Planar Diameter via Metric Compression

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CMU

Joint work with Merav Parter
Weizmann

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Background

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$(1+\varepsilon)$-approx: $O(n/\varepsilon^{O(1)})$ [Chan + Skrepetos 17]

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this talk

- "Simpler" algorithms: amenable to distributed comp + "standard" CONGEST tools
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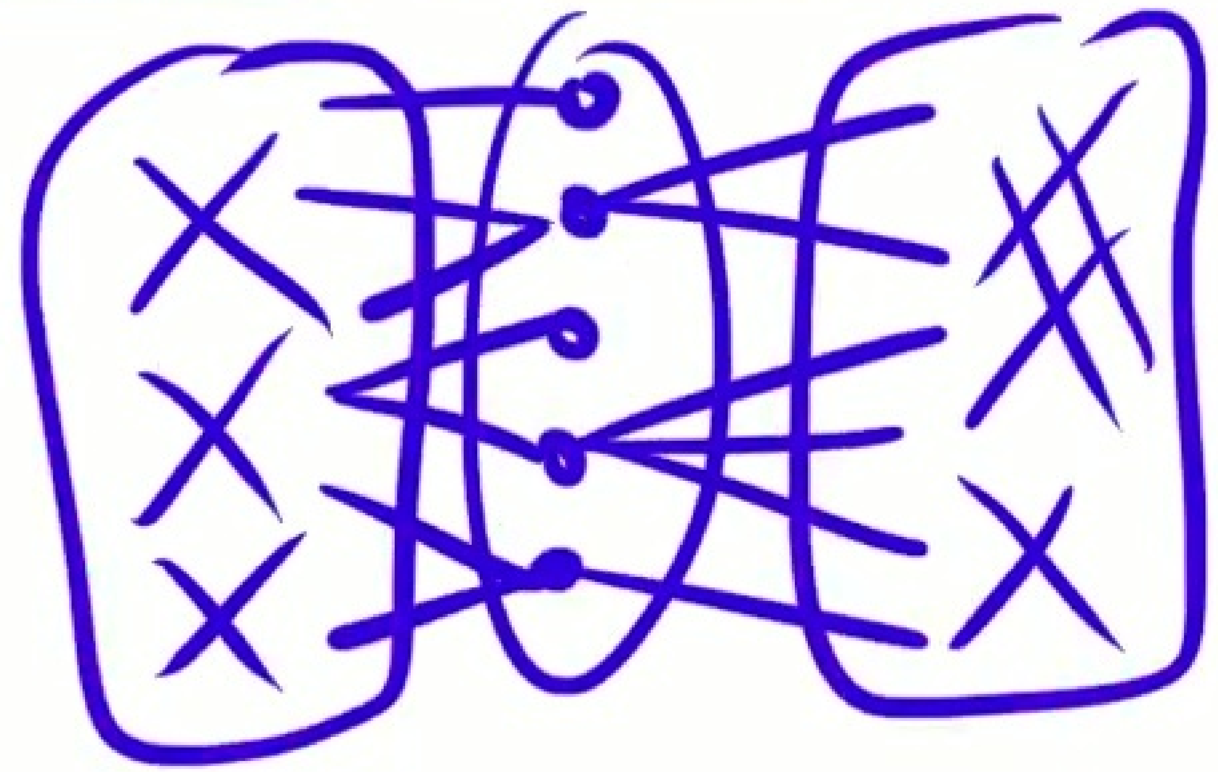
Divide + Conquer

Balanced separator:



