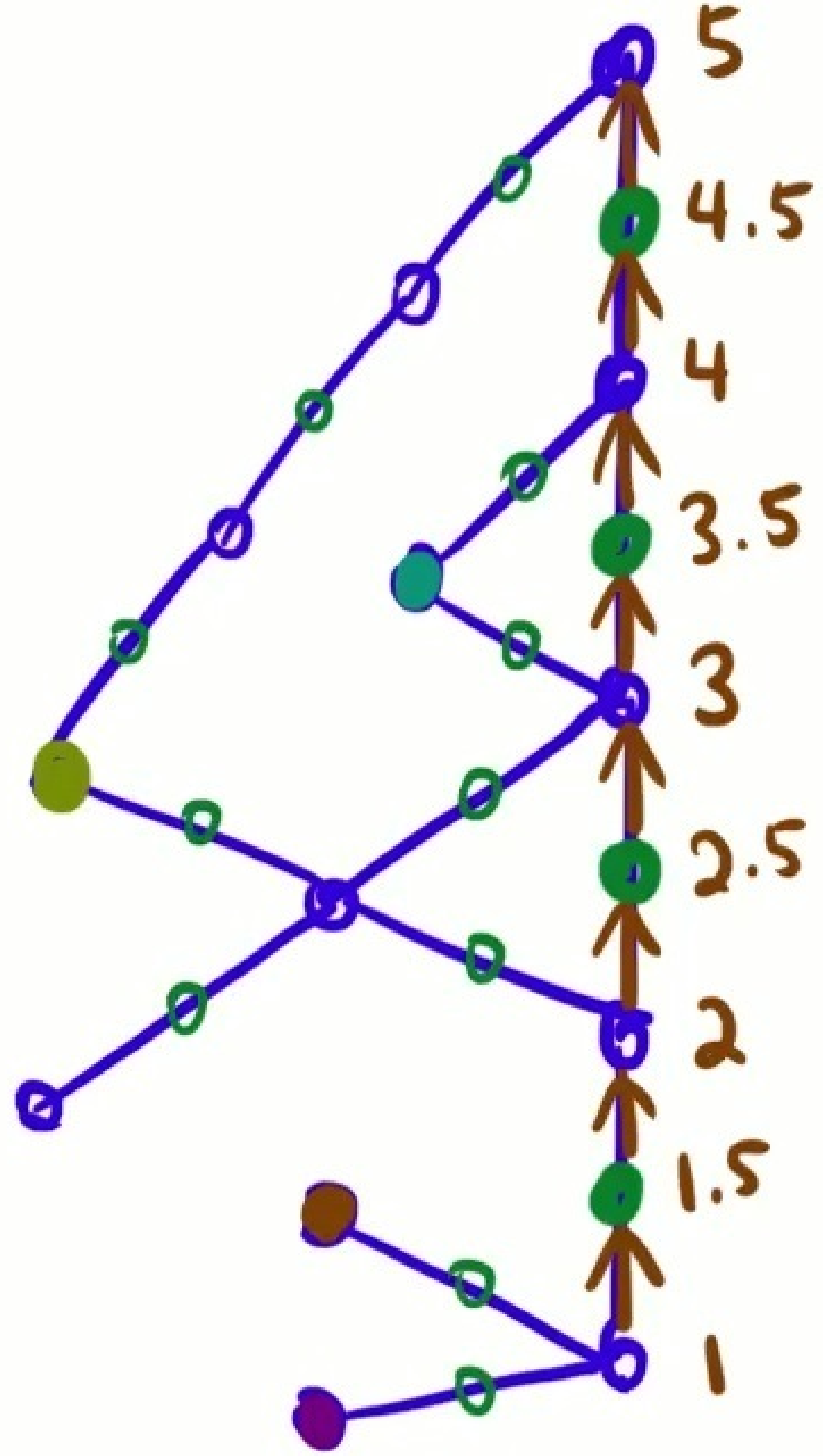
$d=3$: no

↑	↓	↑
↓	↓	↑
↑	↓	↓
↑	↑	↑
↓	↓	↓
↑	↑	↓
↓	↑	↑
↓	↑	↓

$|\mathcal{L}| = O(Dk^3)$



(2)	↑	↑	↓	↑	↓	↓	↓	↓
(3)	↓	↑	↑	↑	↓	↑	↓	↓
(5)	↑	↑	↑	↑	↑	↑	↑	↑
(5)	↑	↑	↑	↑	↑	↑	↑	↑

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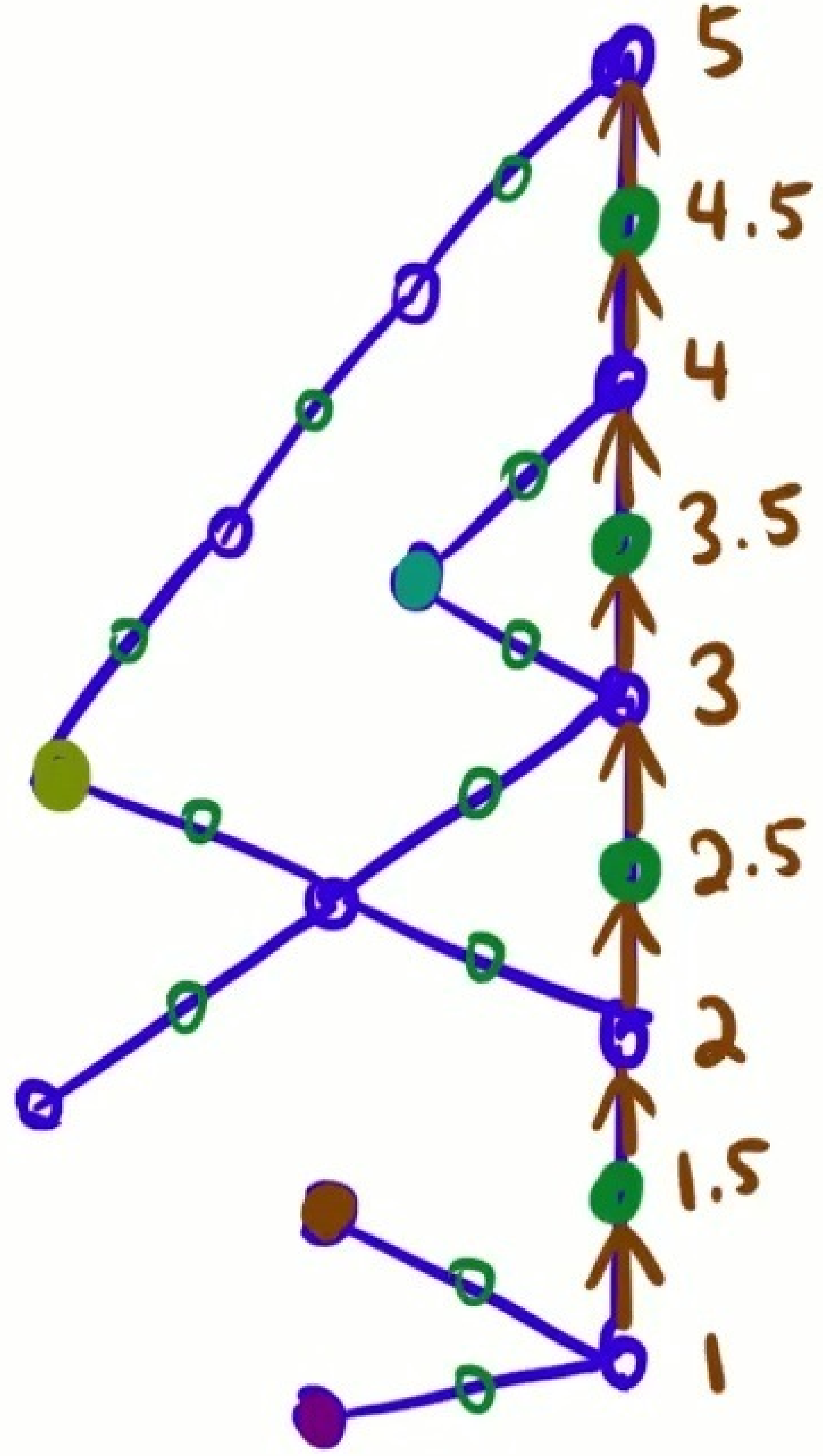
↓	↓	↓
↑	↑	↓
↓	↑	↑
↓	↑	↓

Sauer's Lemma:
If k columns and VC dim $< d$, then $O(k^{d-1})$ distinct rows.

$|\mathcal{L}| = O(Dk^3)$



$$
\begin{array}{c|cccccccc}
(2) & \uparrow & \uparrow & \downarrow & \uparrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(3) & \downarrow & \uparrow & \uparrow & \uparrow & \downarrow & \uparrow & \downarrow & \downarrow \\
(5) & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
(5) & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
\end{array}
$$

$\le O(D)$ choices

How many distinct strings?

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$$
\begin{array}{ccc}
\uparrow & \downarrow & \uparrow \\
\downarrow & \downarrow & \uparrow \\
\uparrow & \downarrow & \downarrow \\
\uparrow & \uparrow & \uparrow \\
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$\ll 2^k$

$|\mathcal{L}| = O(Dk^3)$

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VC dim < 4.

$\Longrightarrow O(k^3)$ distinct rows

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Pf: suppose not: $\exists\ 2^4 \times 4$ submtx.

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Then, there is
2x4 submtx $\begin{matrix}\uparrow & \downarrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \downarrow & \uparrow\end{matrix}$
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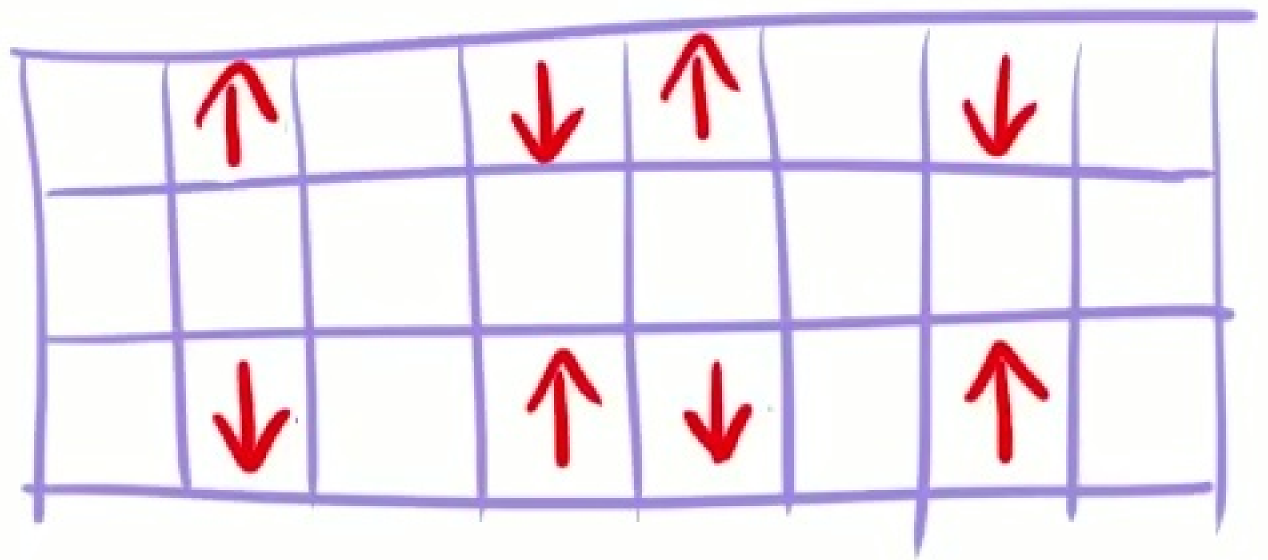

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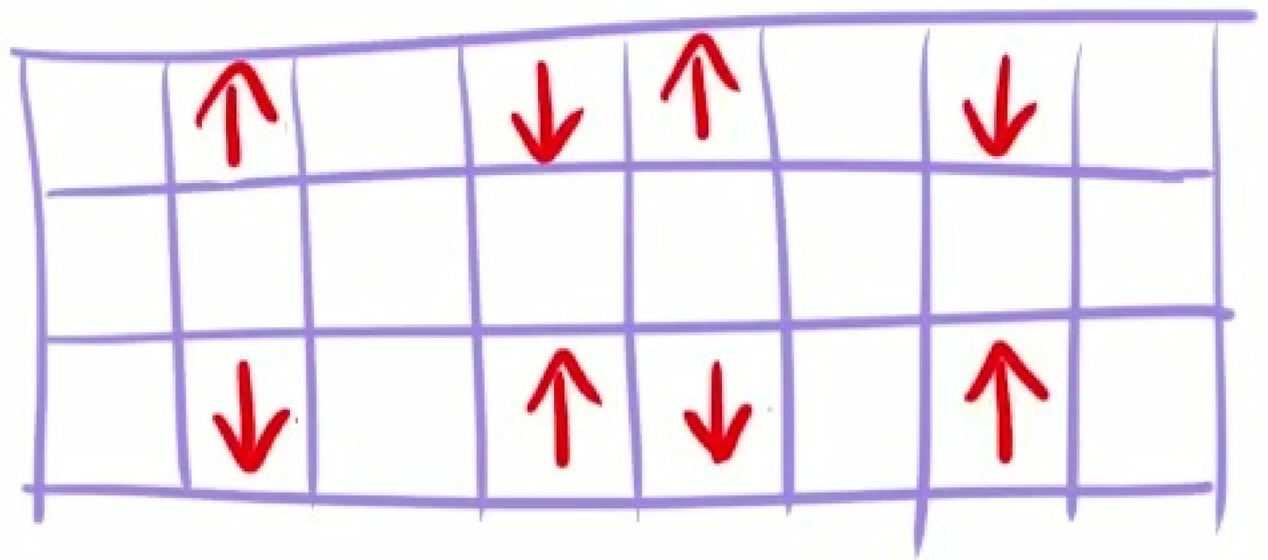

To show: violates planarity!

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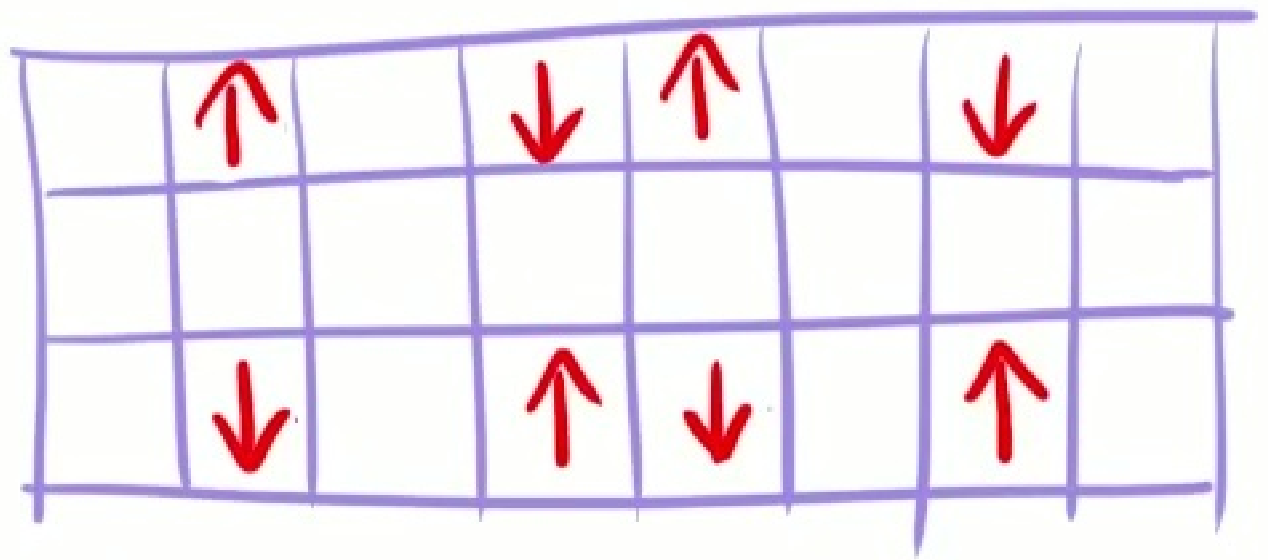

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violates <u>Monge property</u>

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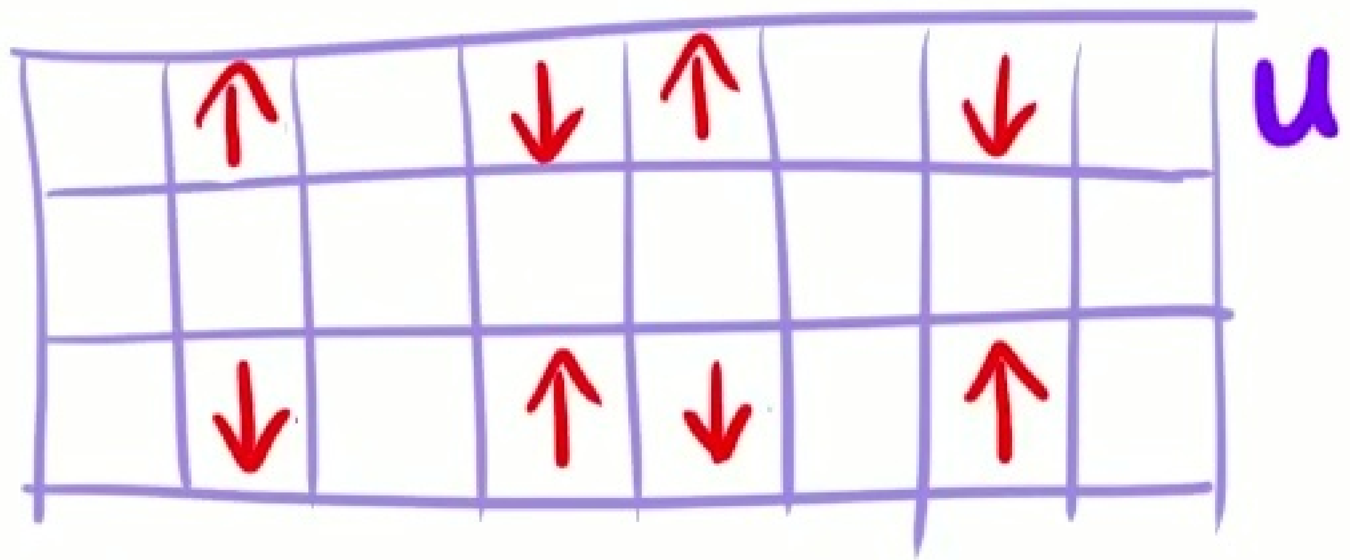

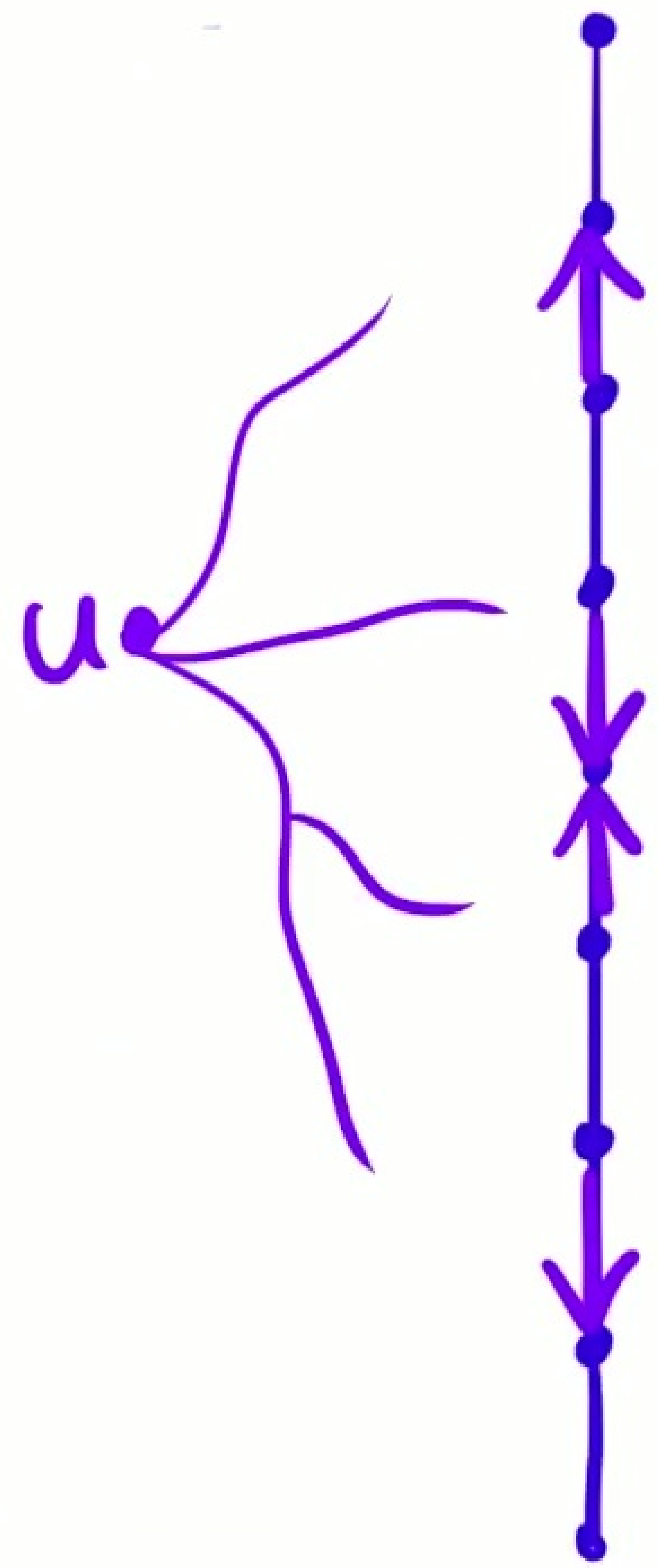

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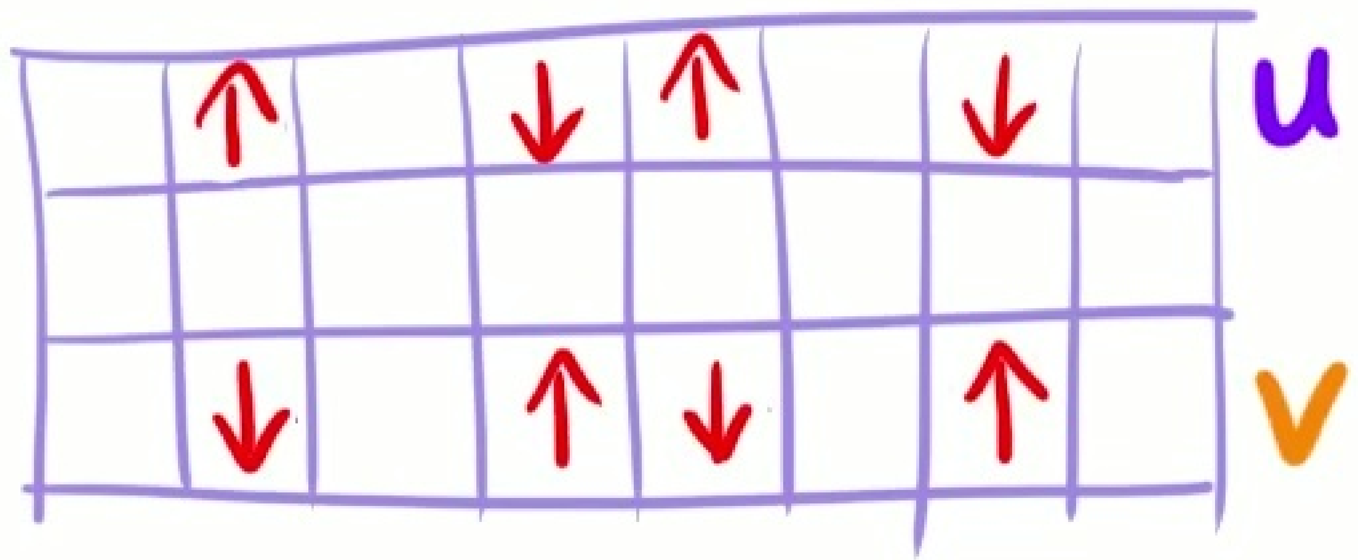

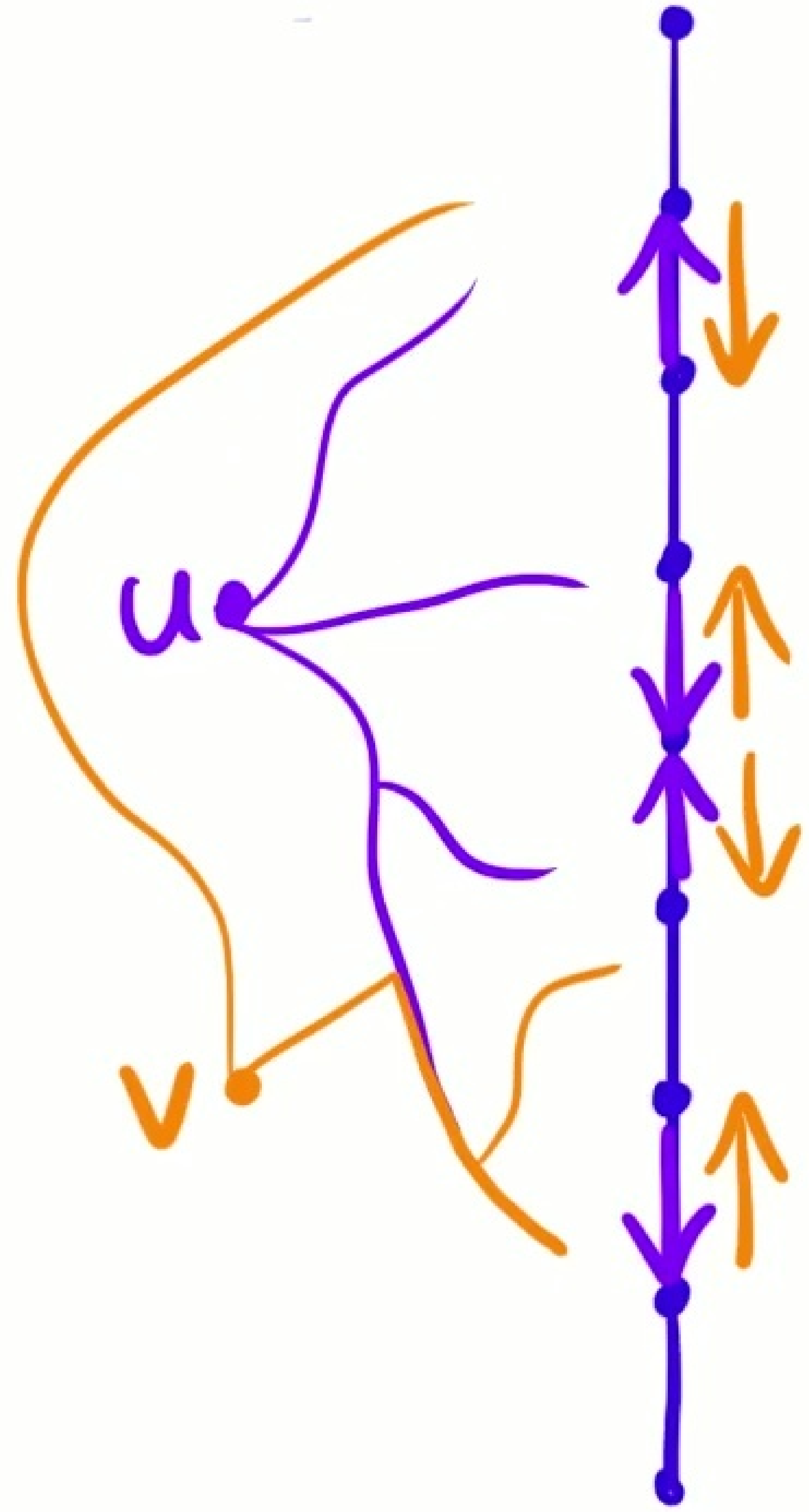

To show: violates planarity!

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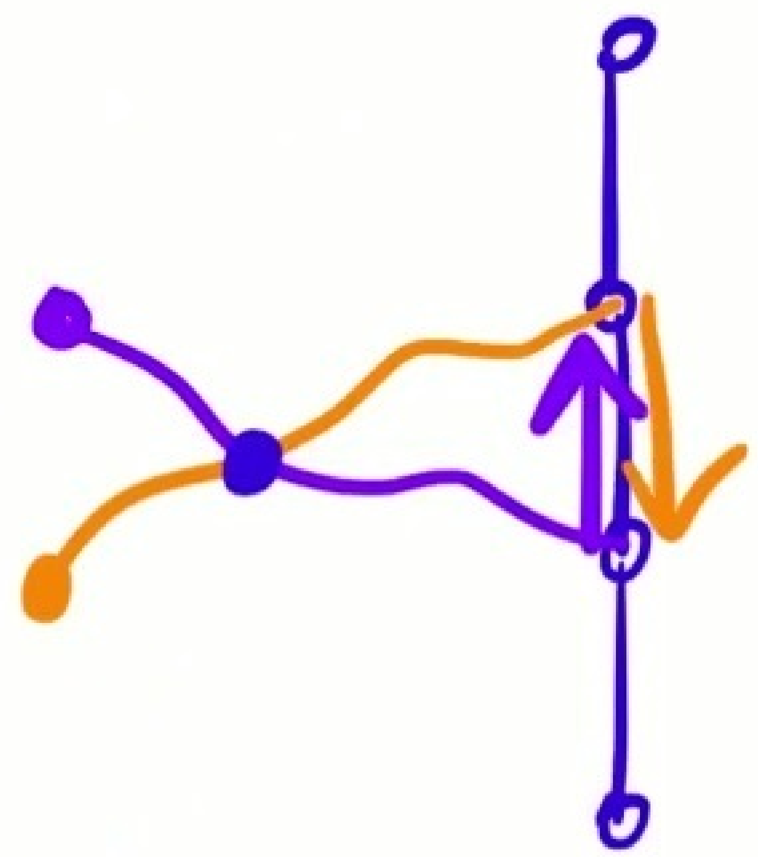
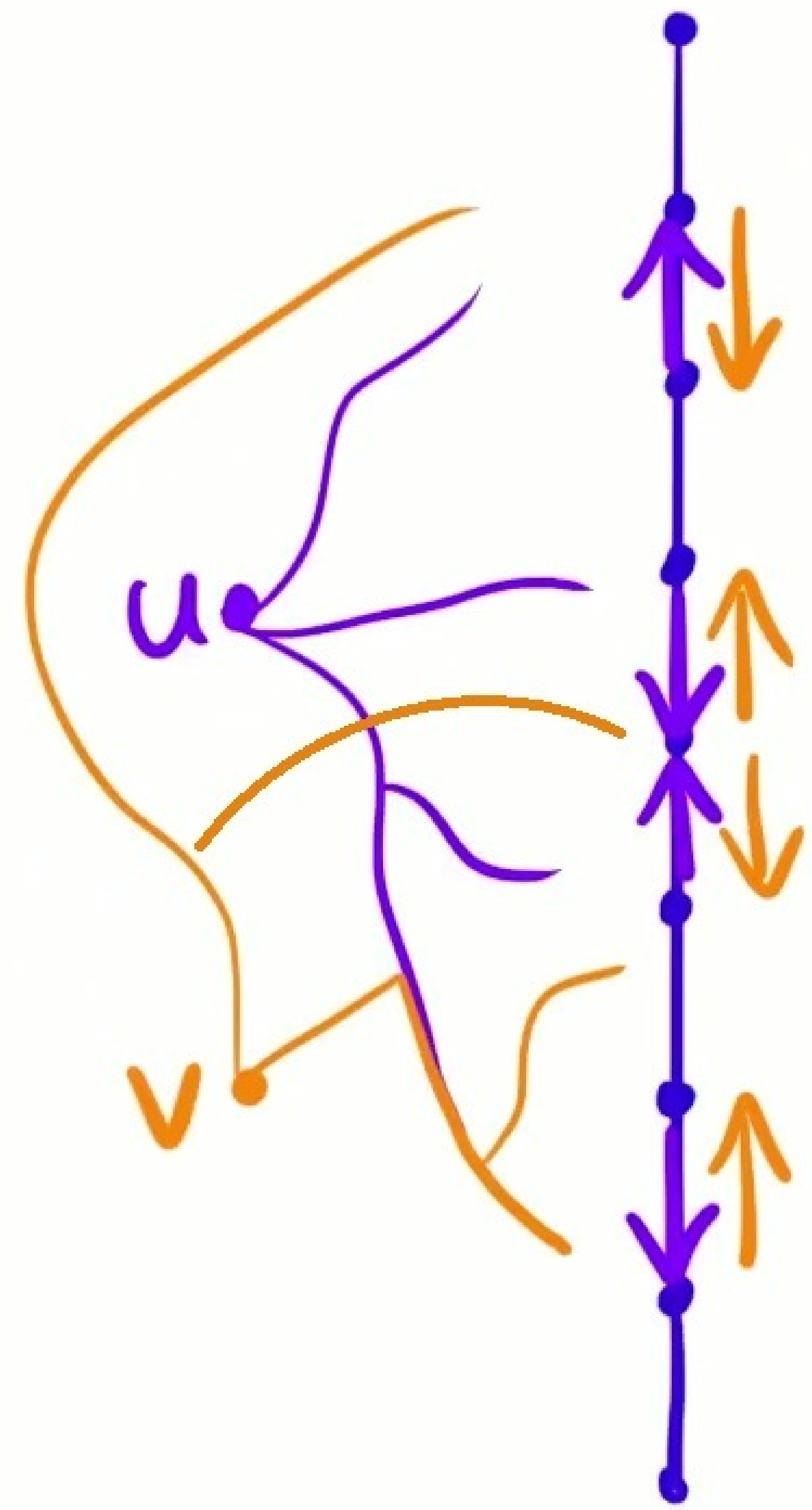
Monge property:

Cannot have:





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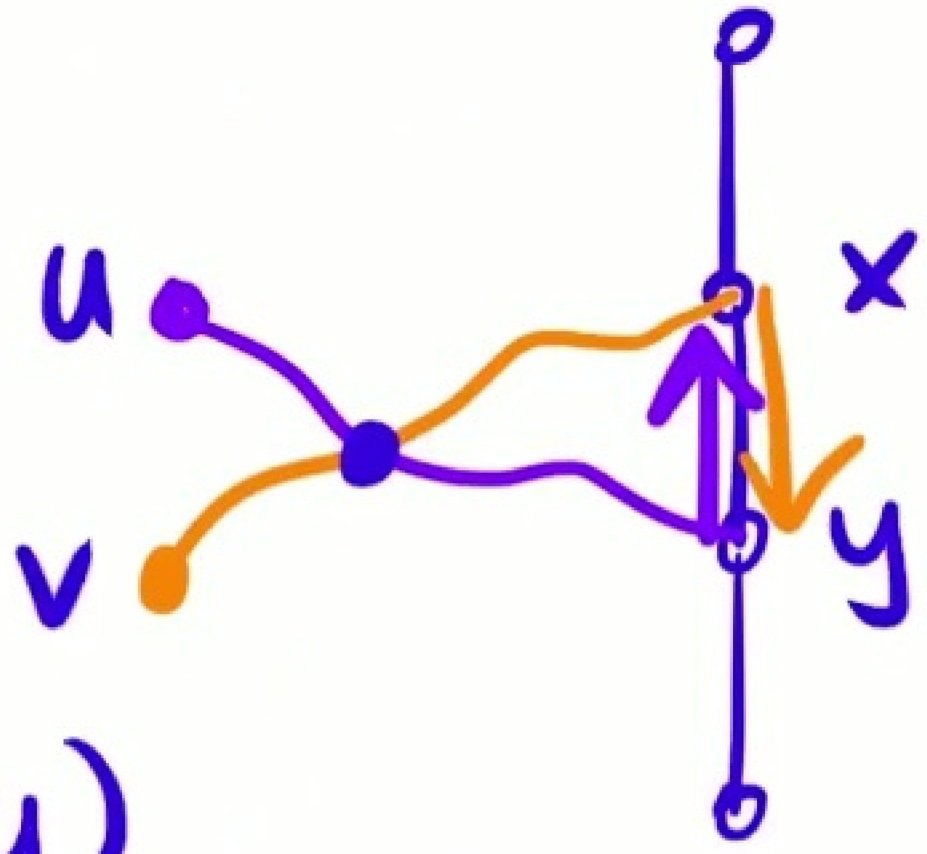
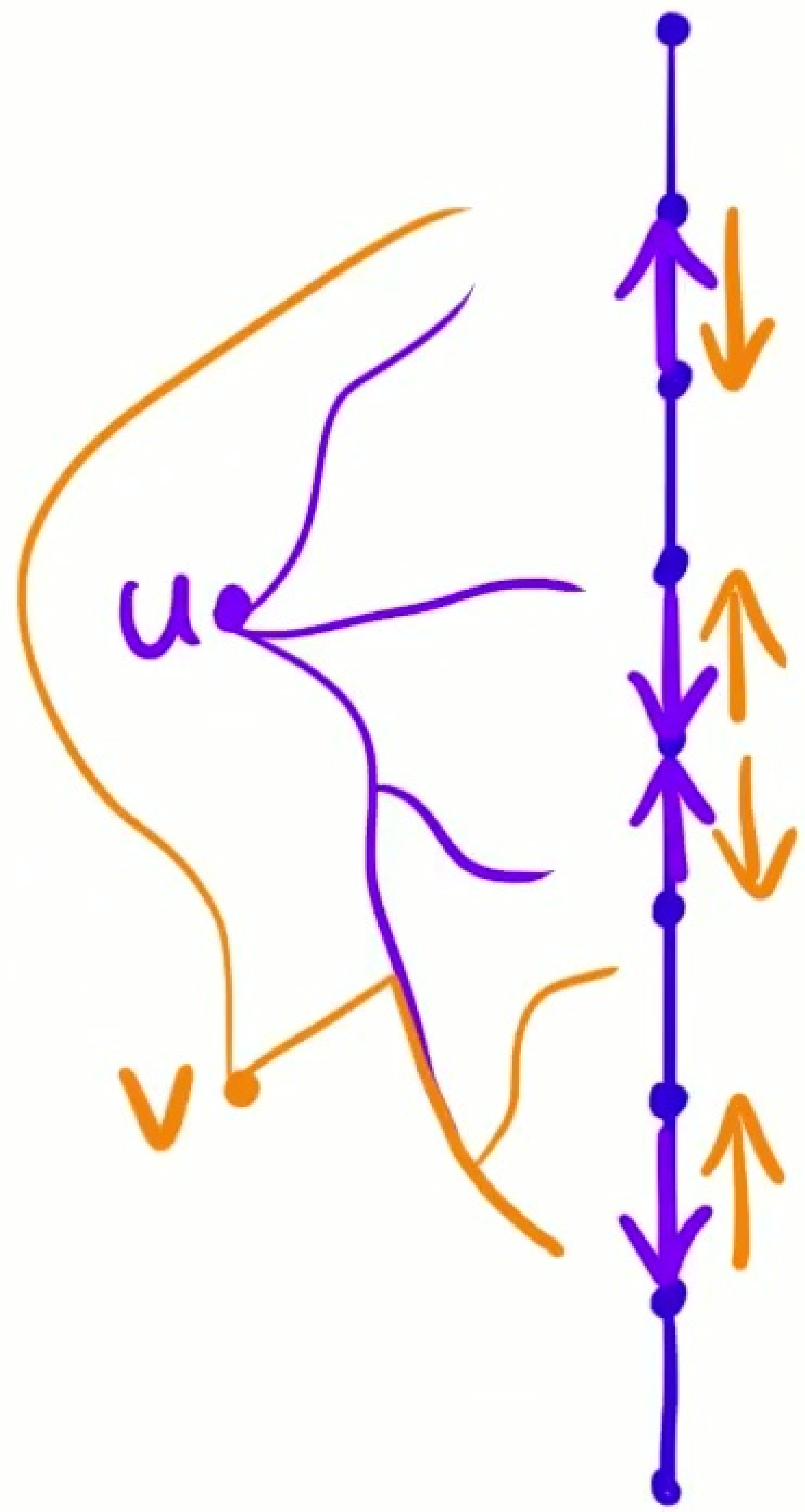


Pf: $d(u,x) > d(u,y)$

$d(v,y) > d(v,x)$



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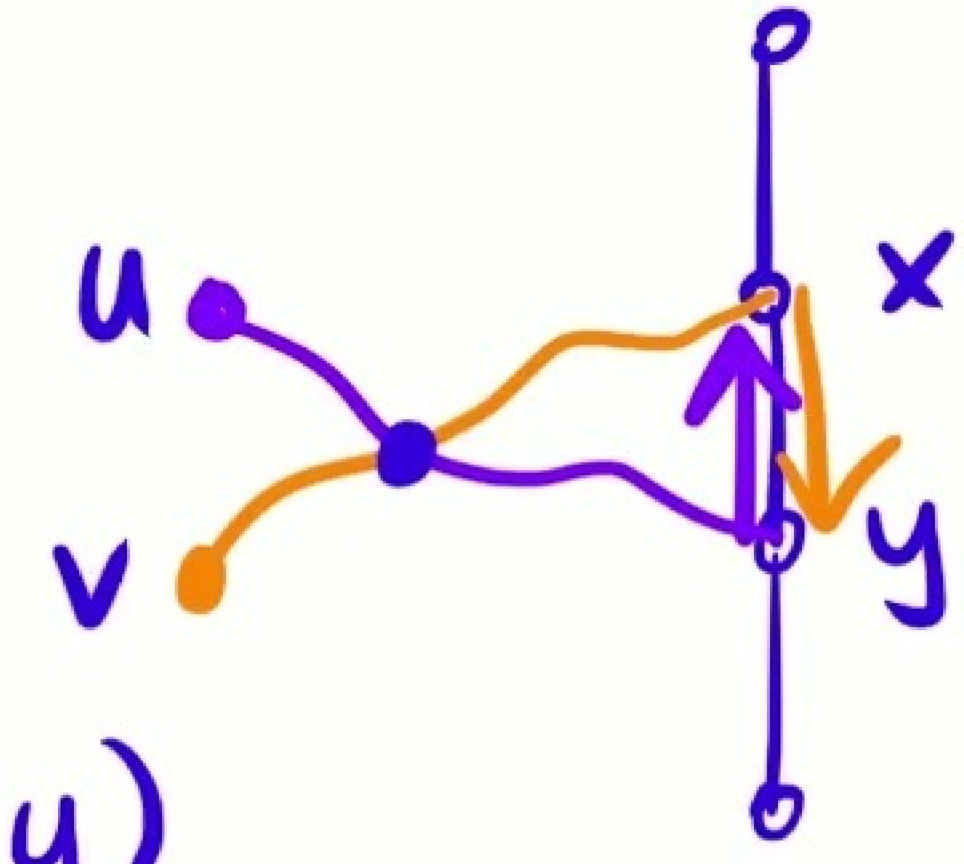
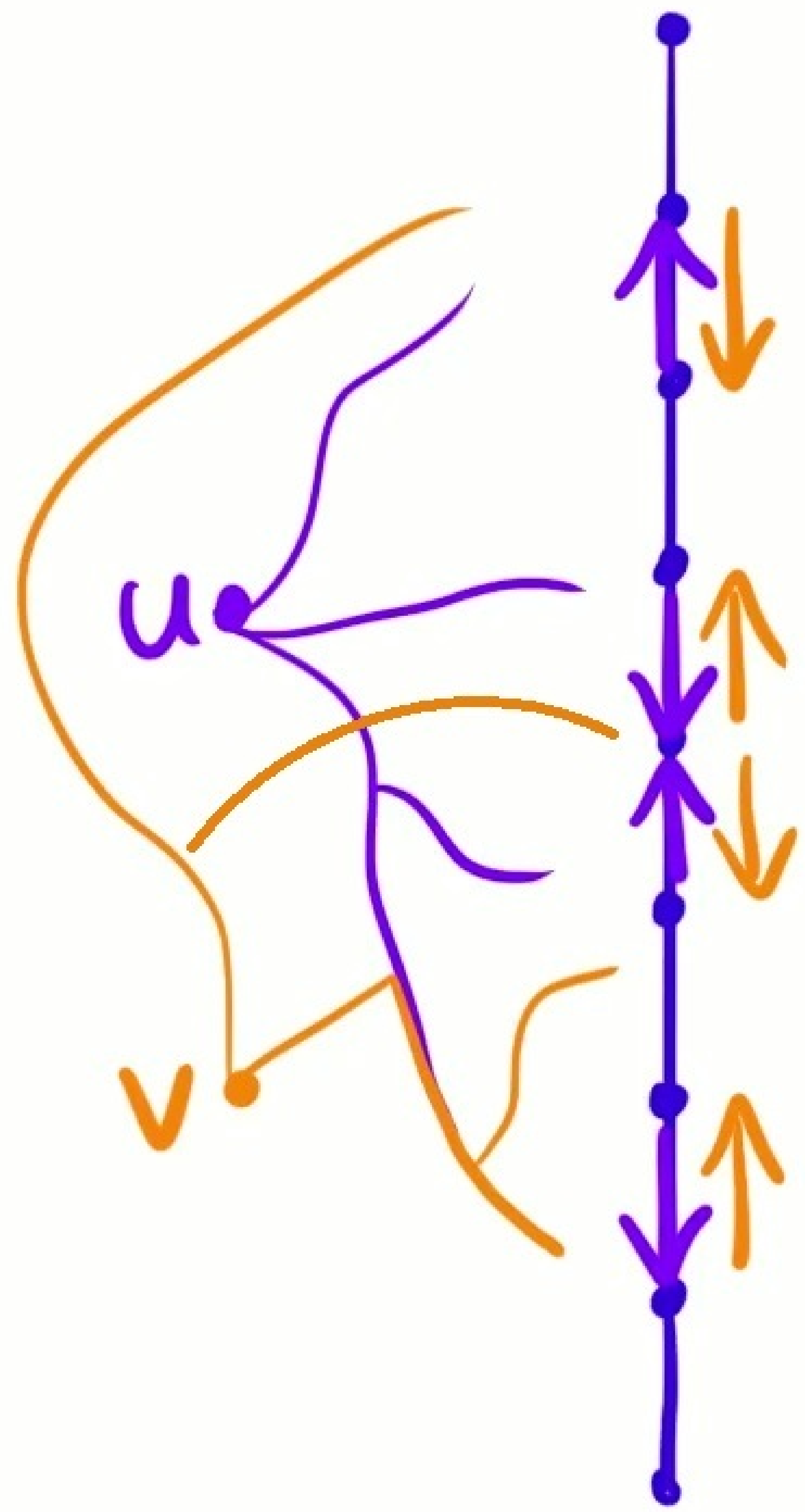




Pf: $d(u,x) > d(u,y)$

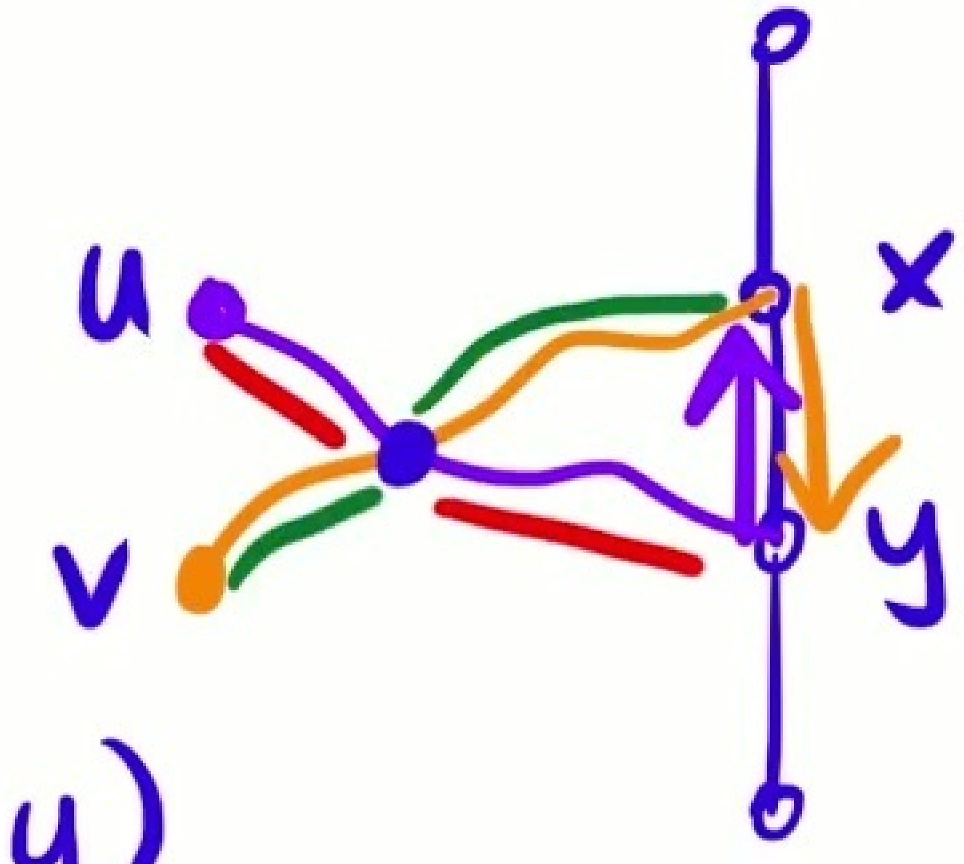
$d(v,y) > d(v,x)$

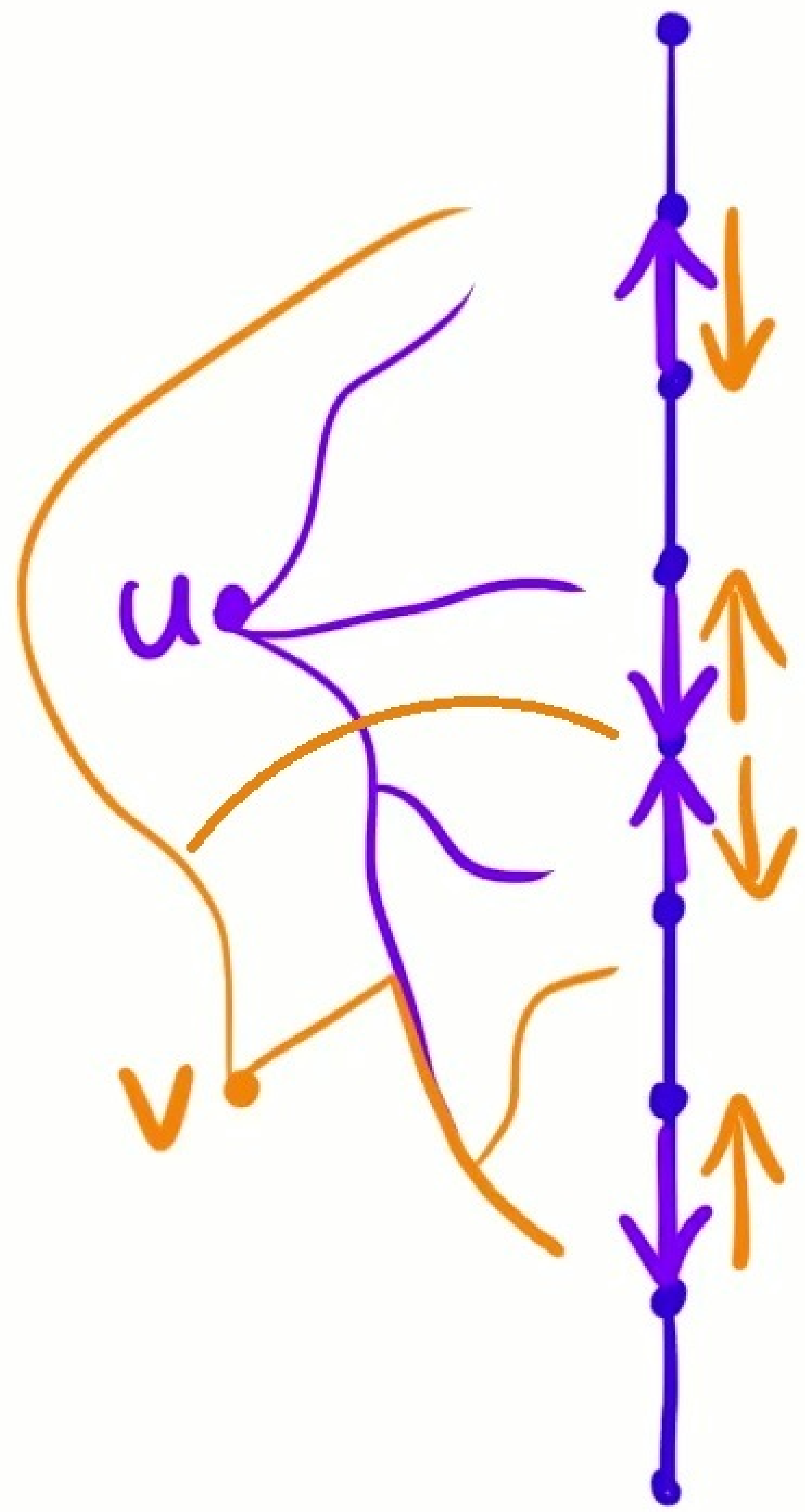
$\Rightarrow d(u,x) + d(v,y) > d(u,y) + d(v,x)$