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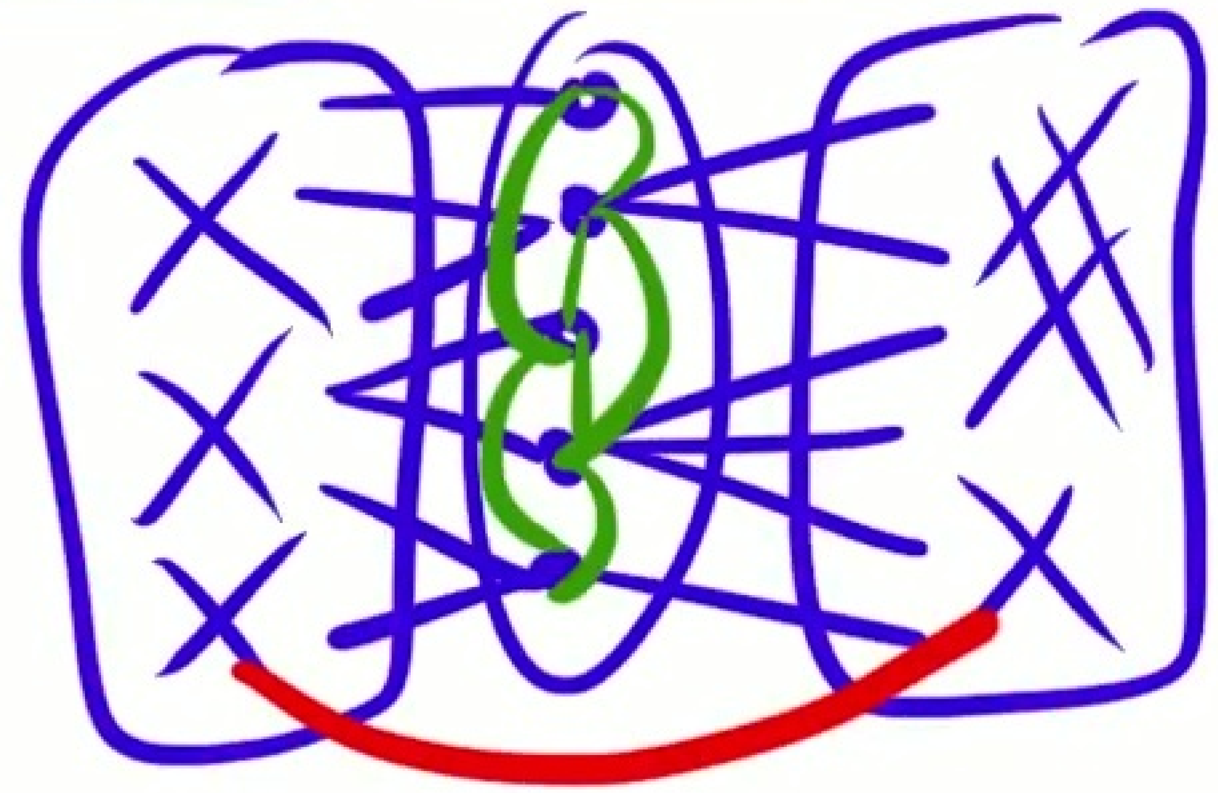

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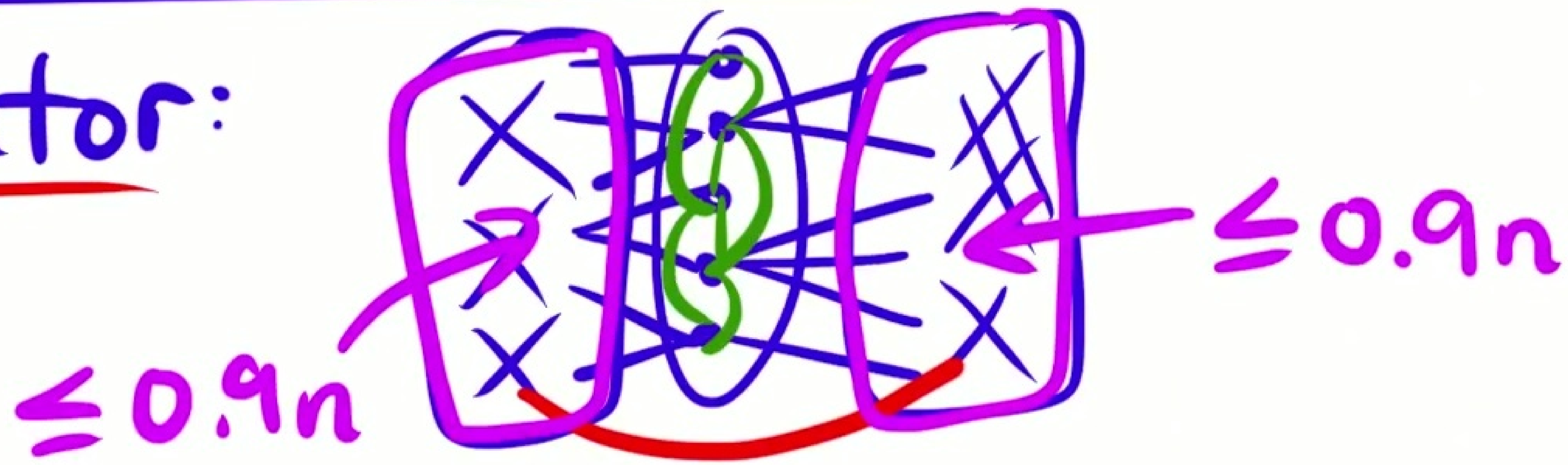