$|\mathcal{L}| = O(Dk^3)$

Monge property:

Cannot have:





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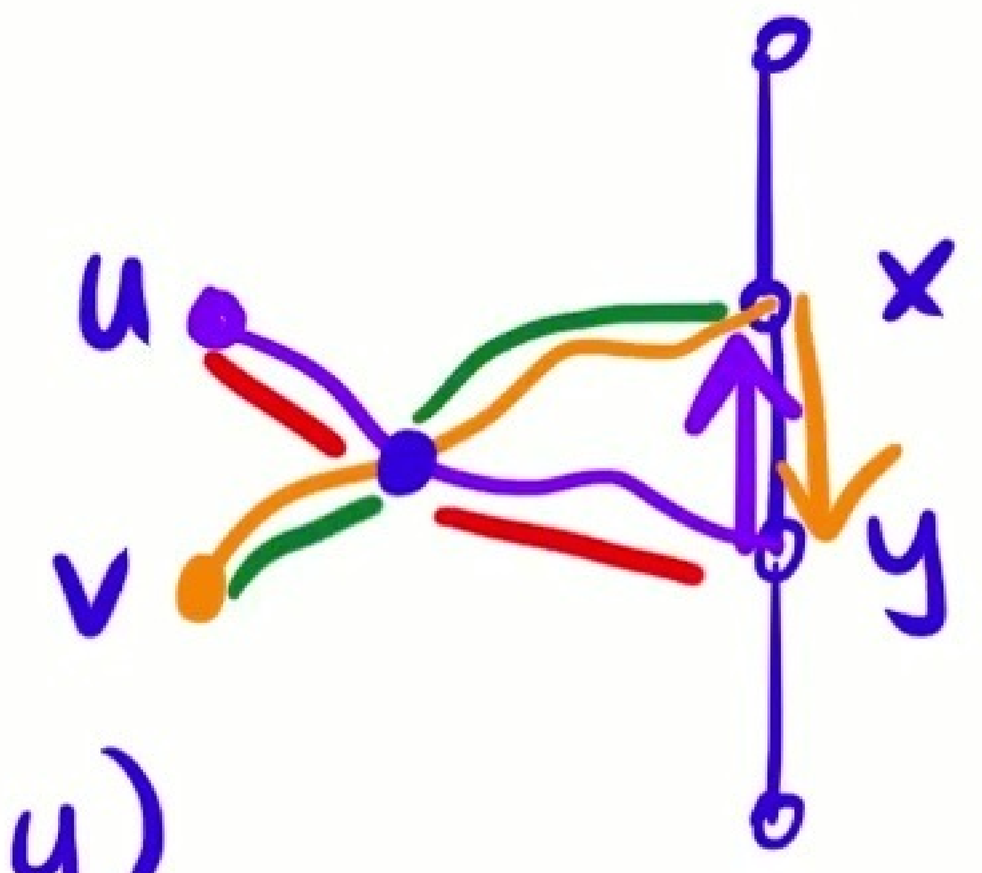
Cannot have:



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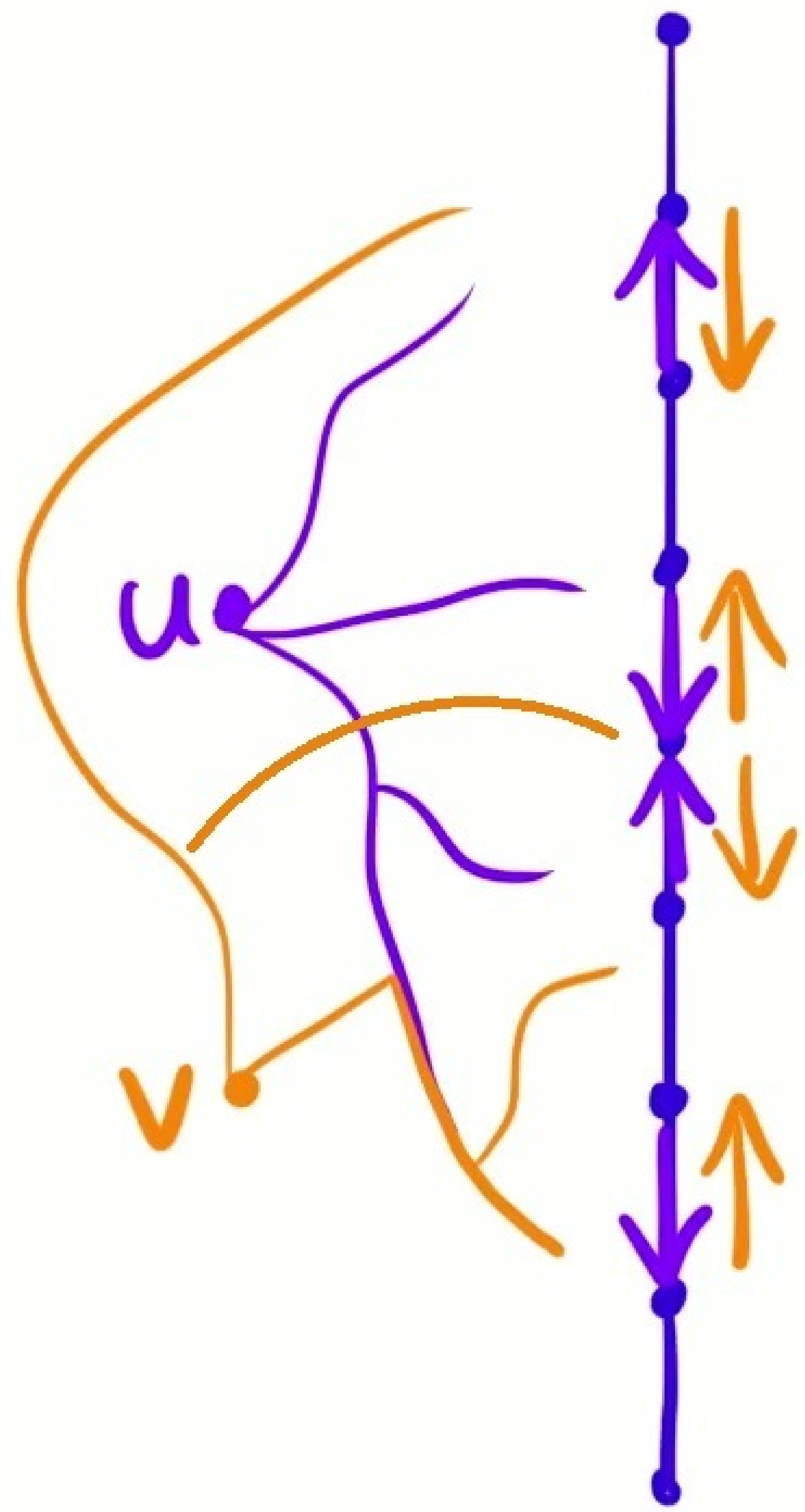
$\Rightarrow d(u,x) + d(v,y) > d(u,y) + d(v,x)$

Contradiction.



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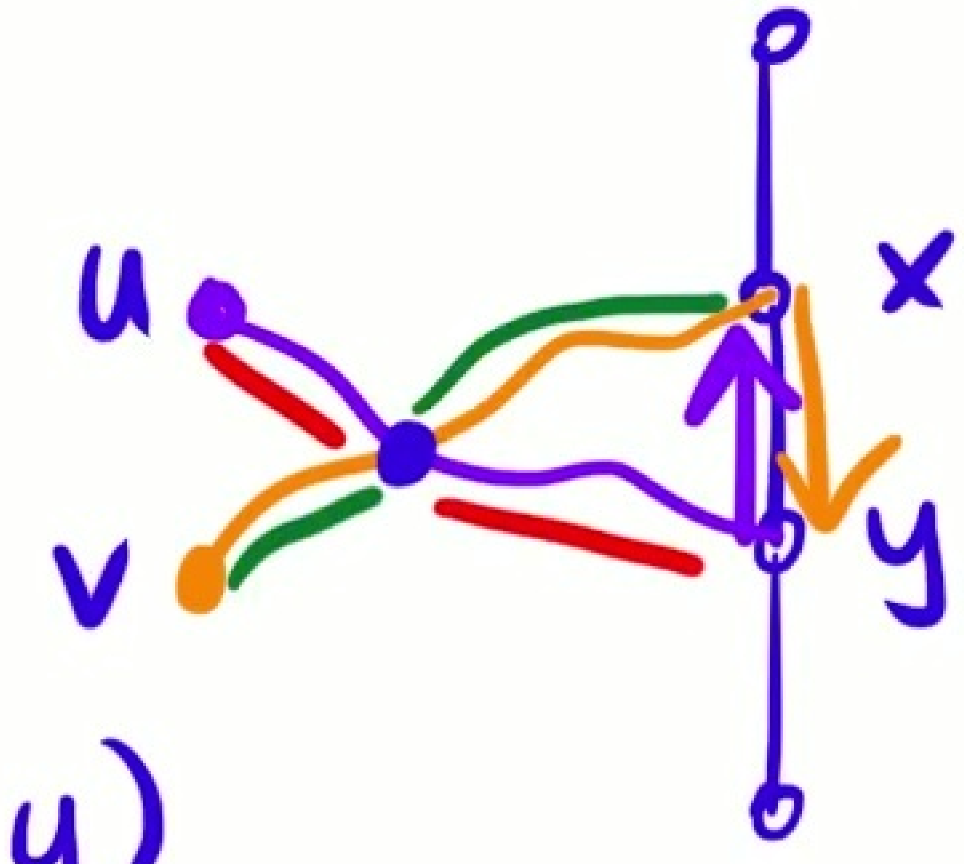
Cannot have:



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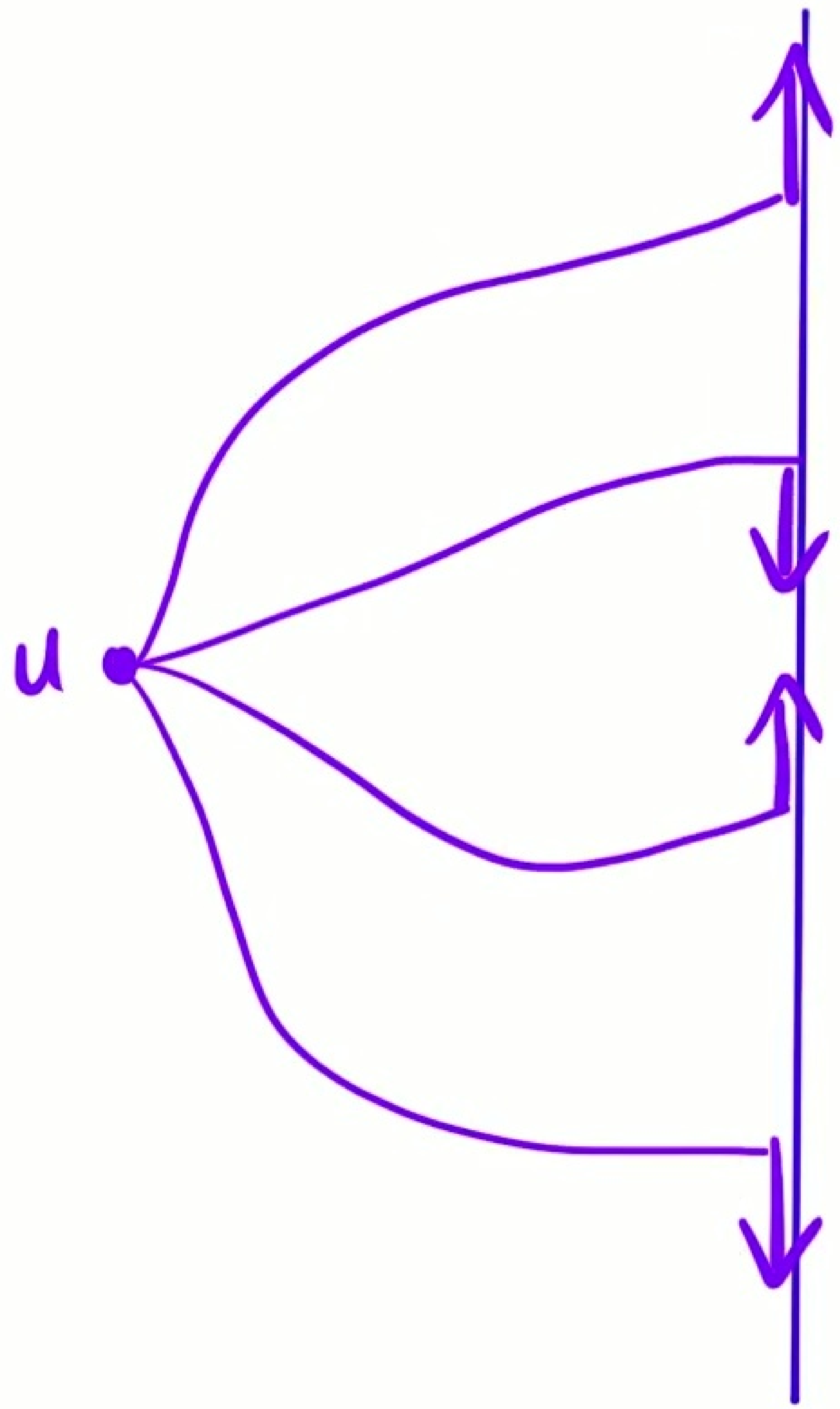

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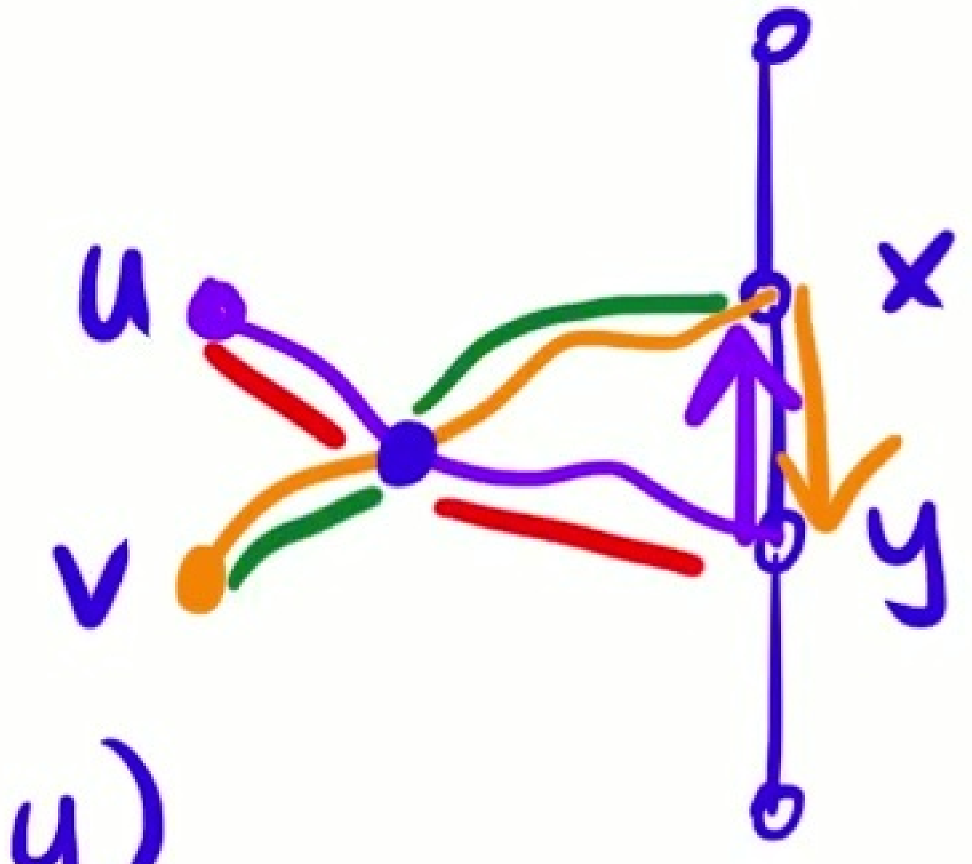
Cannot have:



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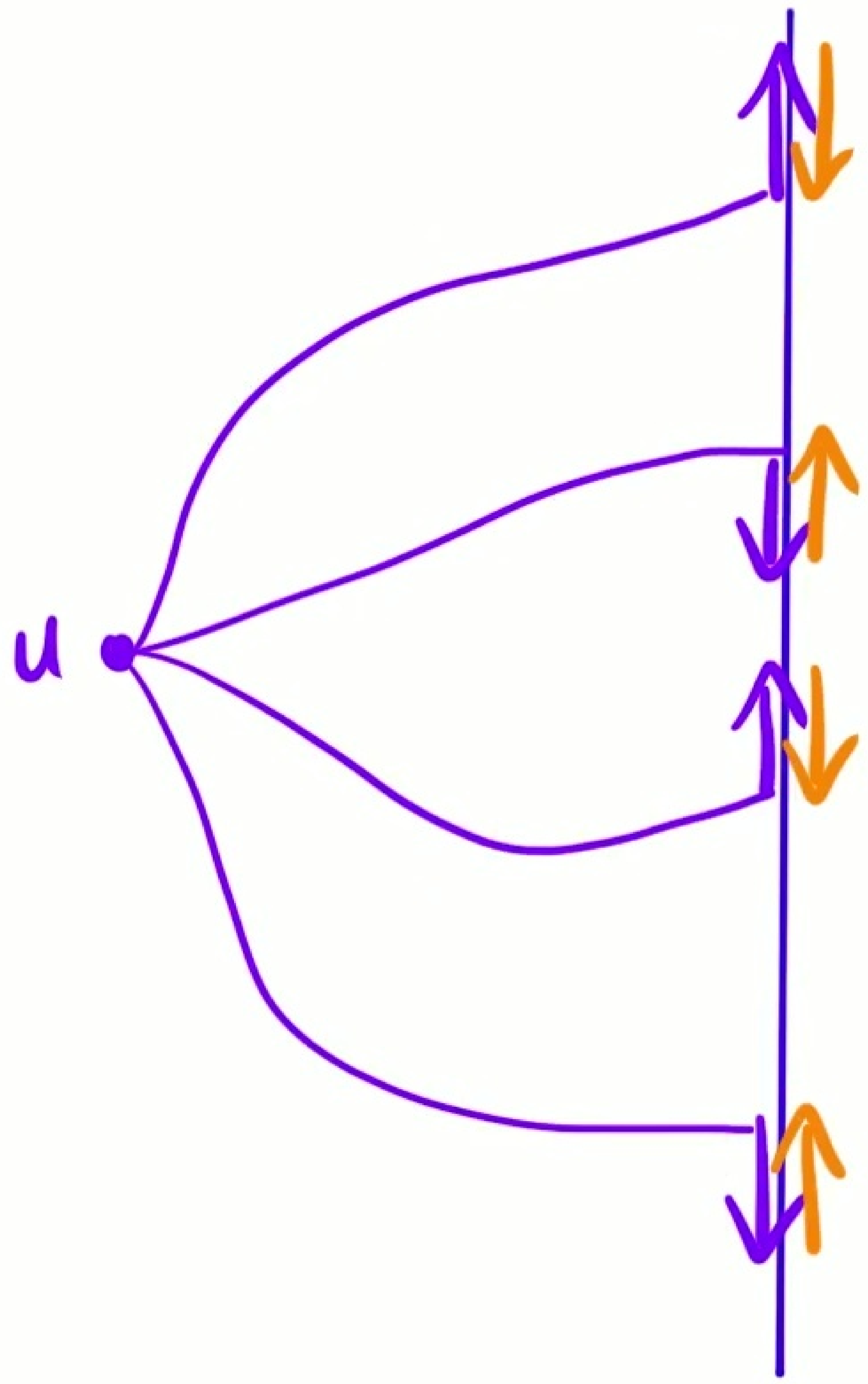

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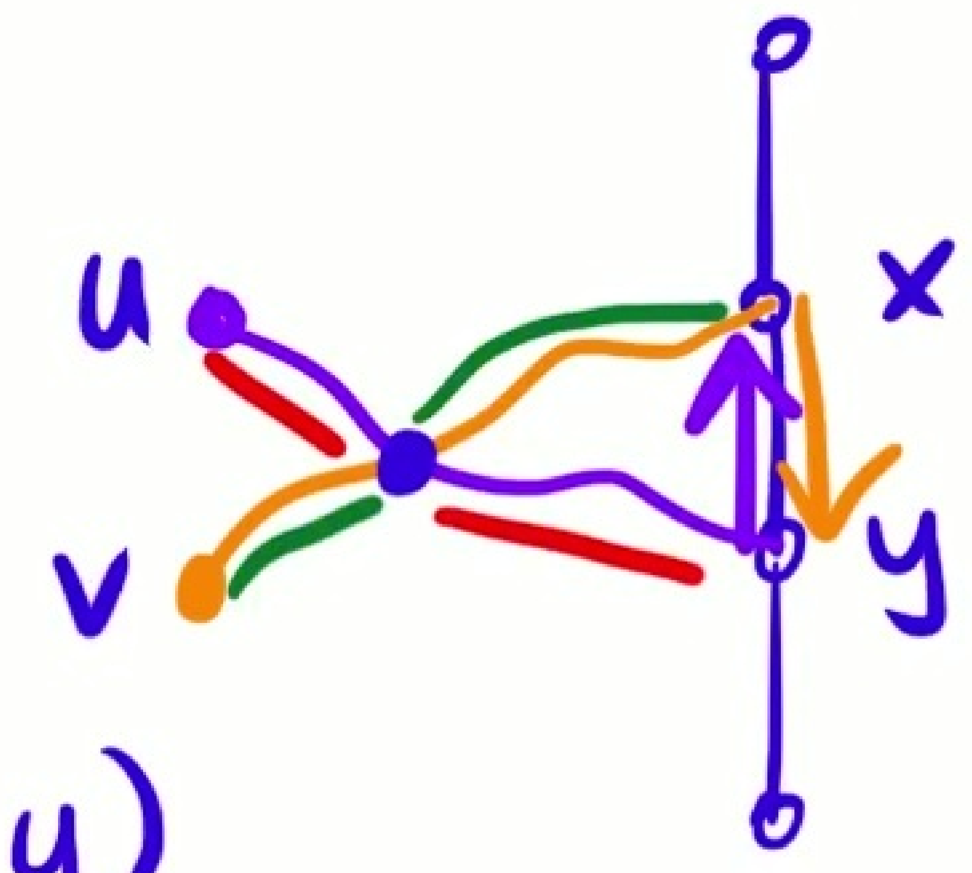

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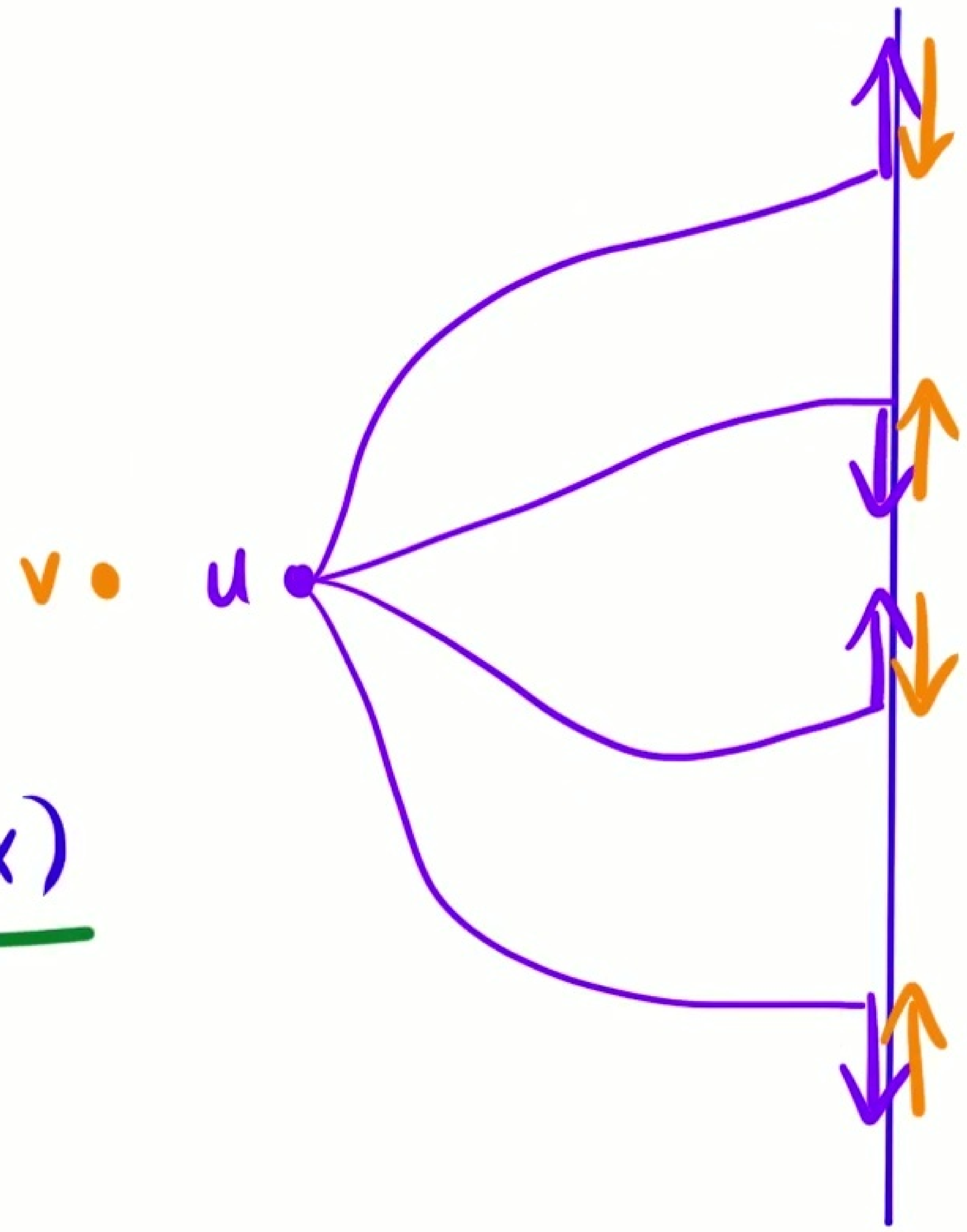

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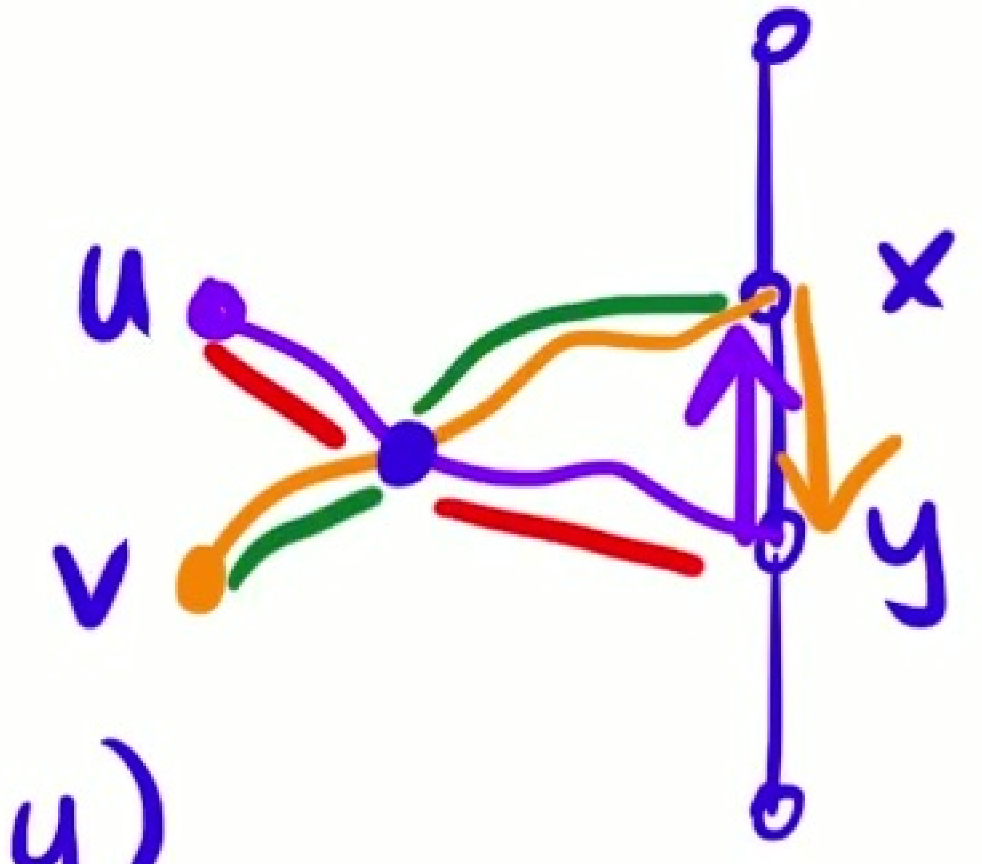

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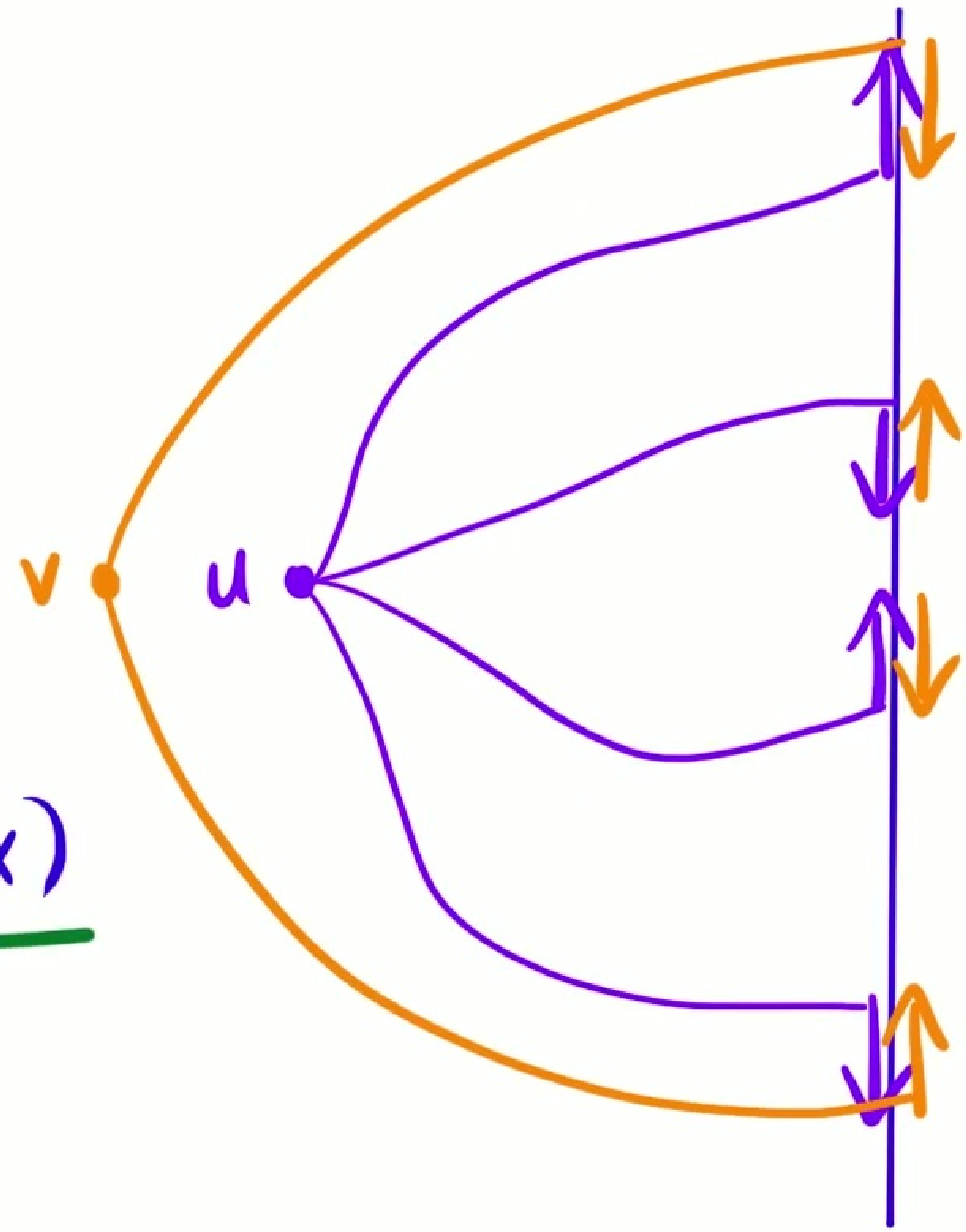

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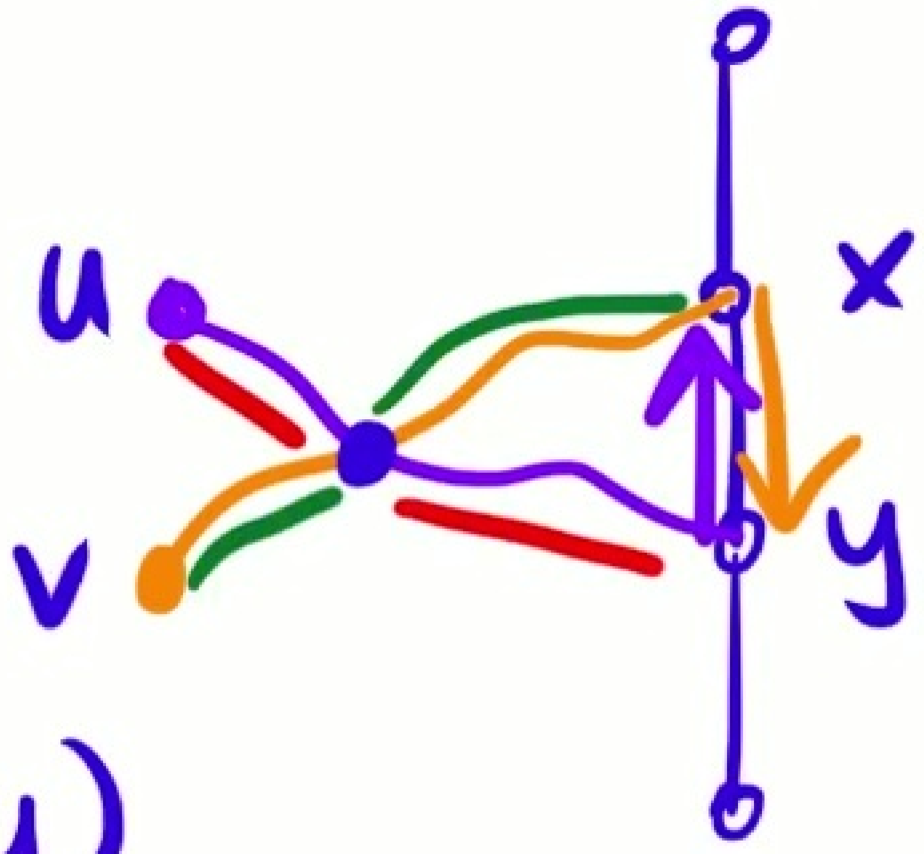

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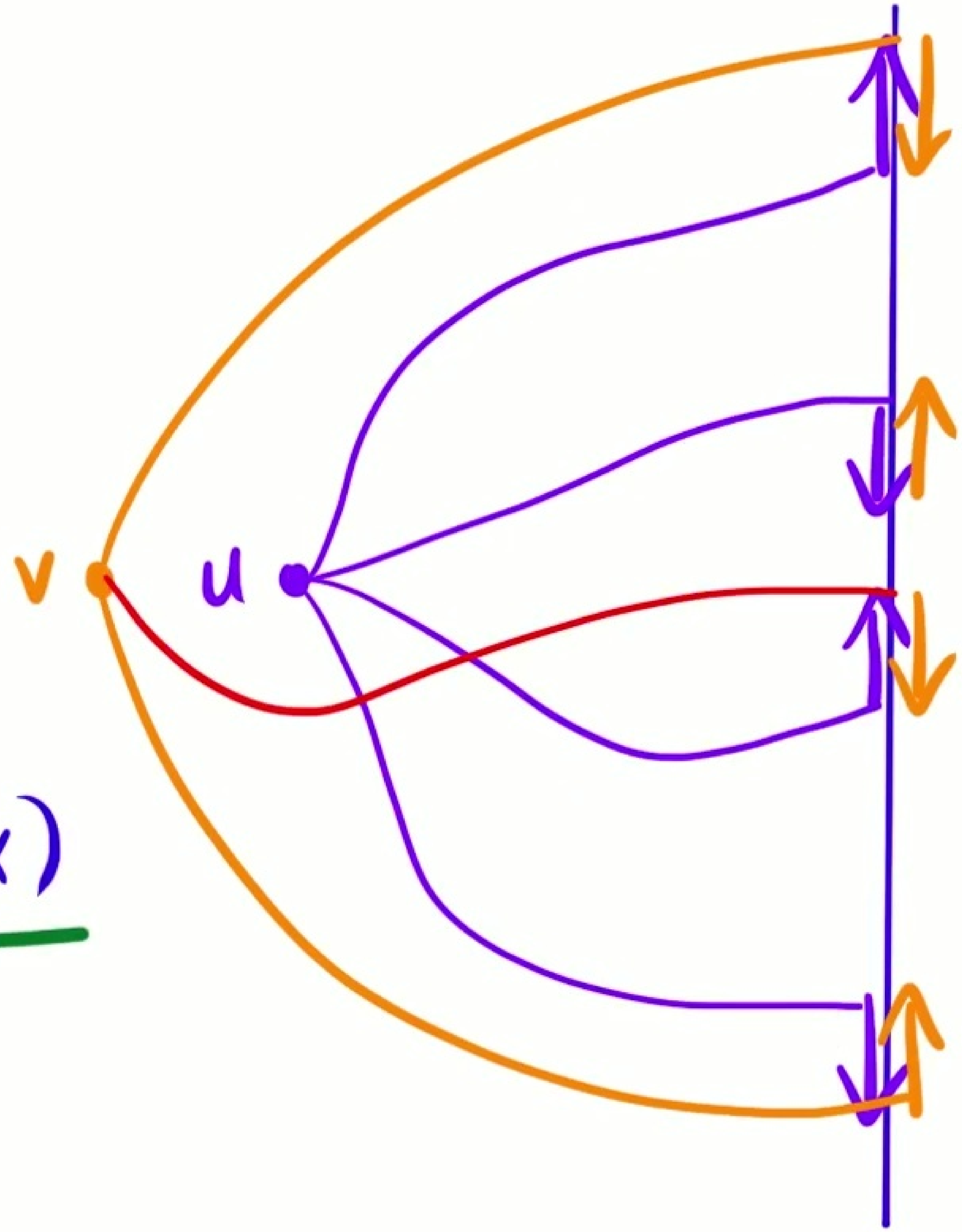

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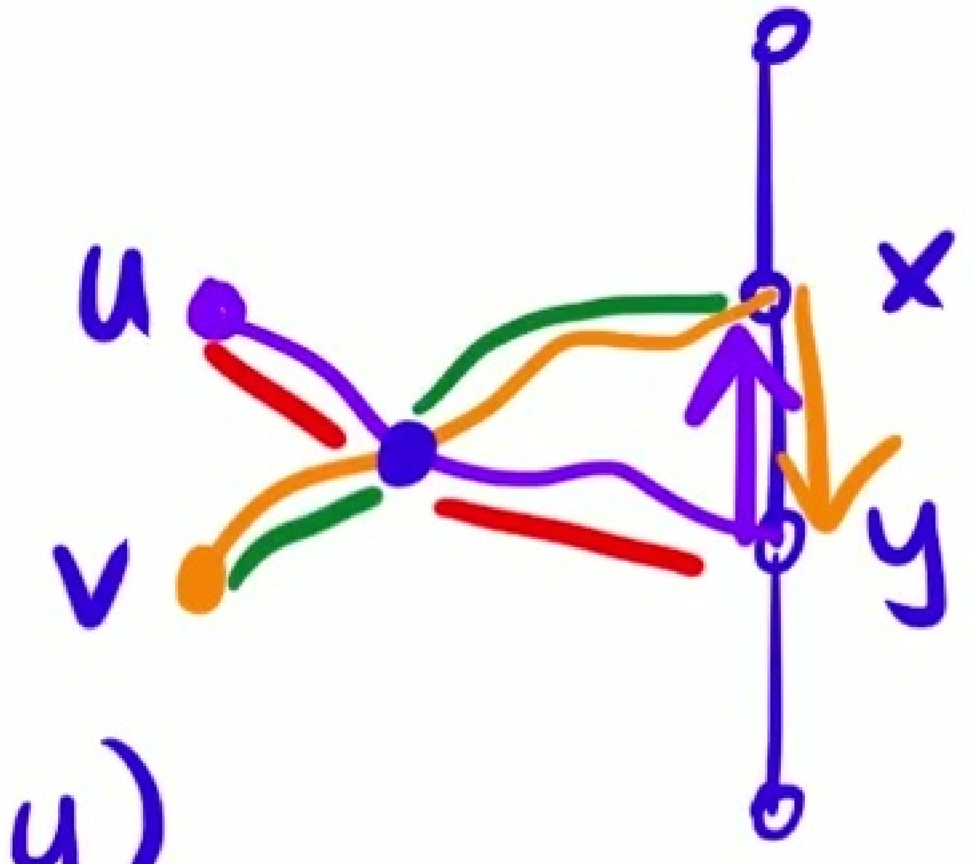

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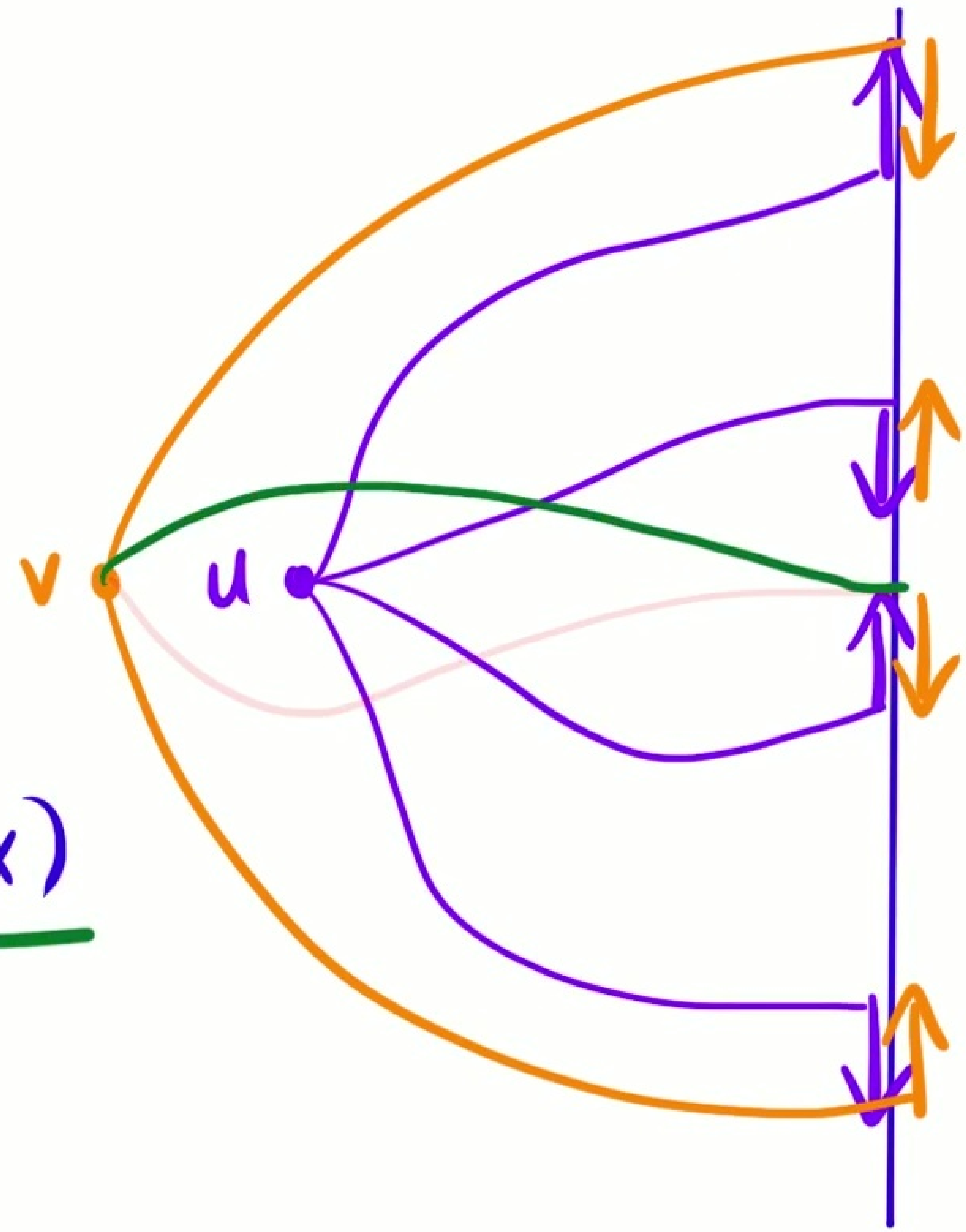

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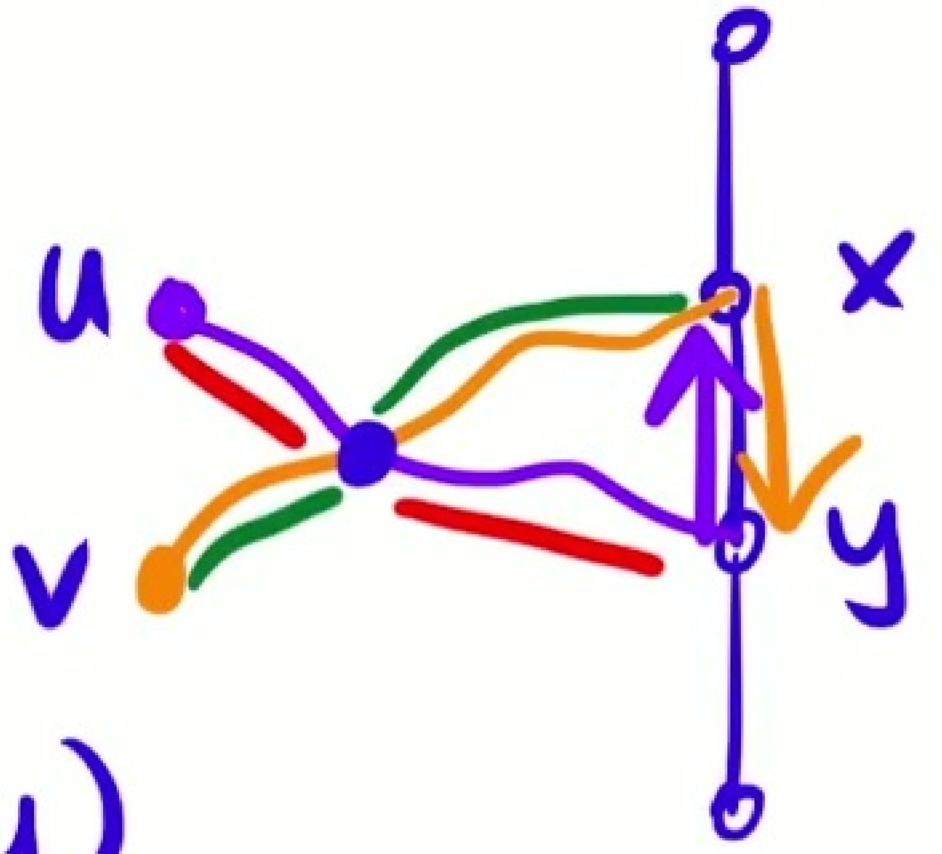