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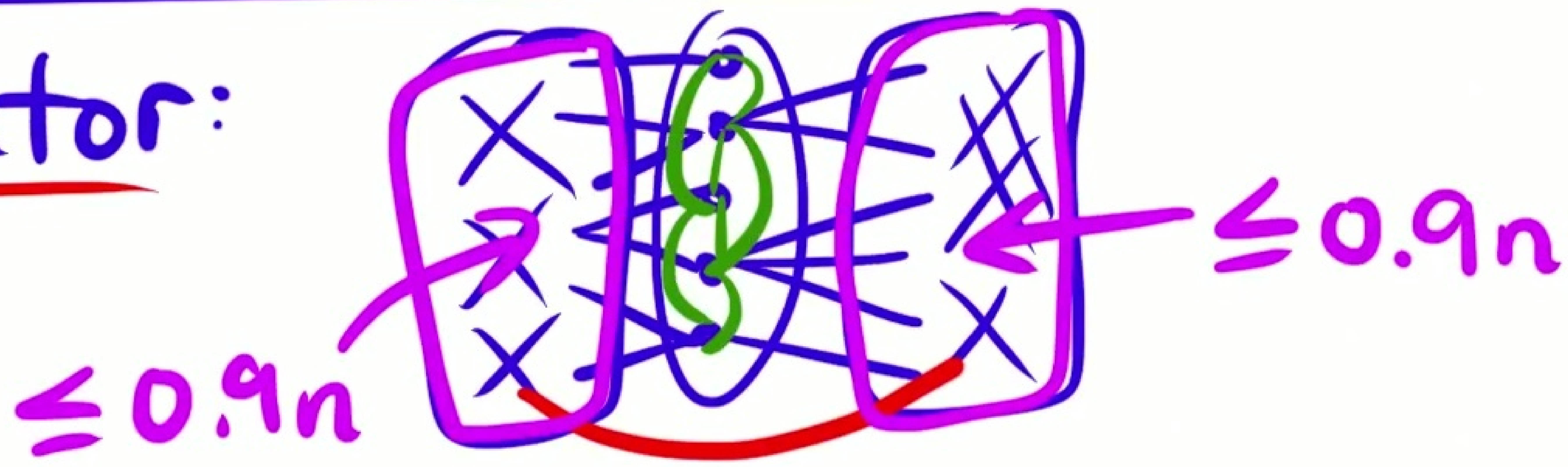

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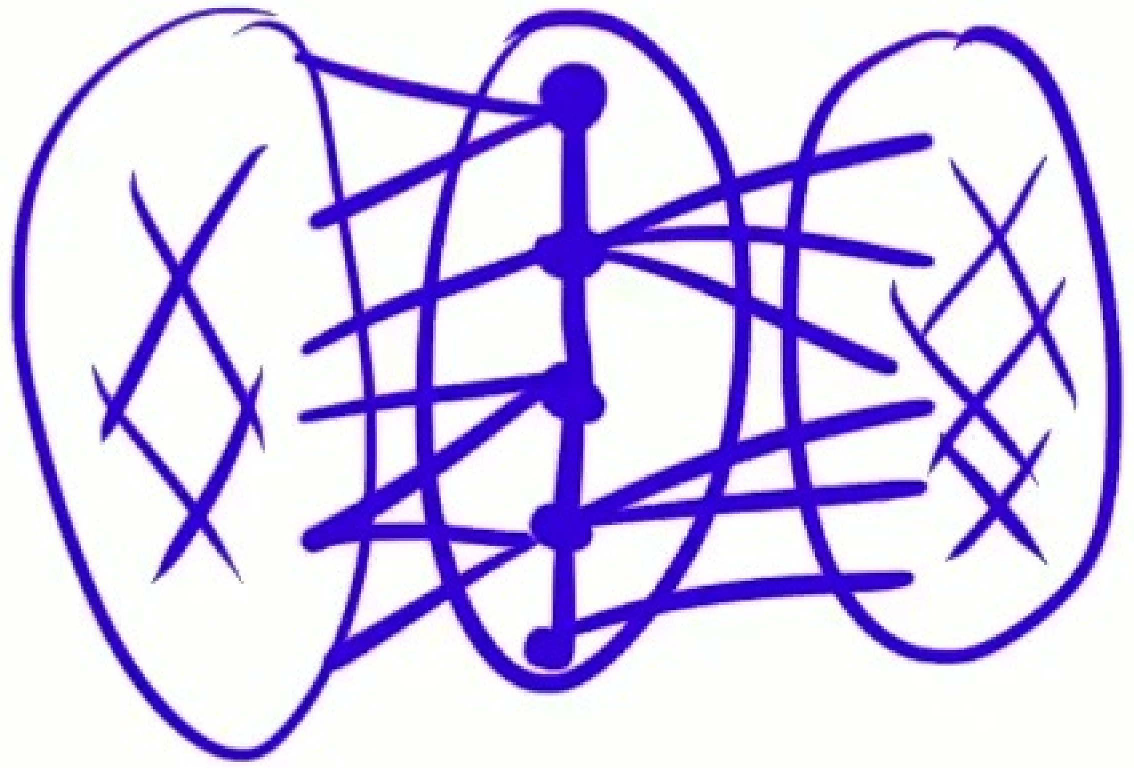

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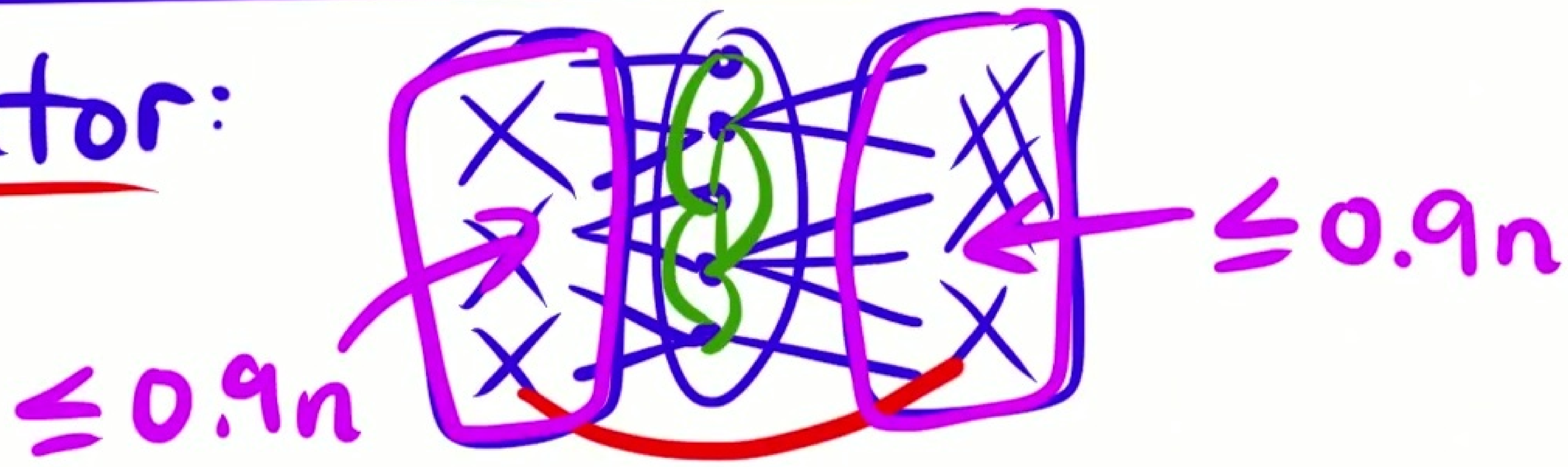