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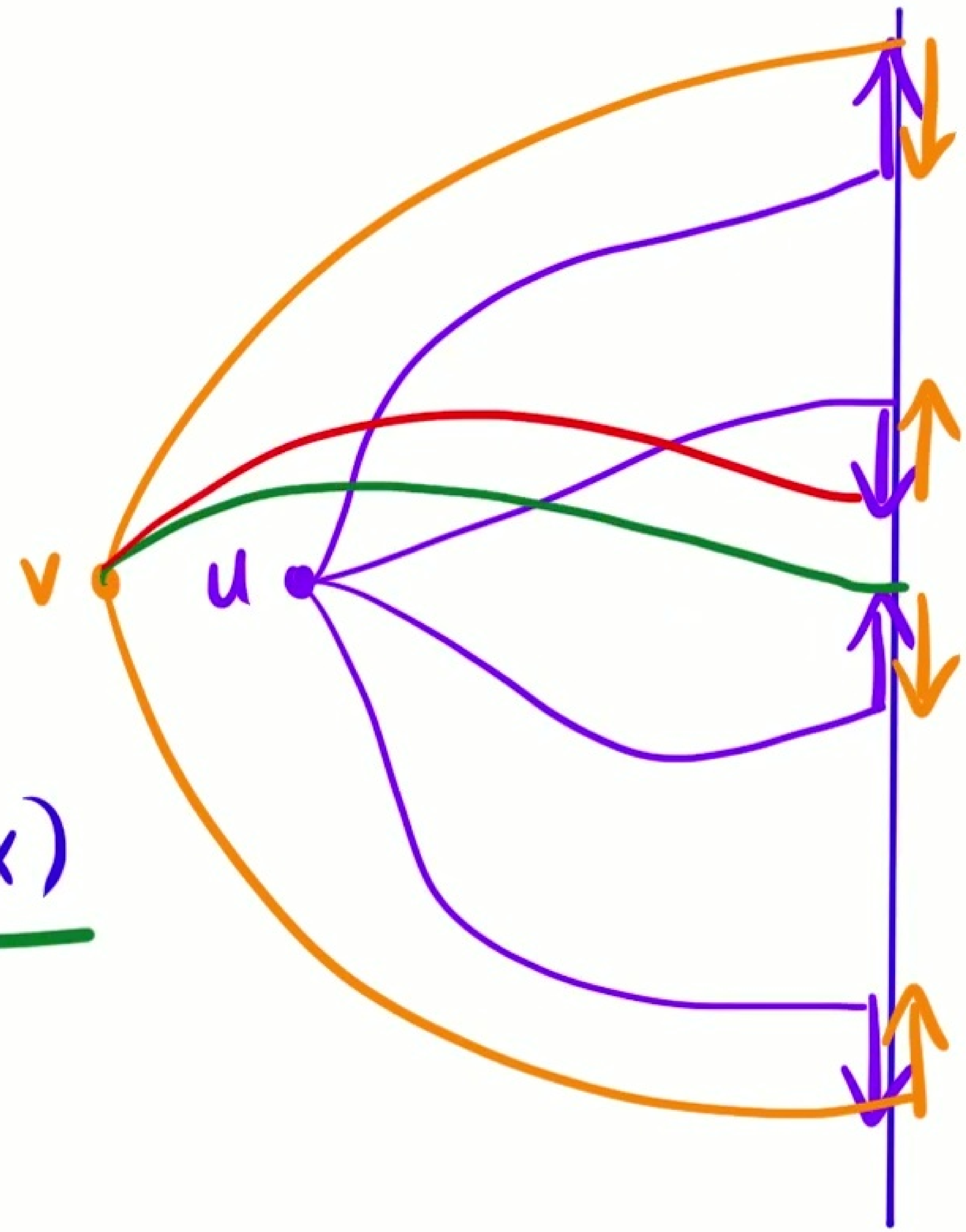

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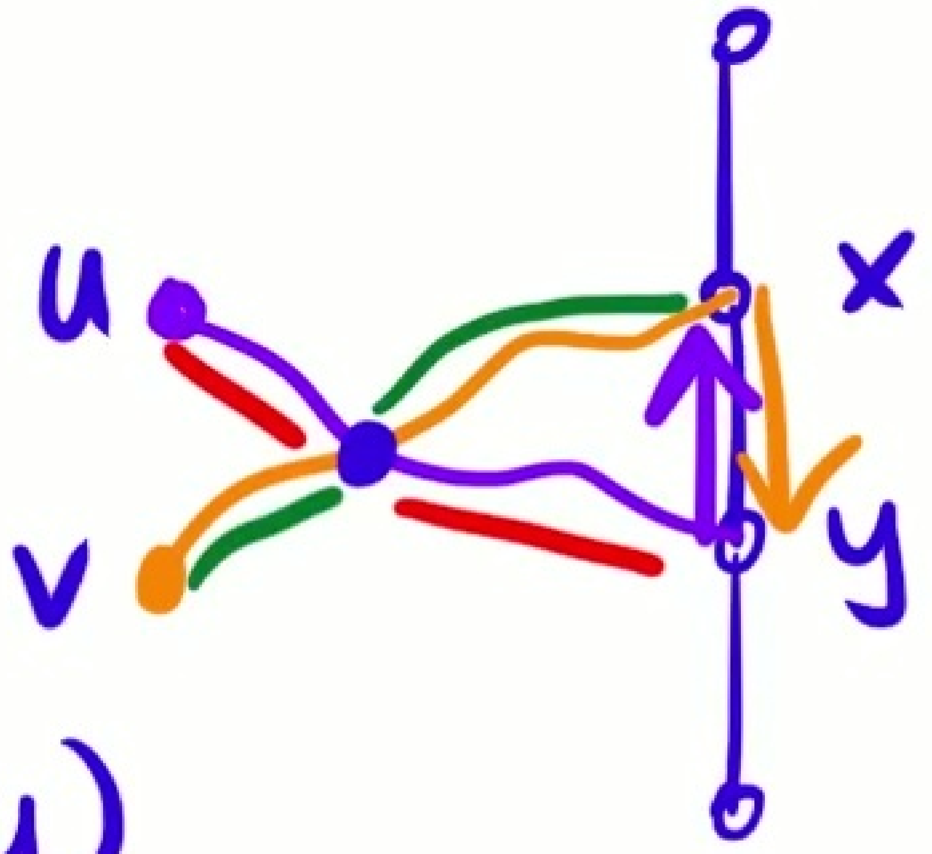

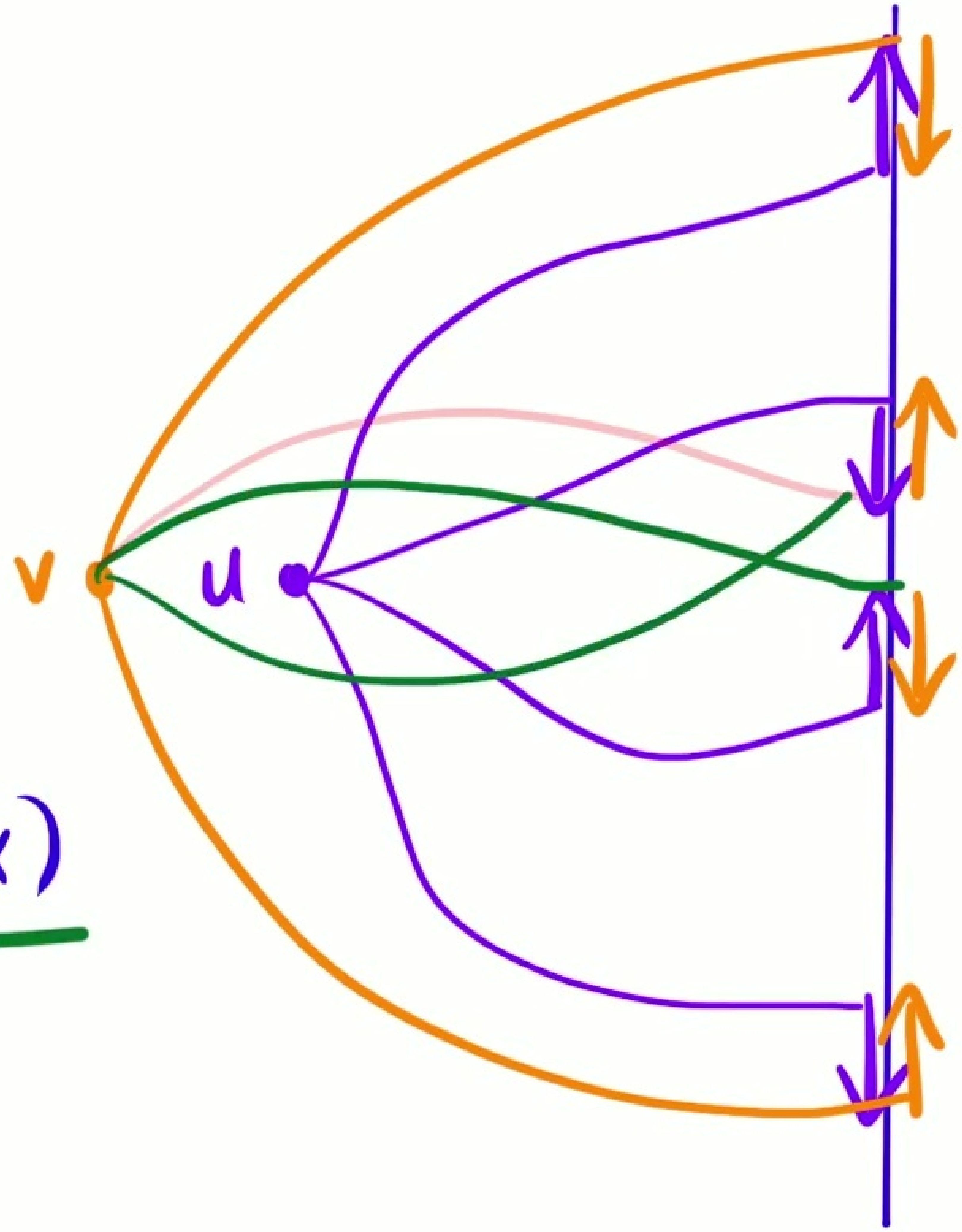

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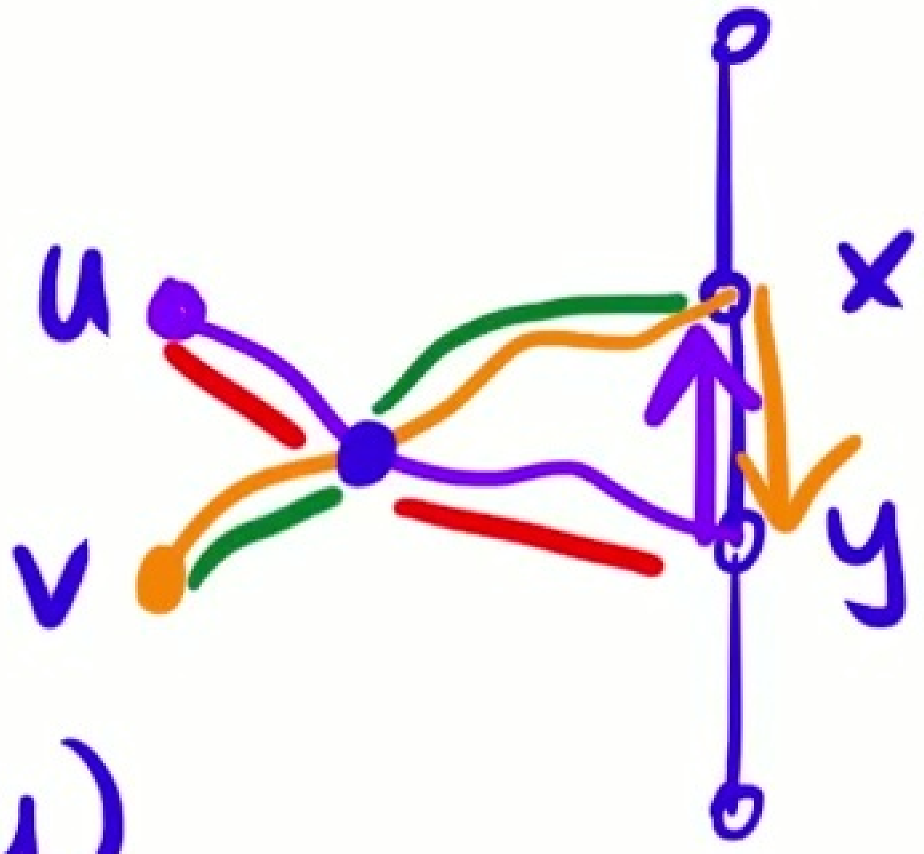

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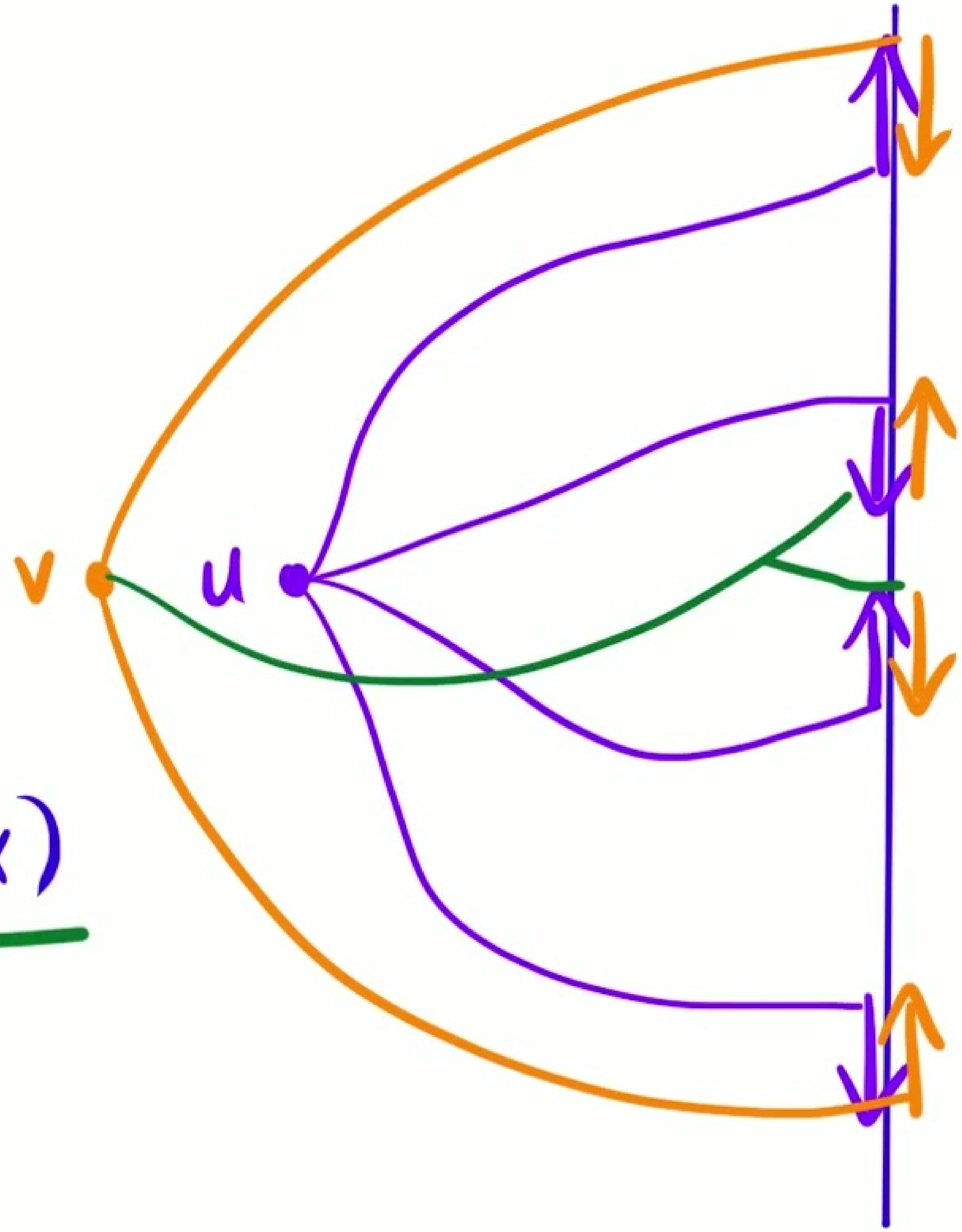

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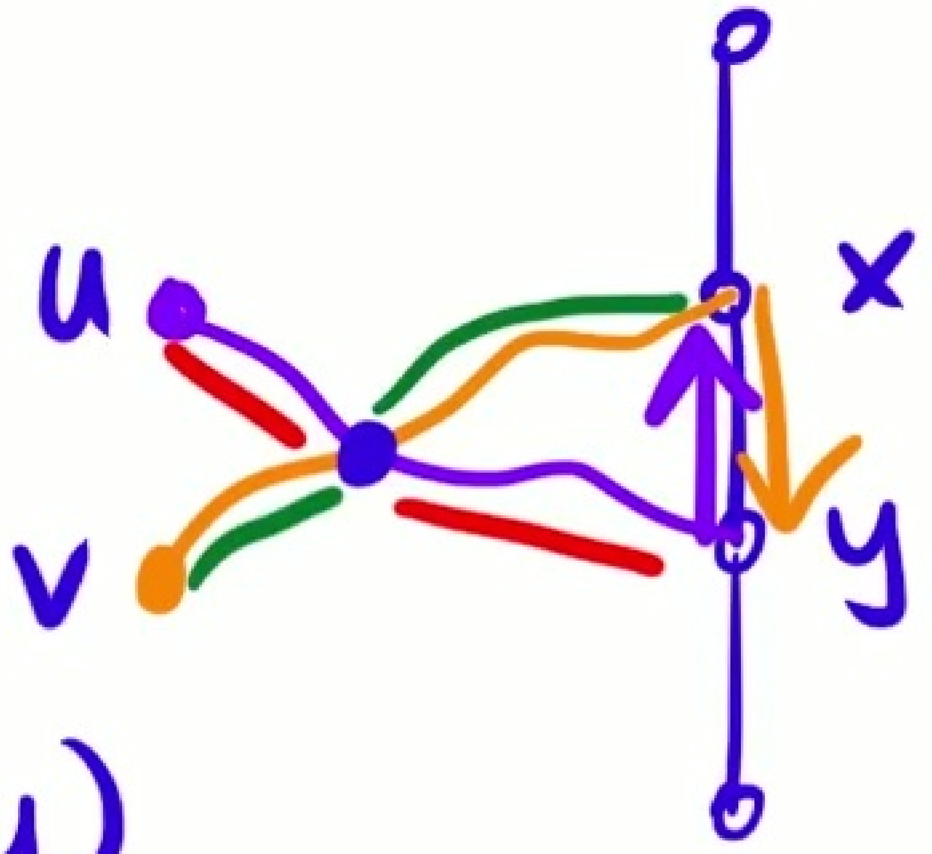

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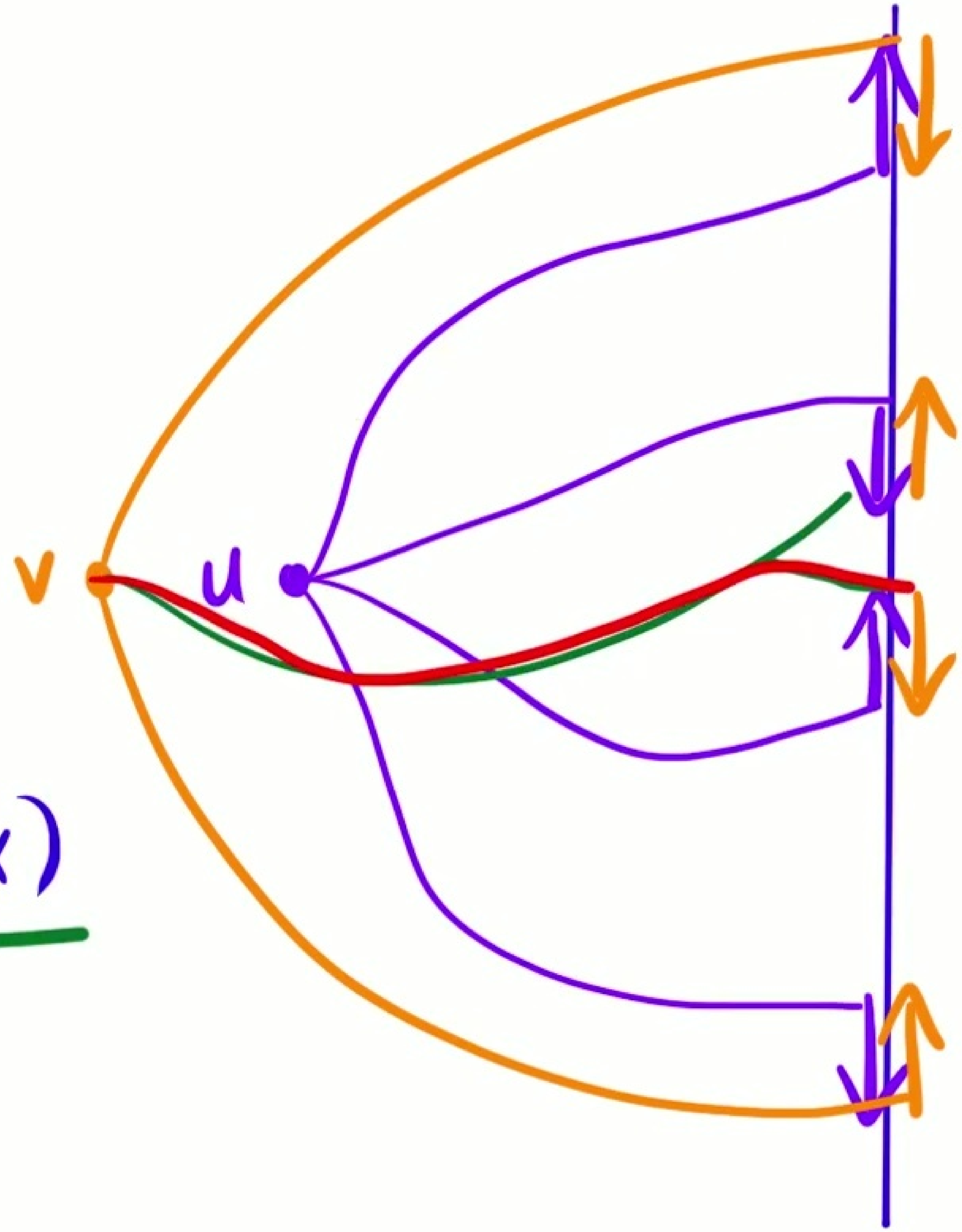

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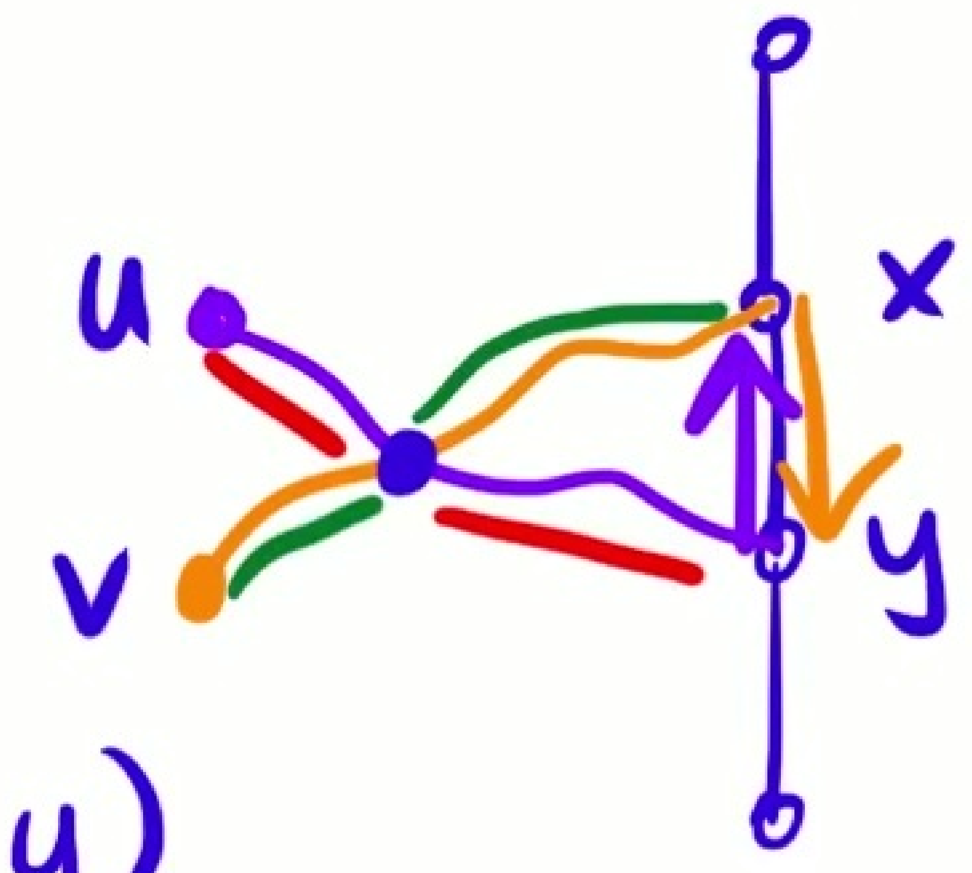