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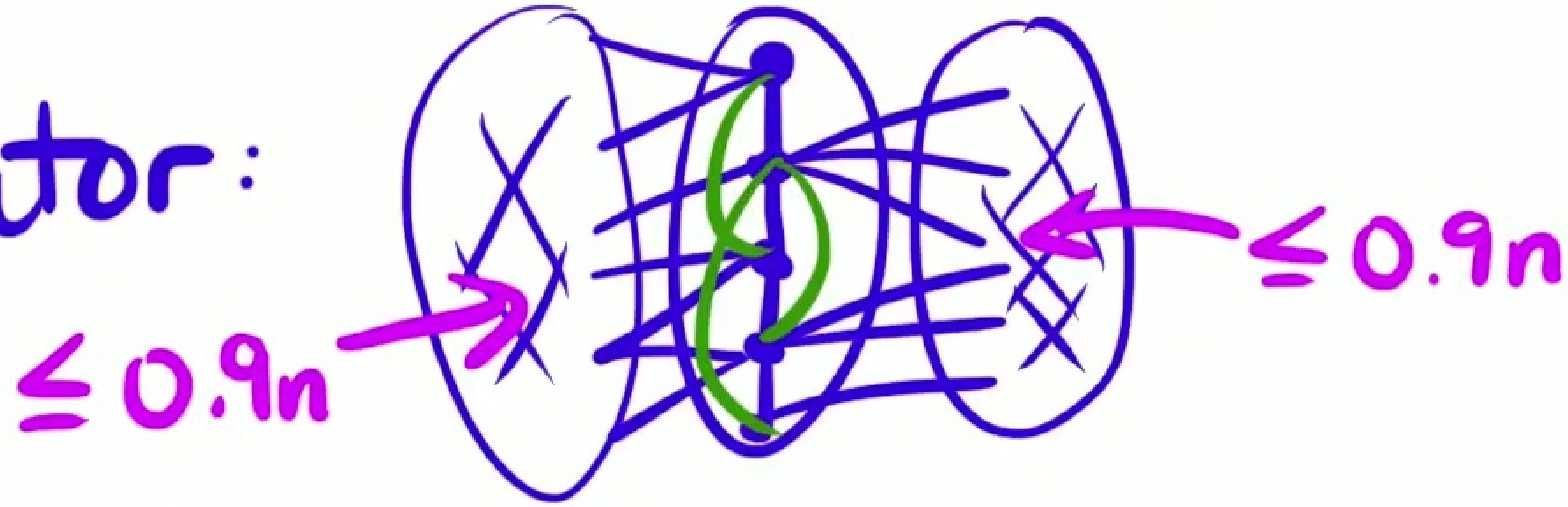