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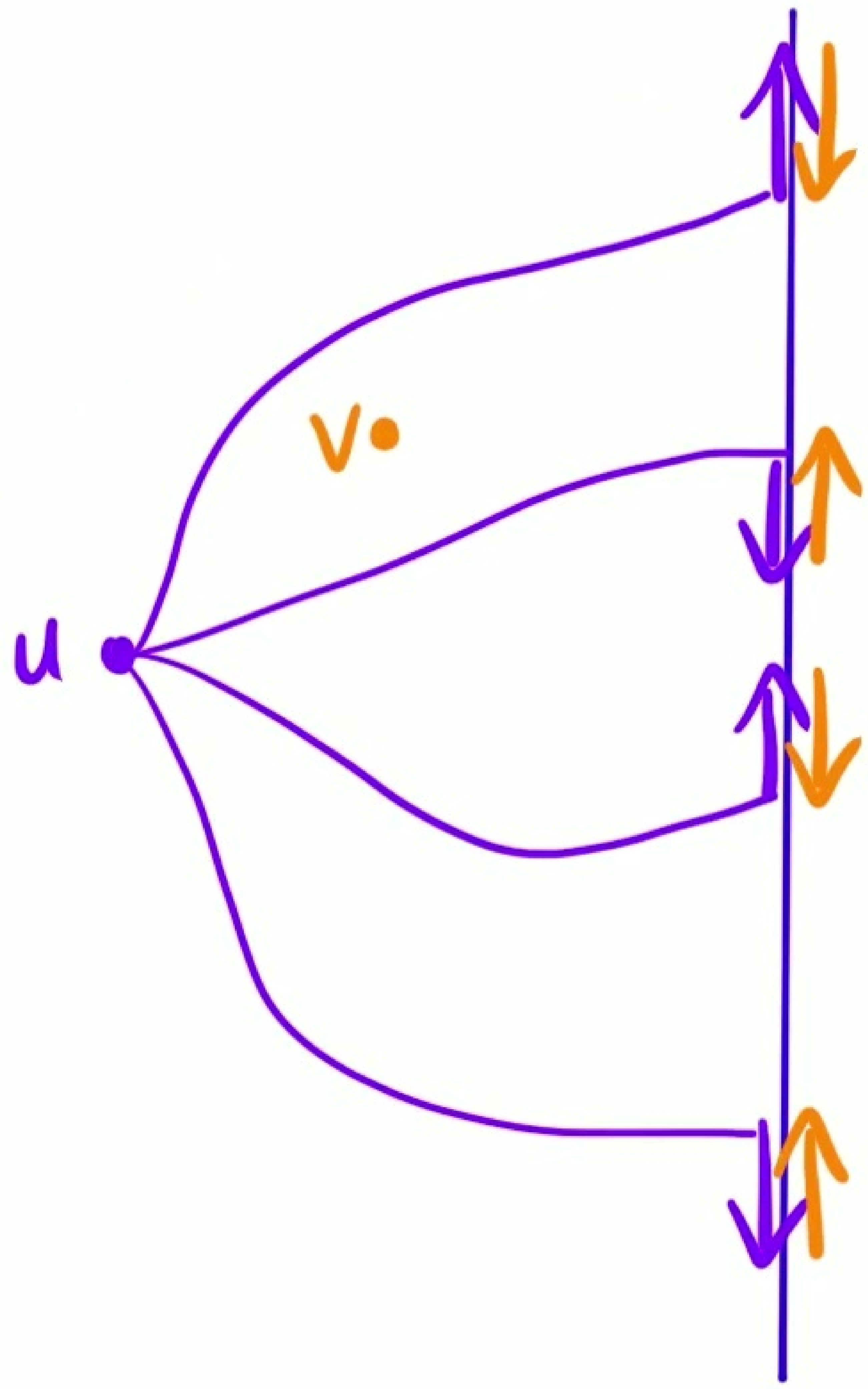

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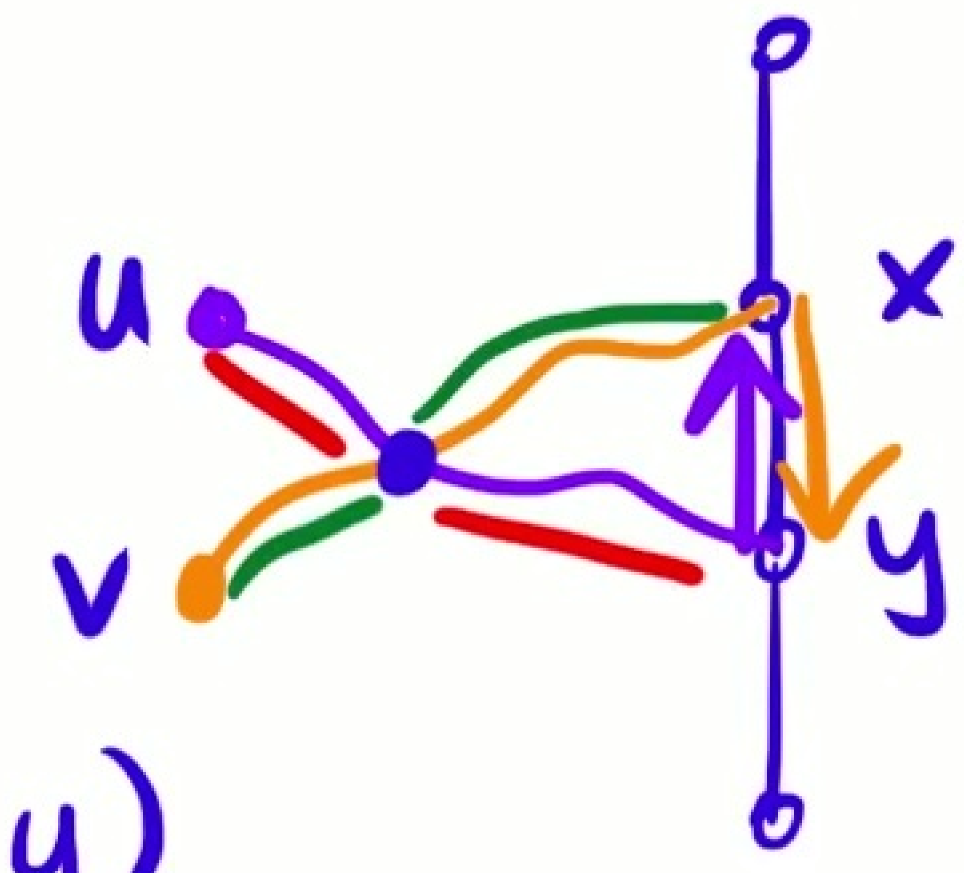

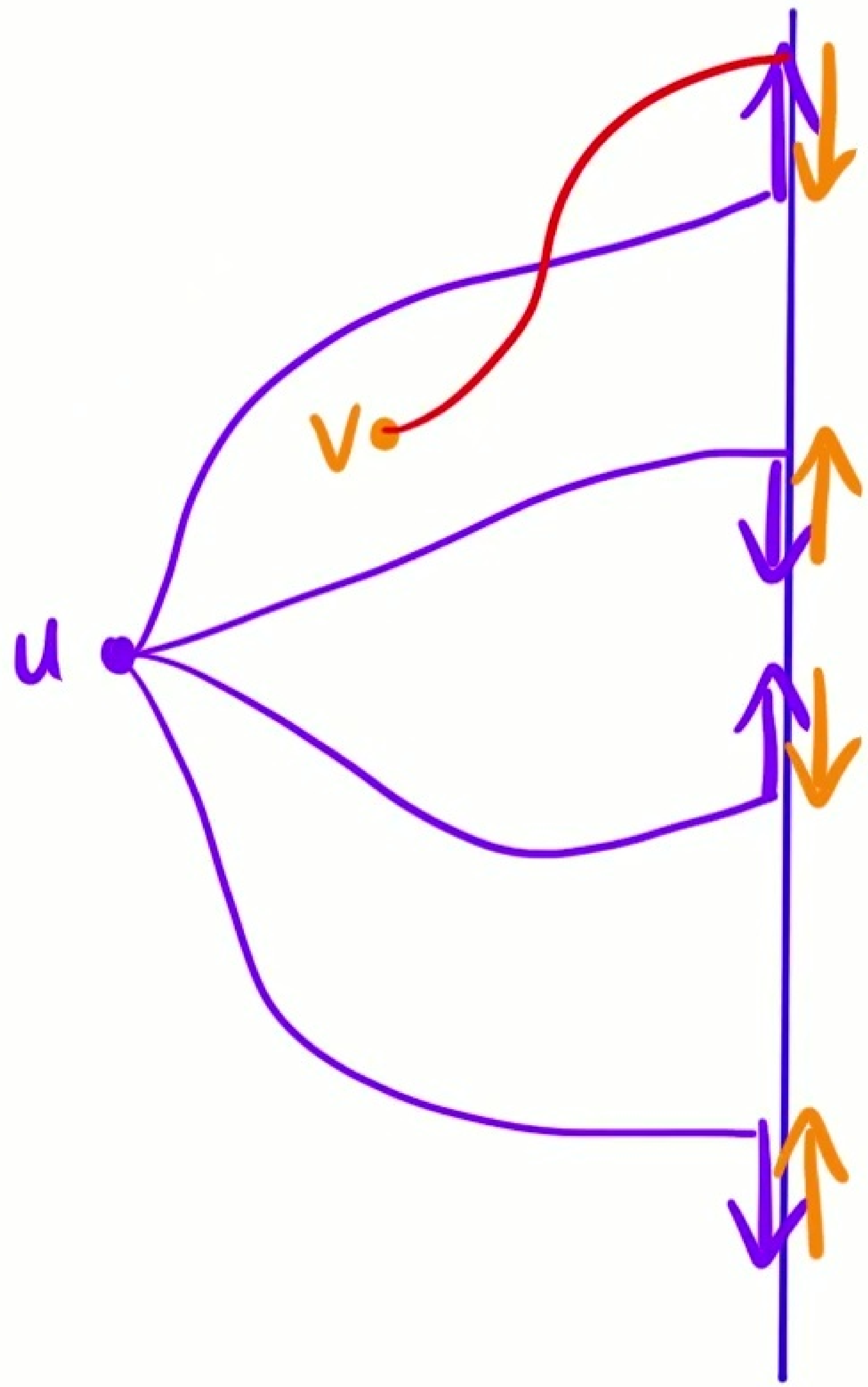

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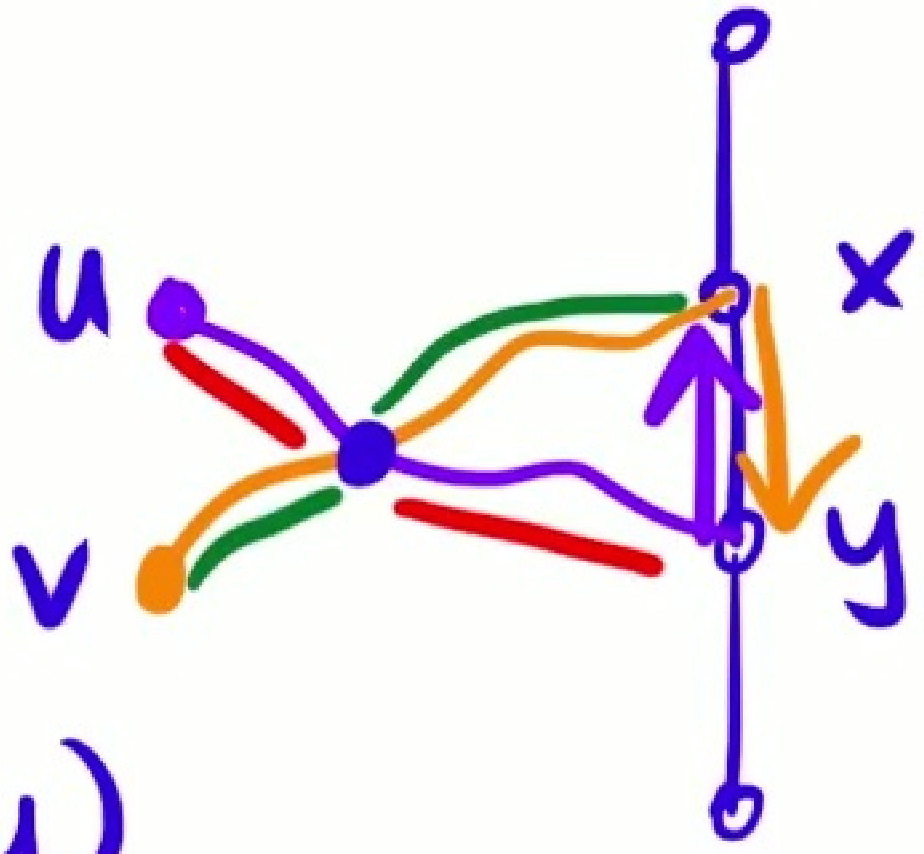

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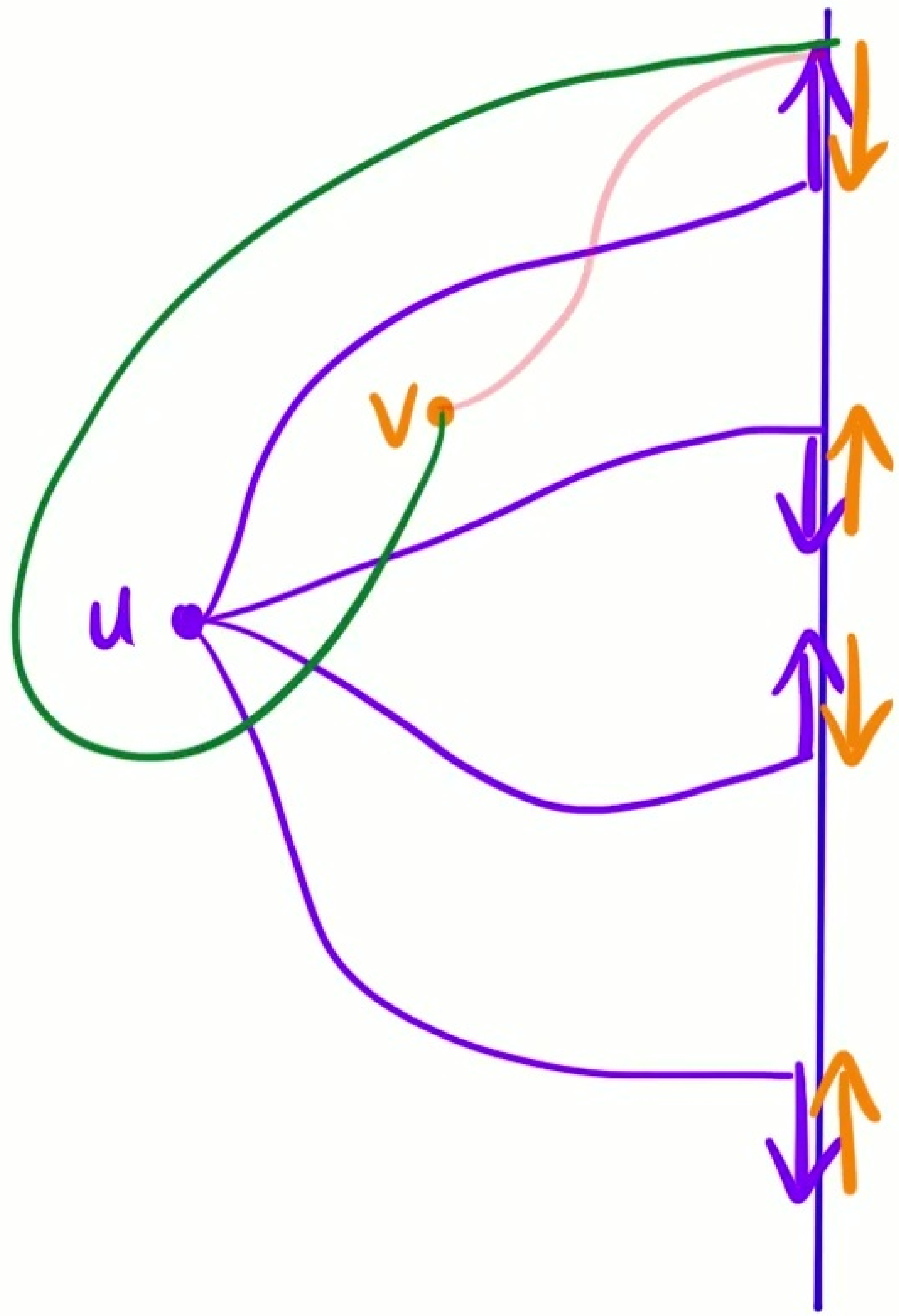

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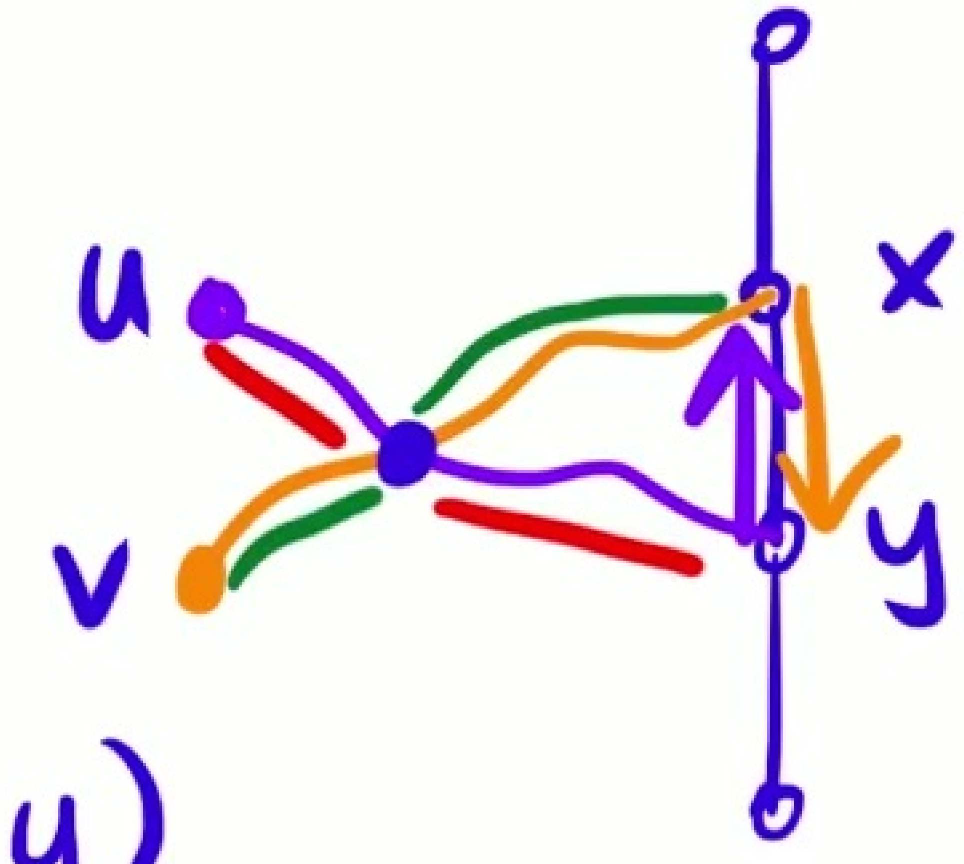

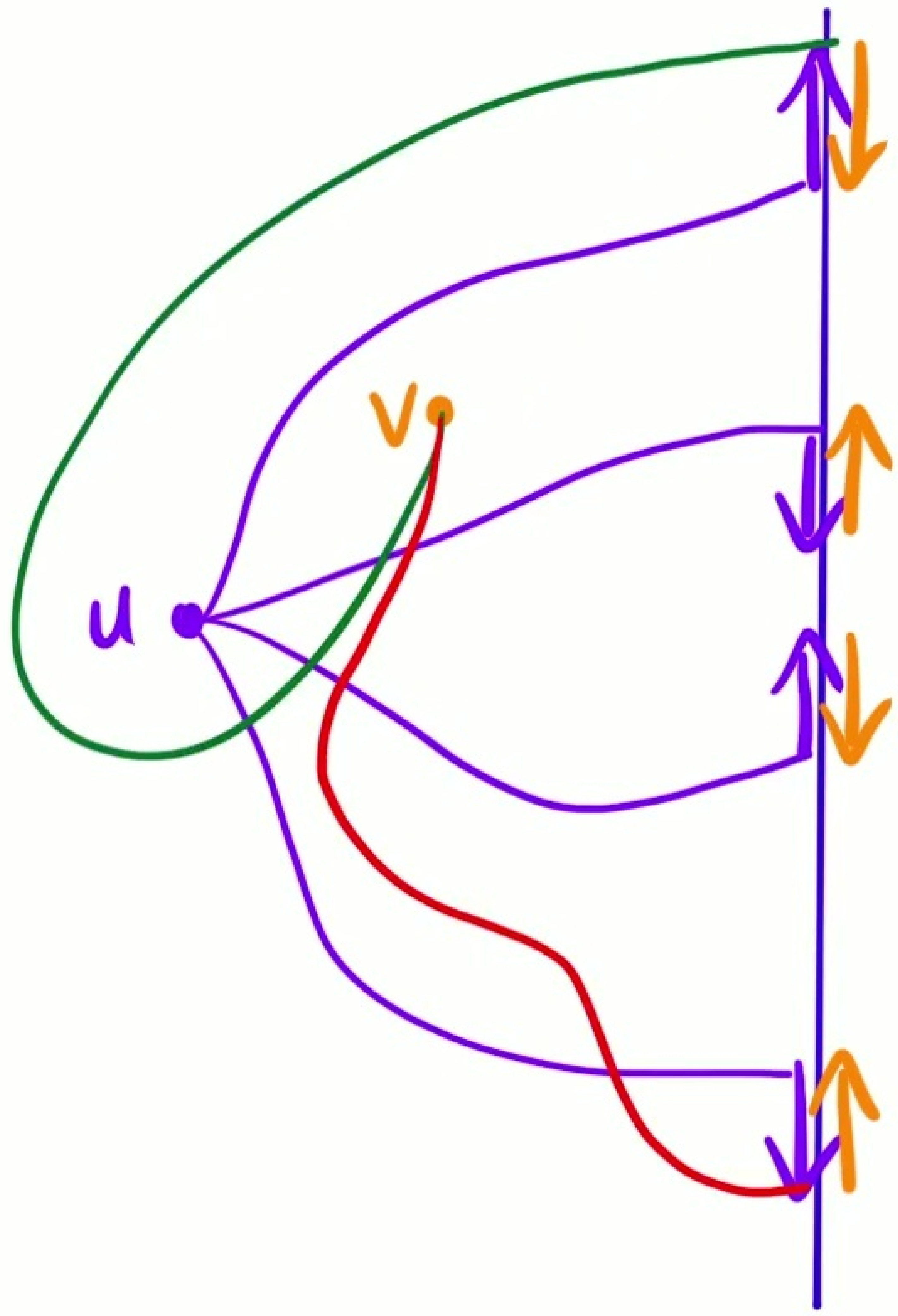

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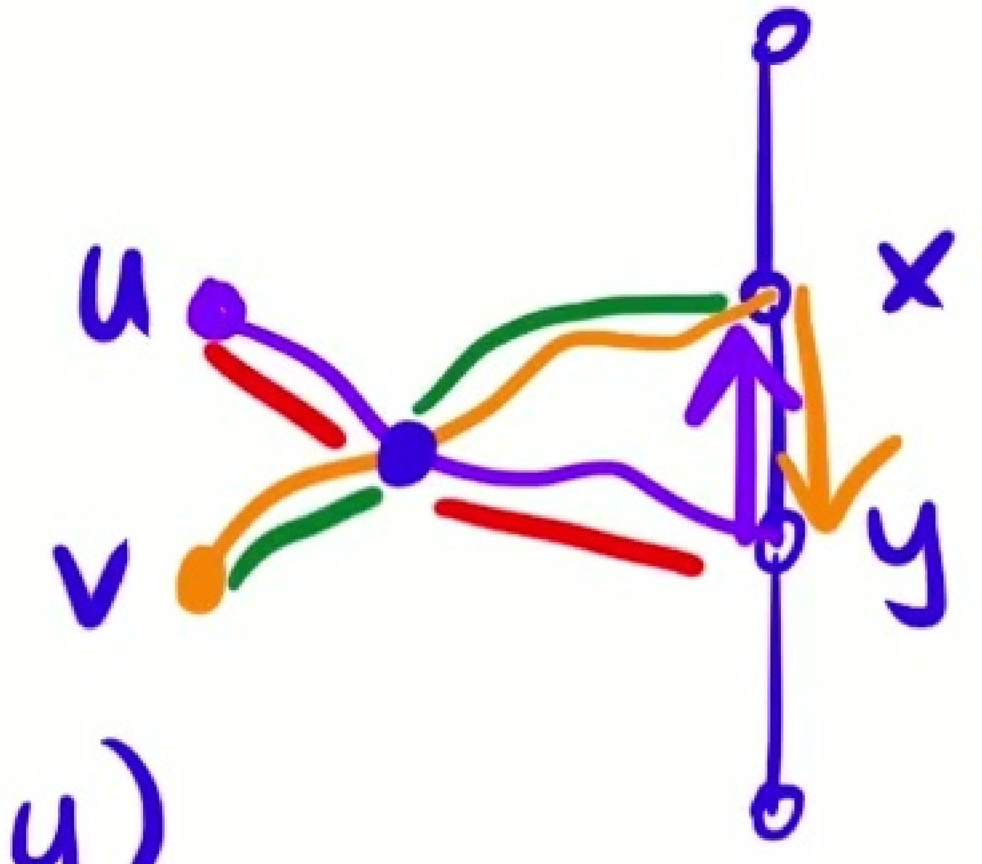

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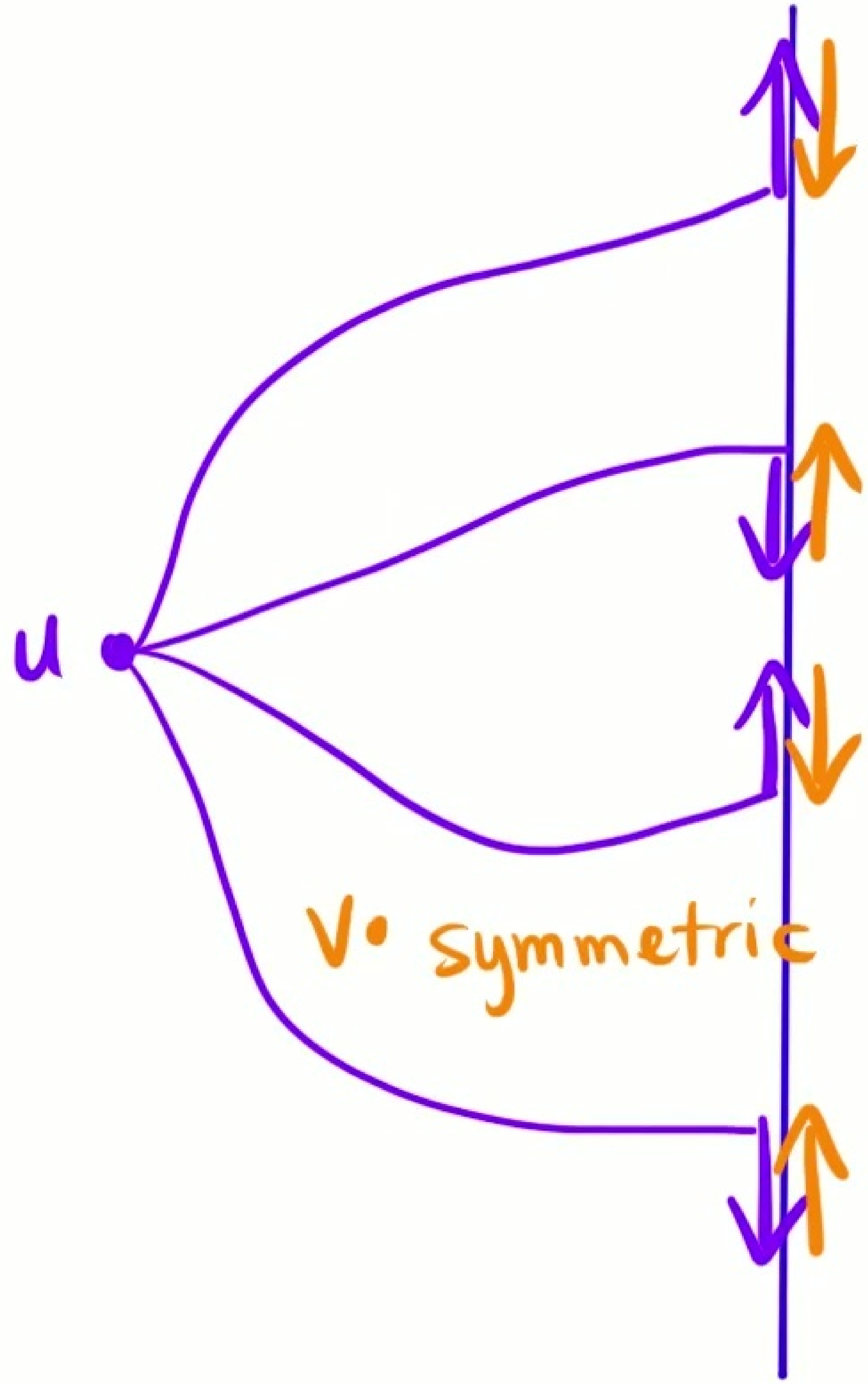

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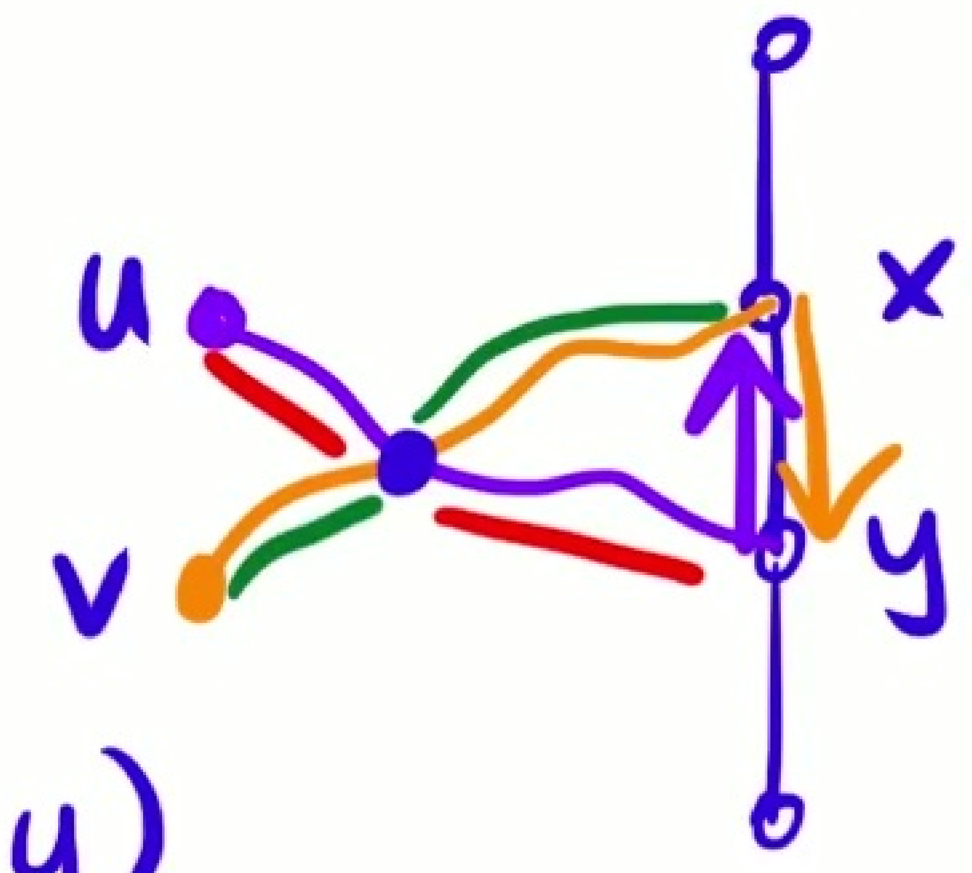

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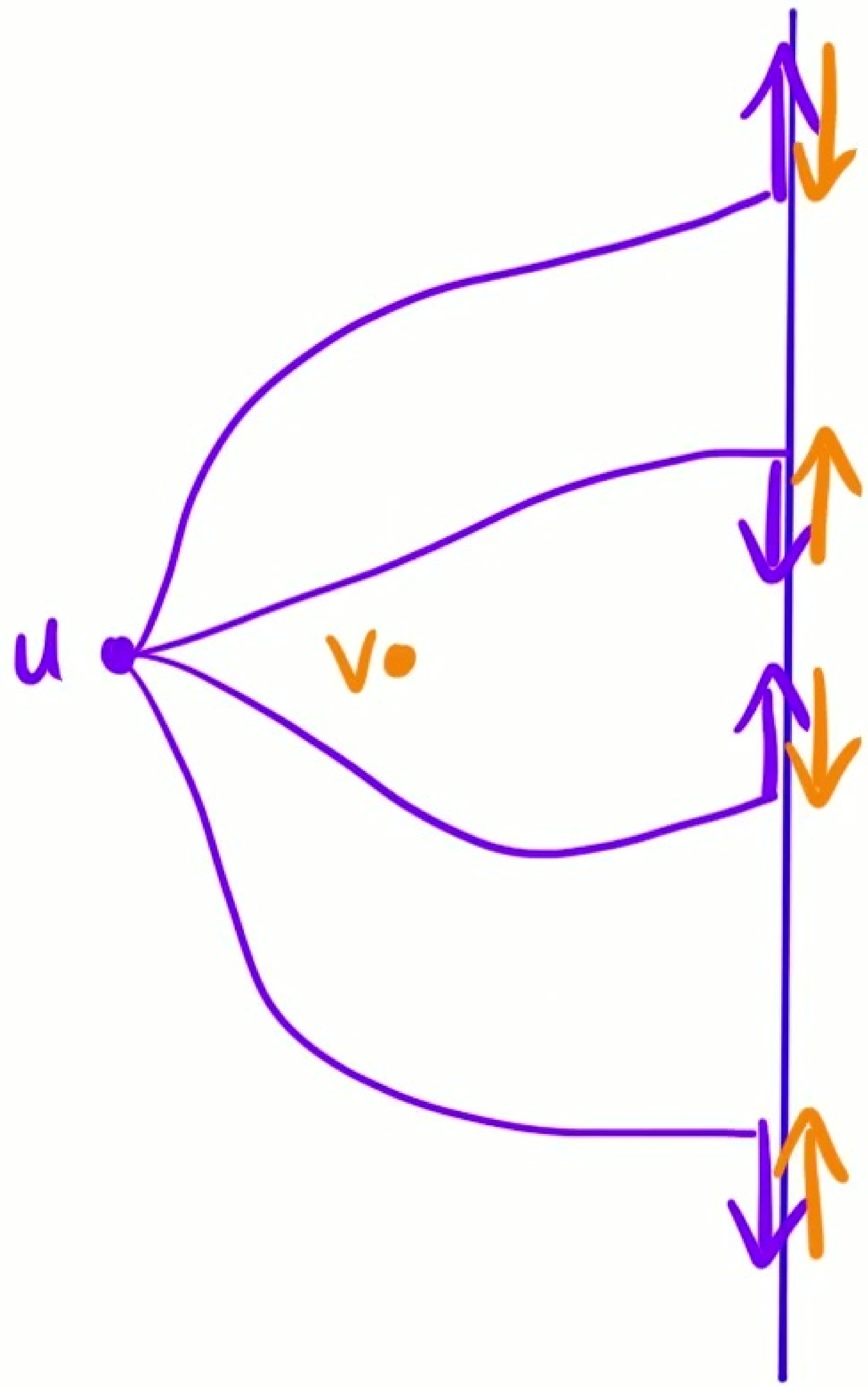

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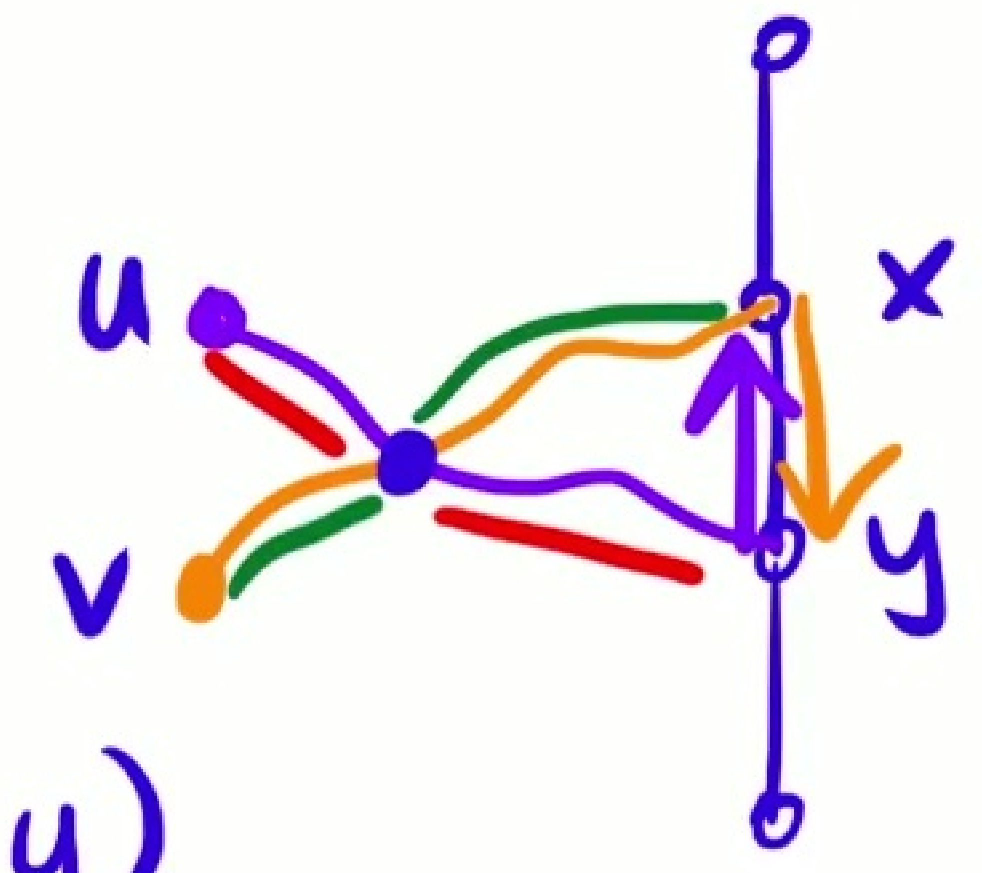

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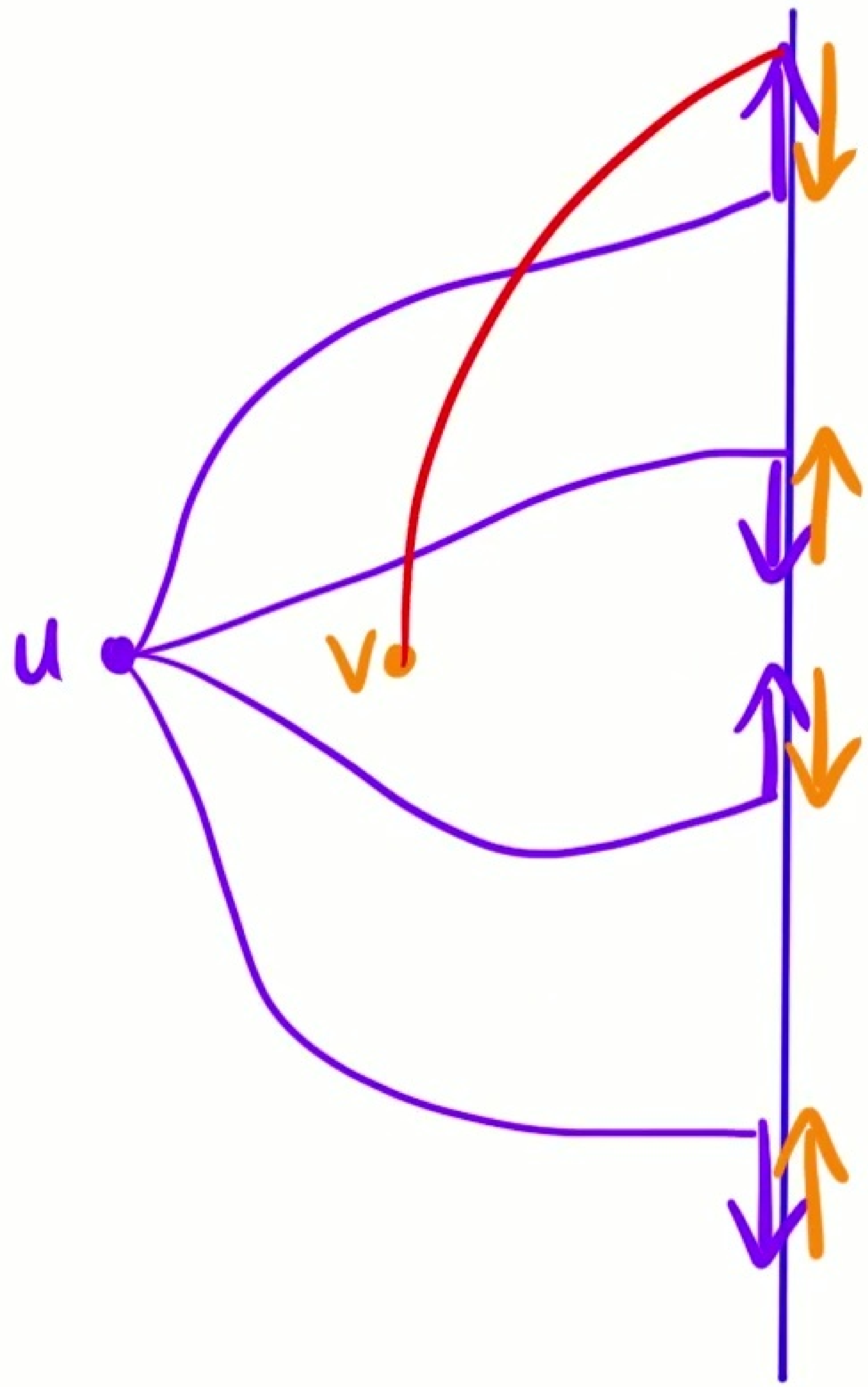

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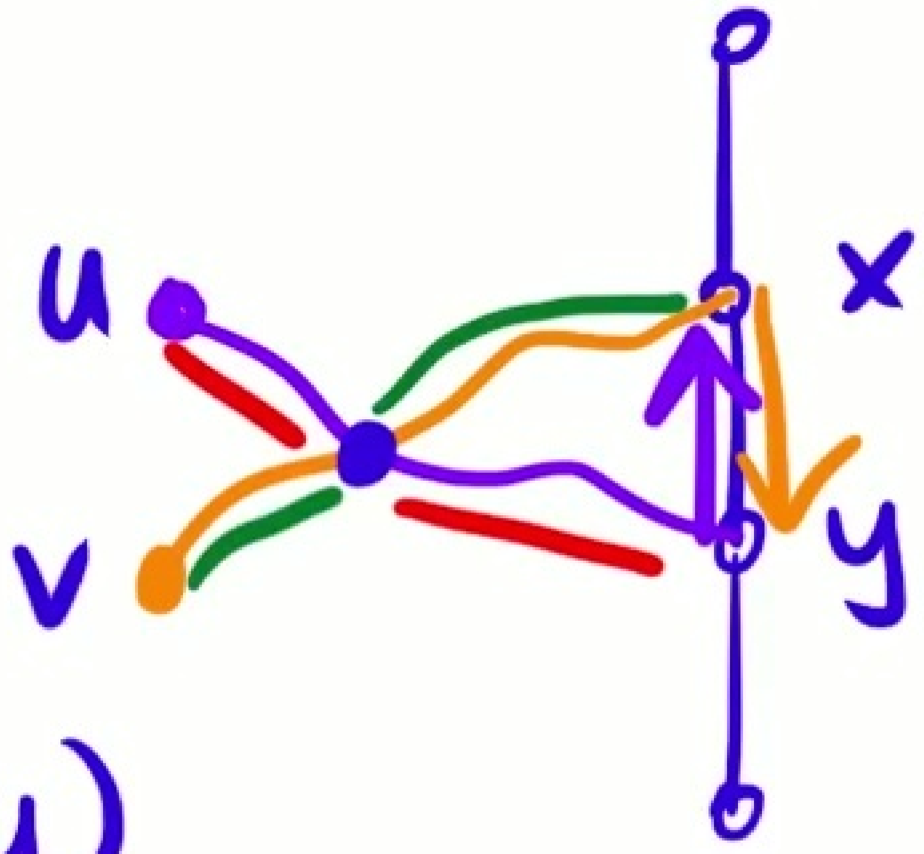

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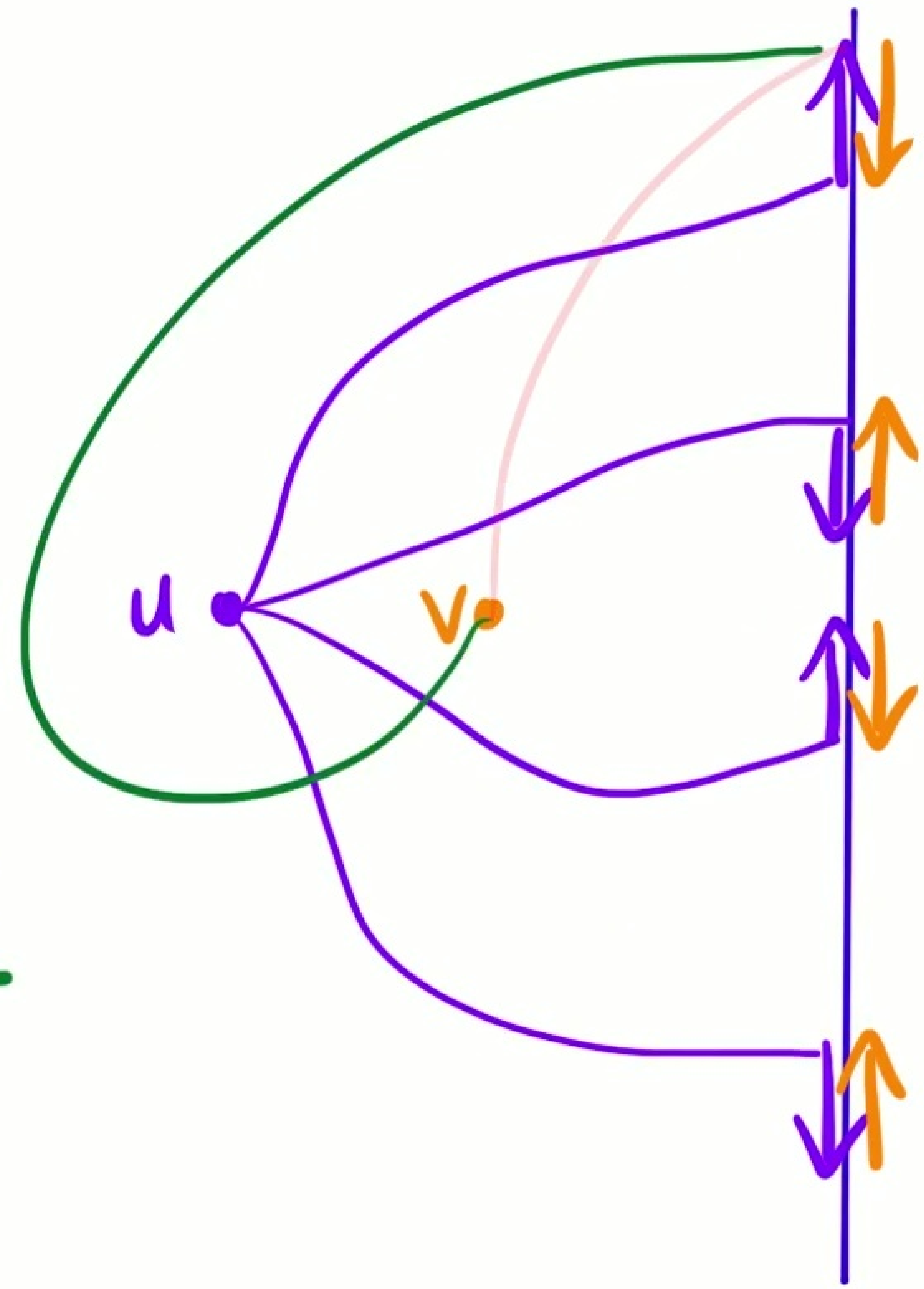

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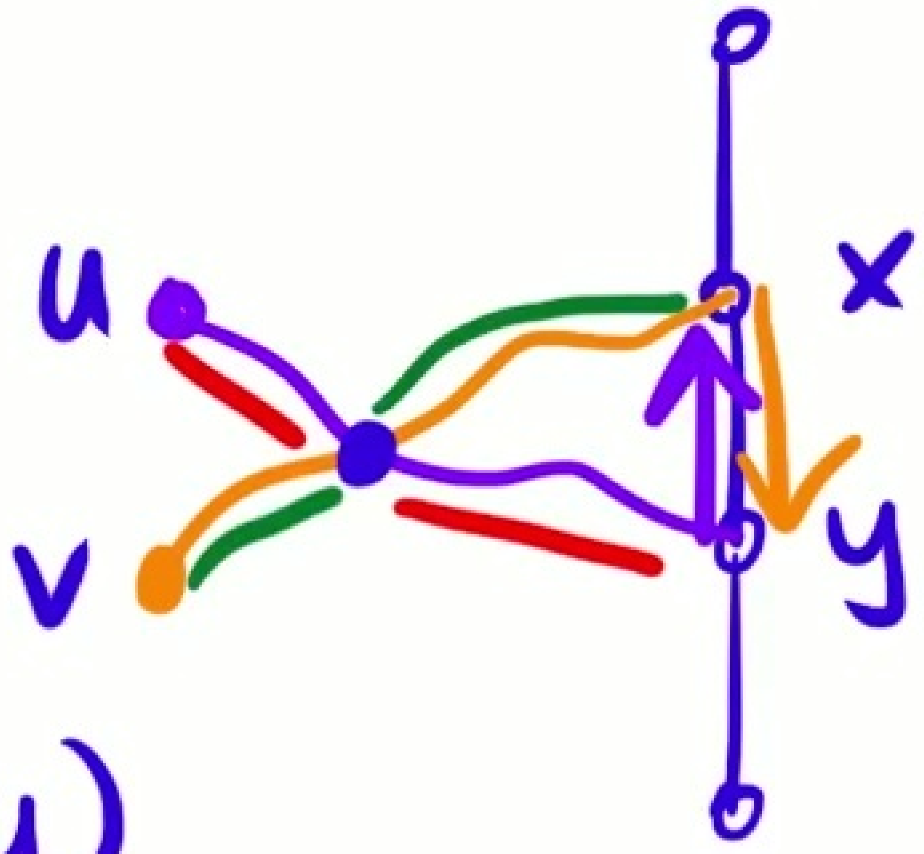

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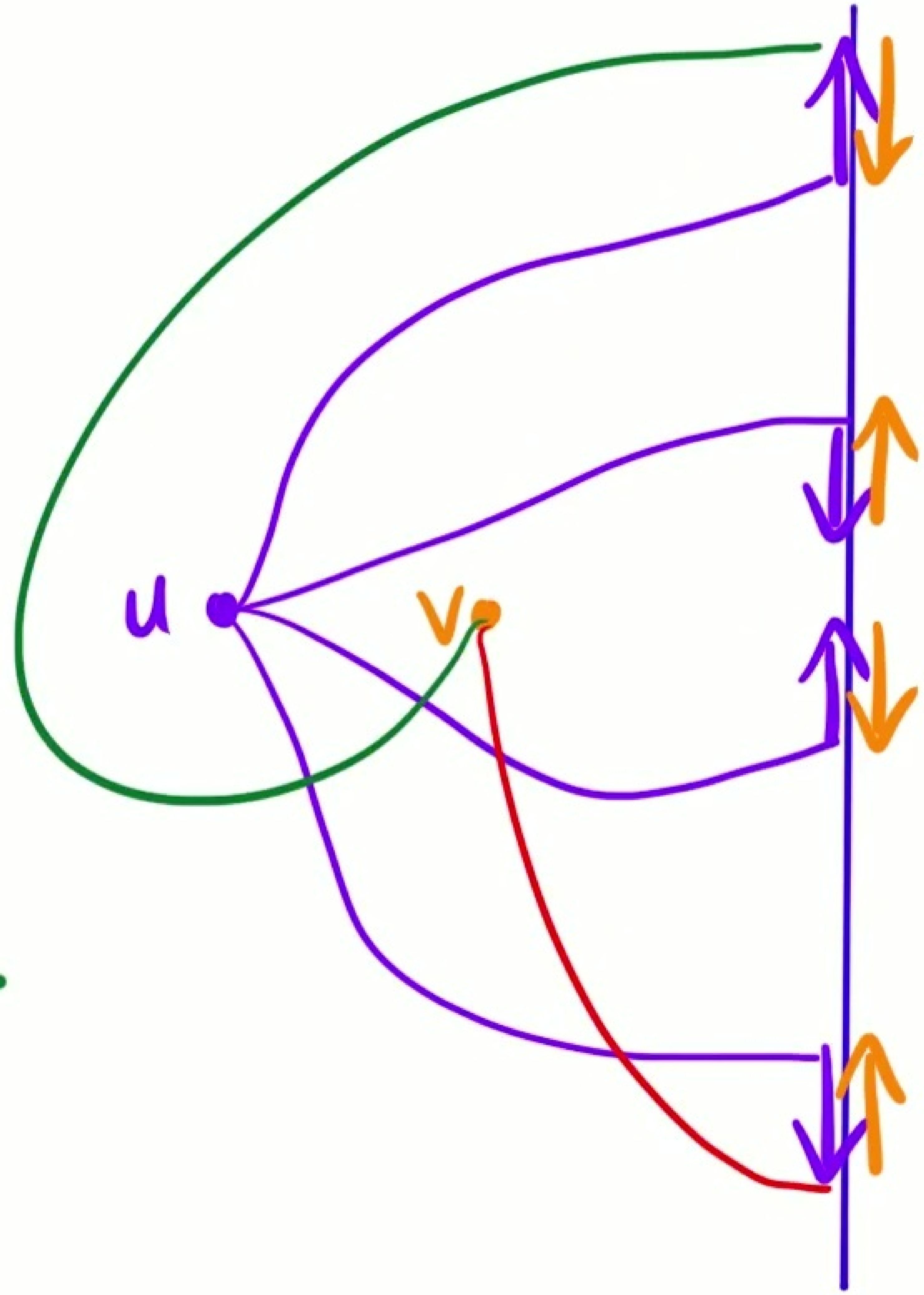

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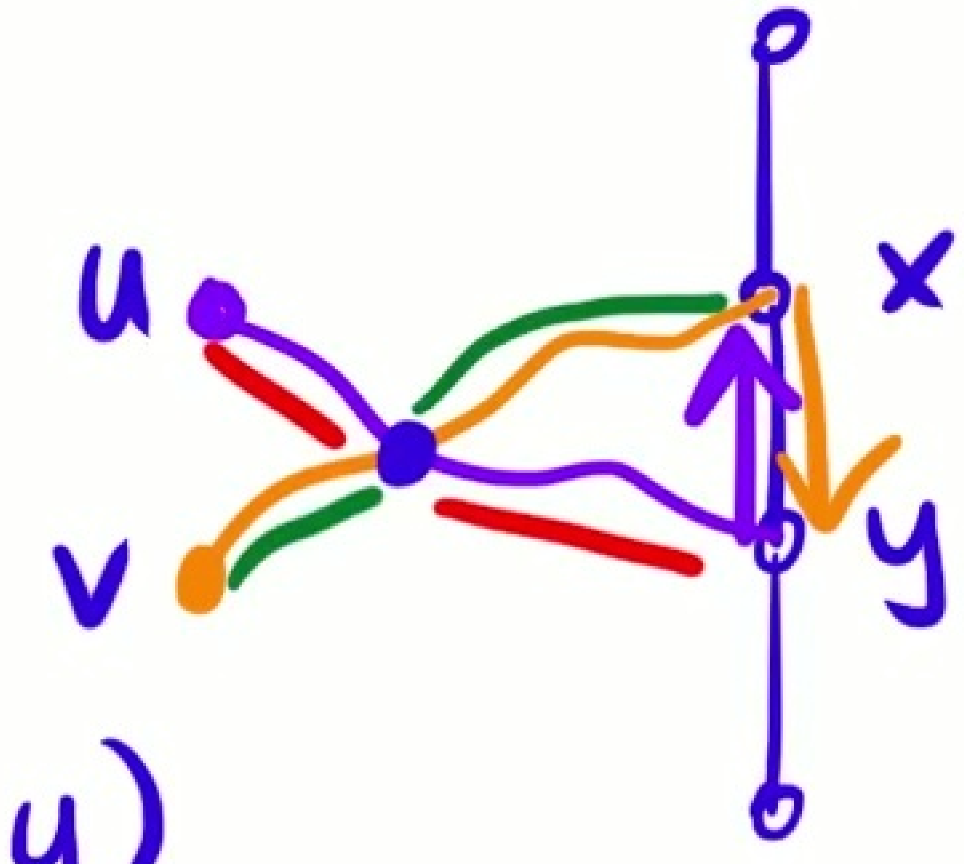

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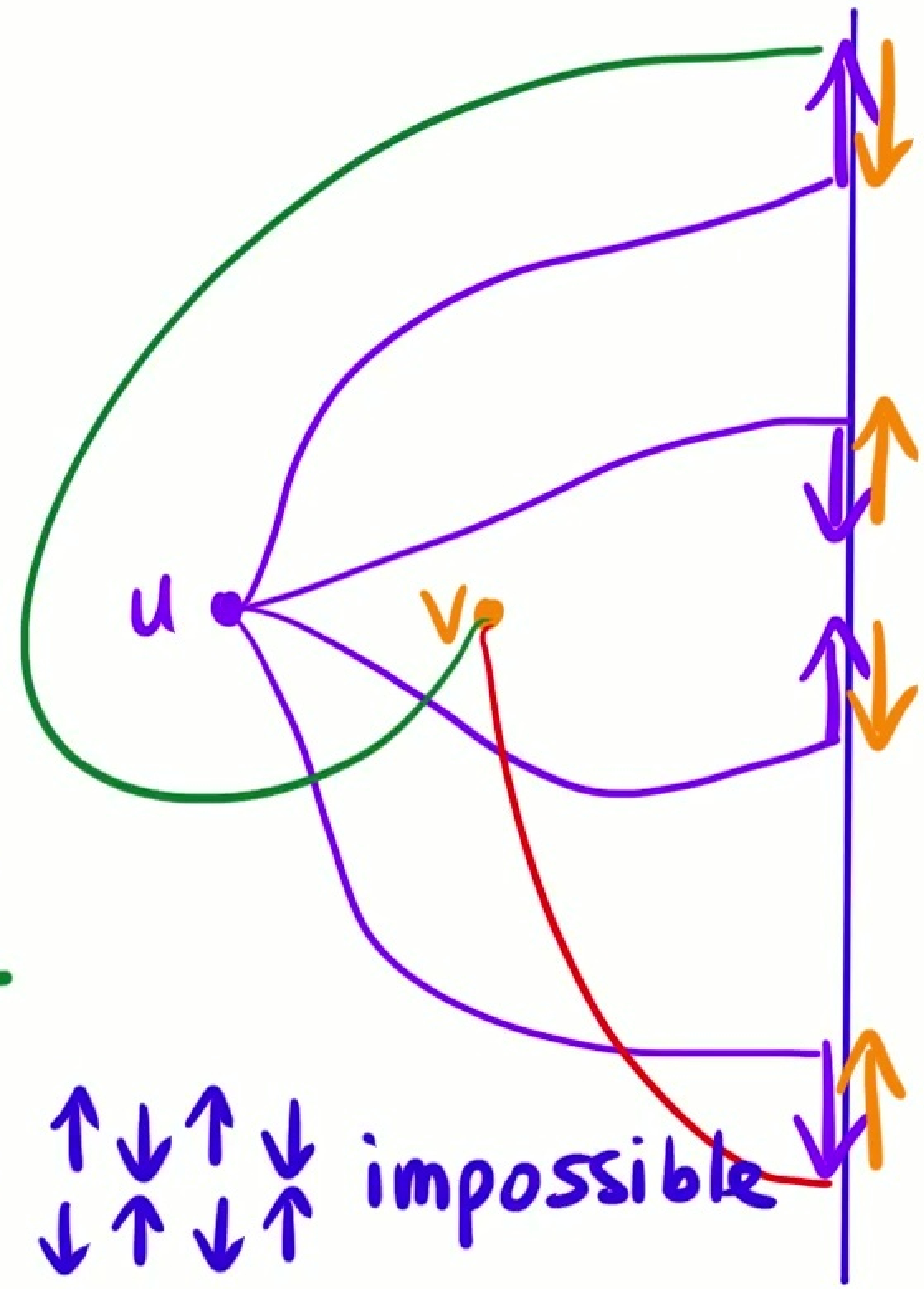

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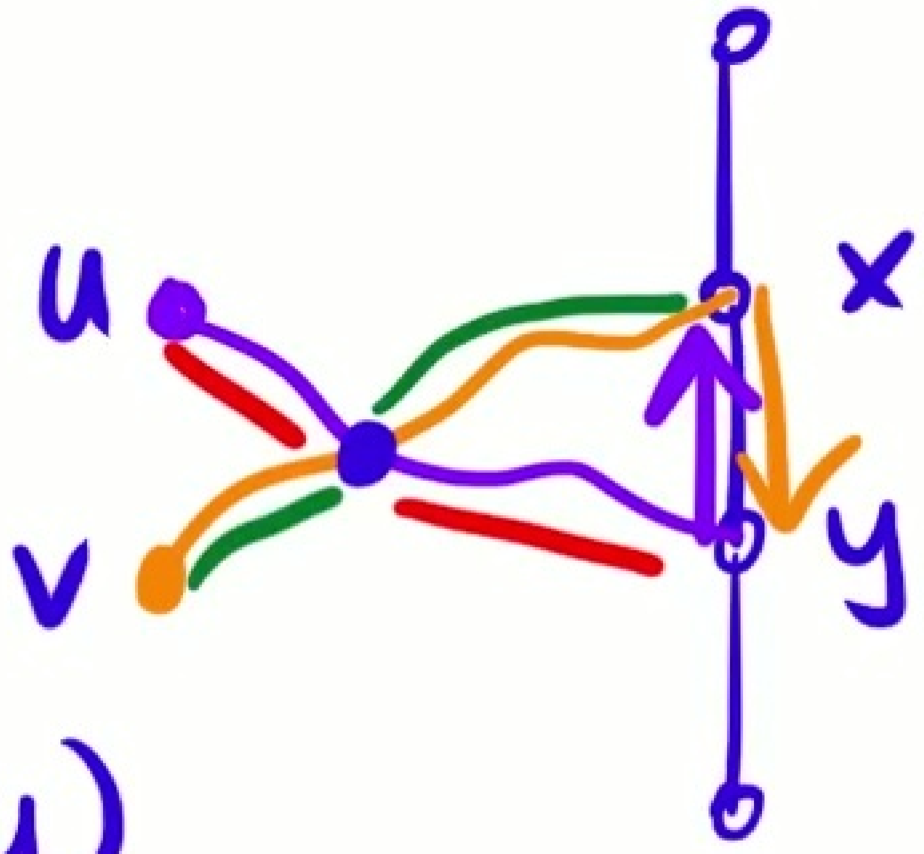

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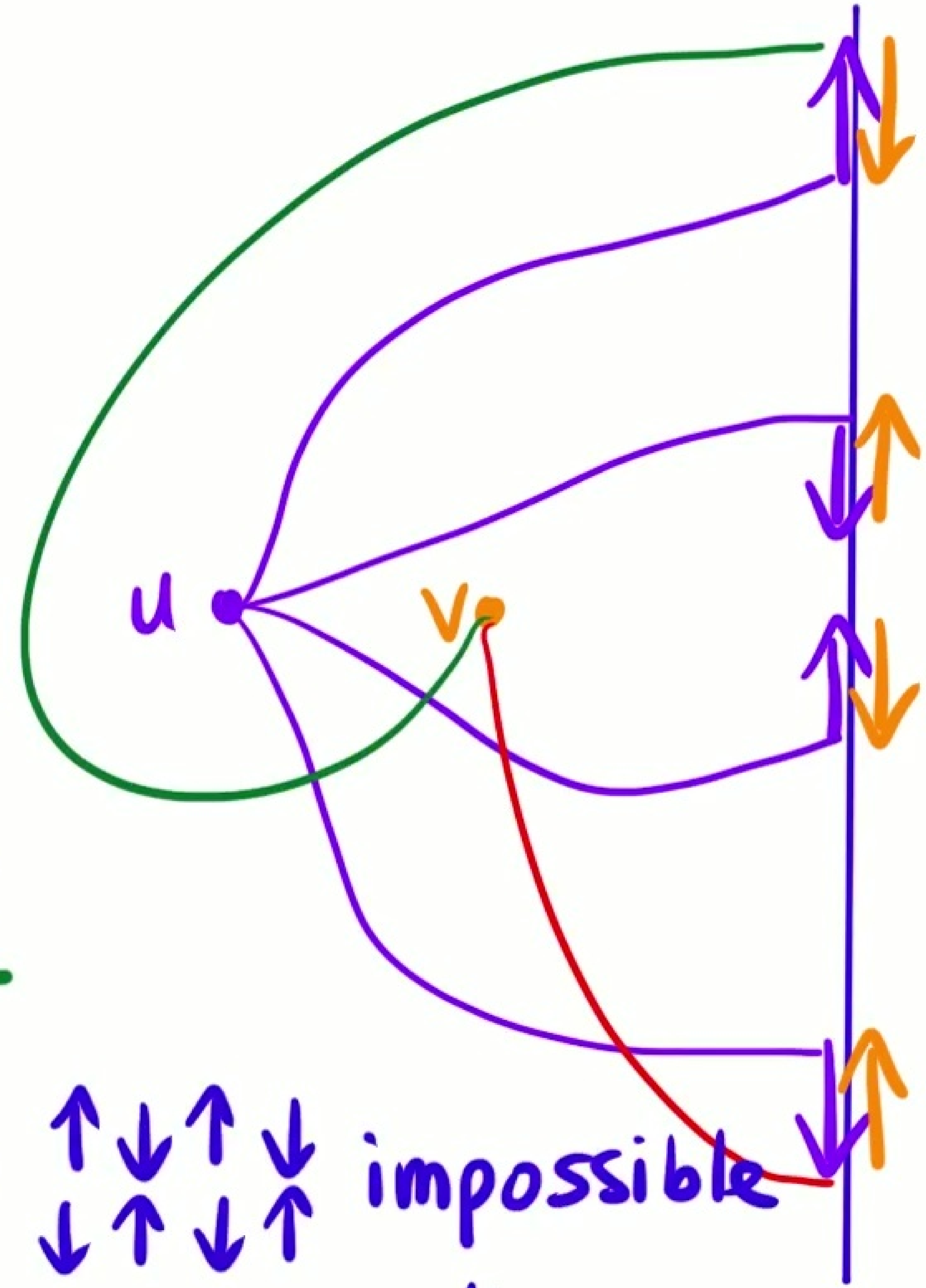

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independent of n or k

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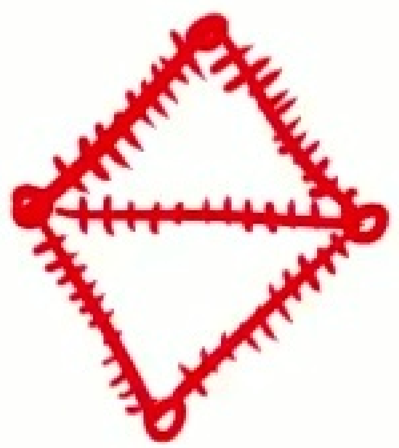
Ex: 2D manifold (sheet of paper) in high-dimensional space:

$$N(s) = O(1/s^2) \Rightarrow \lim_{s\to 0} \frac{\log O(1/s^2)}{\log(1/s)} = 2$$

Set of vectors: $\lim_{\varepsilon\to 0} \dfrac{\log N(\varepsilon\Delta)}{\log(1/\varepsilon\Delta)} = \lim_{\varepsilon\to 0} \dfrac{\log (k/\varepsilon)^{O(1)}}{\log(1/\varepsilon\Delta)} = \lim_{\varepsilon\to 0} \dfrac{O(\log k + \log 1/\varepsilon)}{\log 1/\varepsilon + \log 1/\Delta} = O(1).$

independent of n or k

Subdivide edges infinitely: , looks like O(1)-dimensional manifold in $\mathbb{R}^k$?

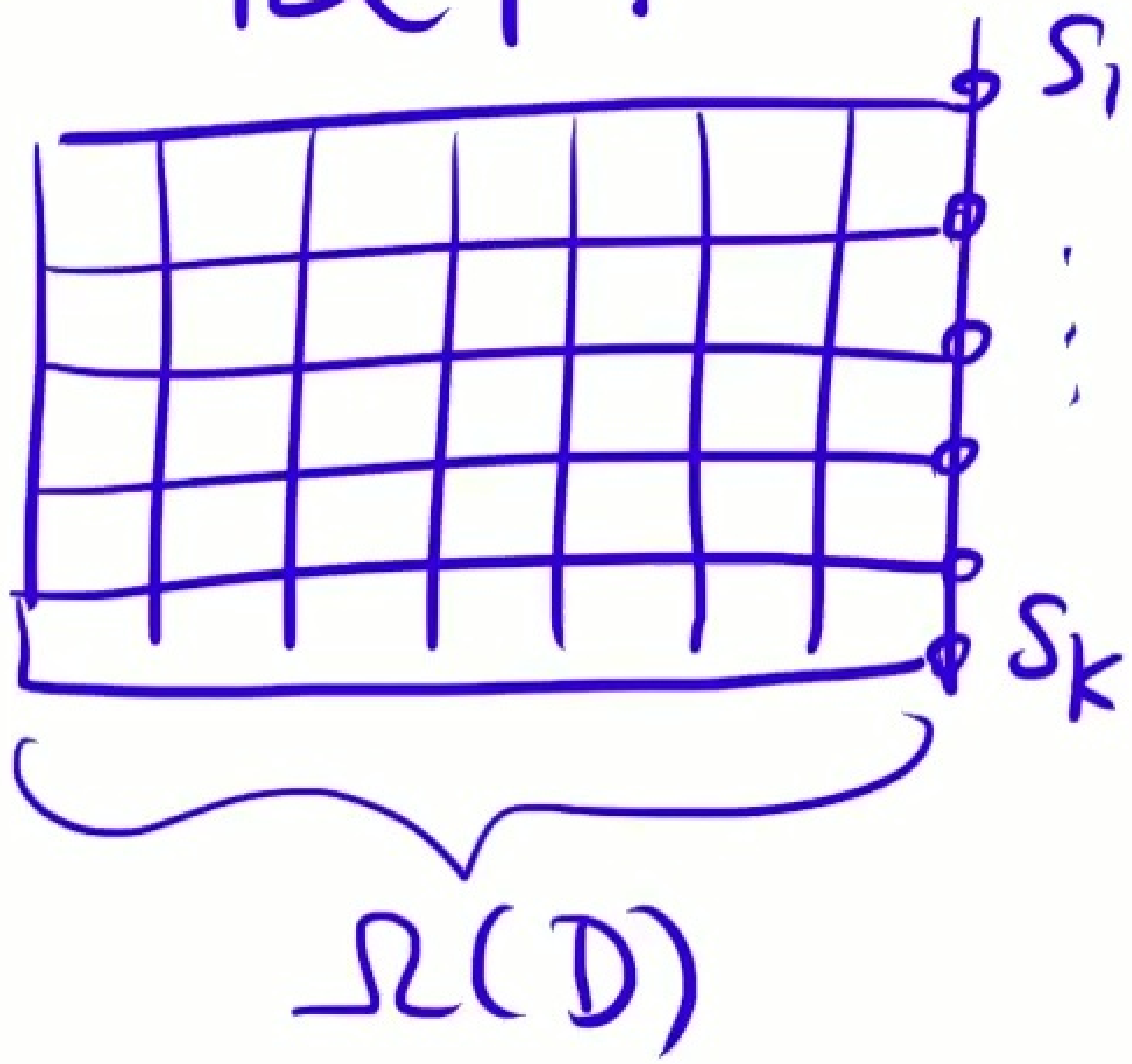
Future directions?

Optimal bound for $|\mathcal{L}|$?

Future directions?

Optimal bound for $|\mathcal{L}|$?

$|\mathcal{L}| = \Omega(KD)$:

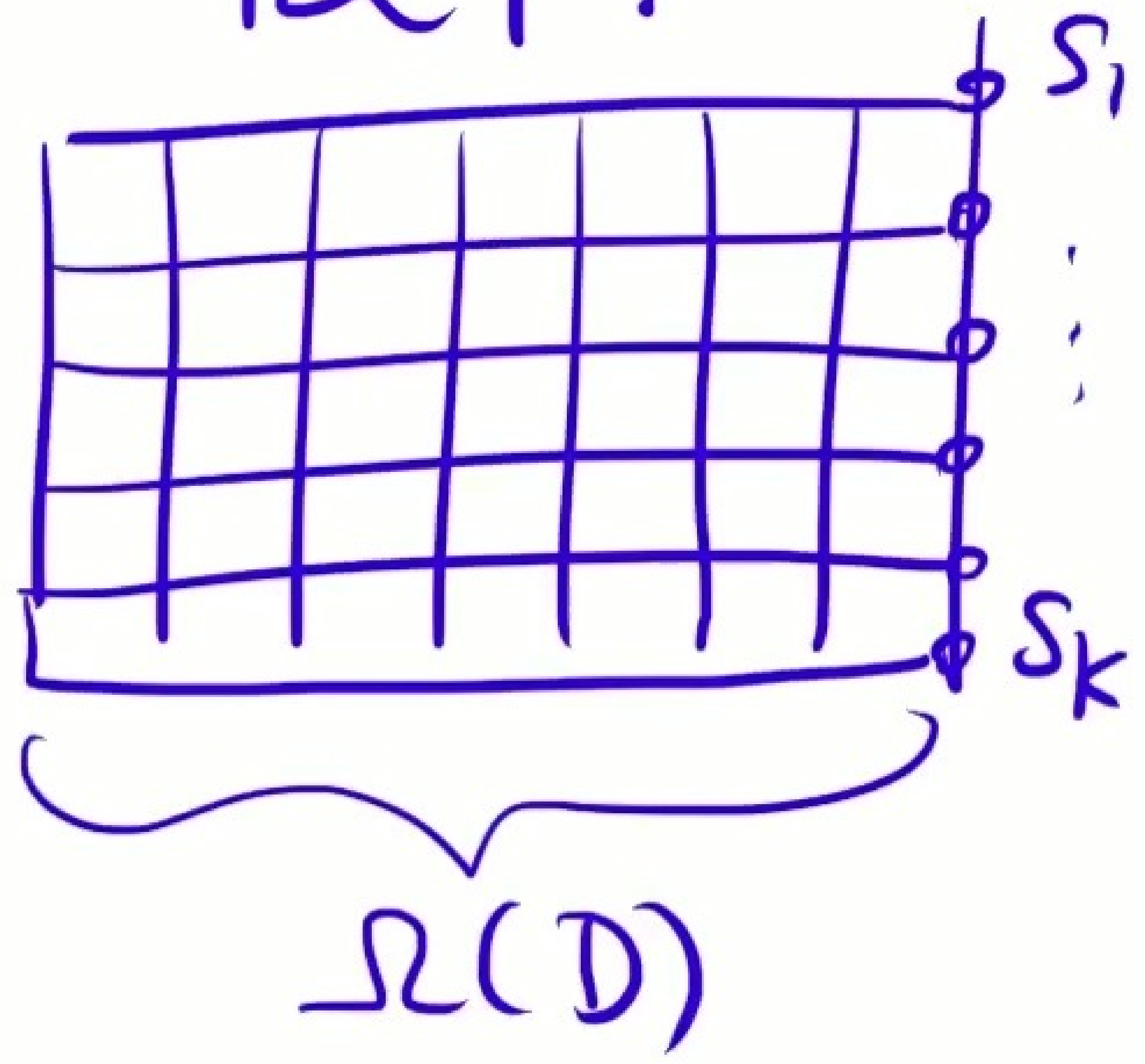


$|\mathcal{L}| = O(K^3 D)$

Future directions?

Optimal bound for $|\mathcal{L}|$?

$|\mathcal{L}| = \Omega(kD)$:



$|\mathcal{L}| = O(k^3 D)$

Bounds for bounded genus? Minor-free?

- $O(k^{O(g)}D)$ seems possible with same techniques