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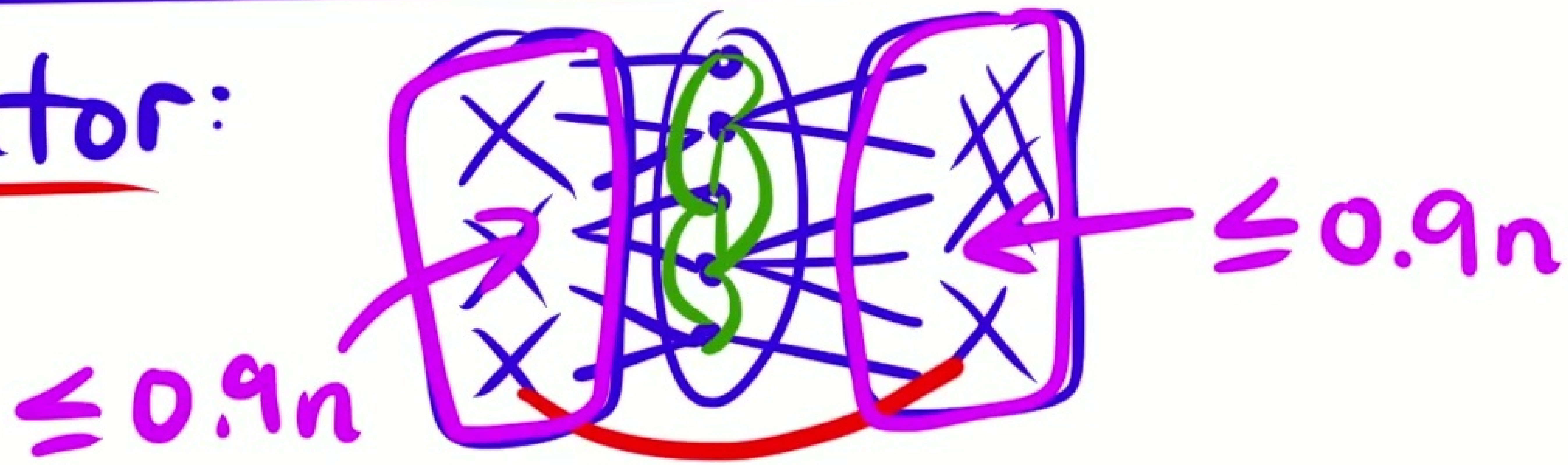

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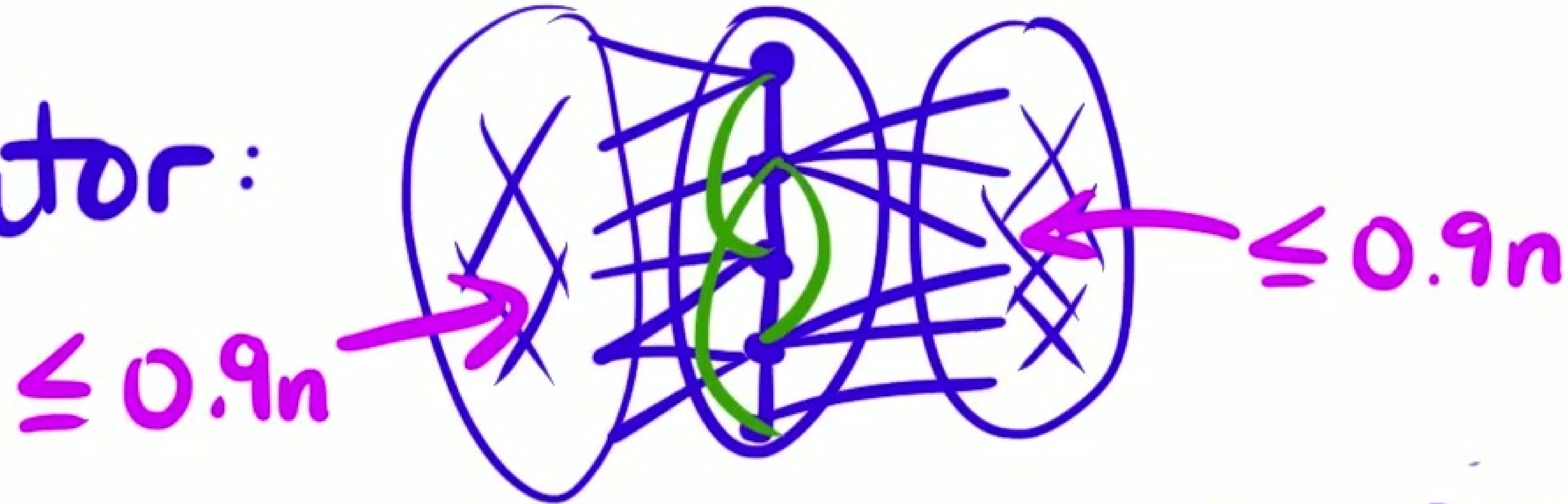
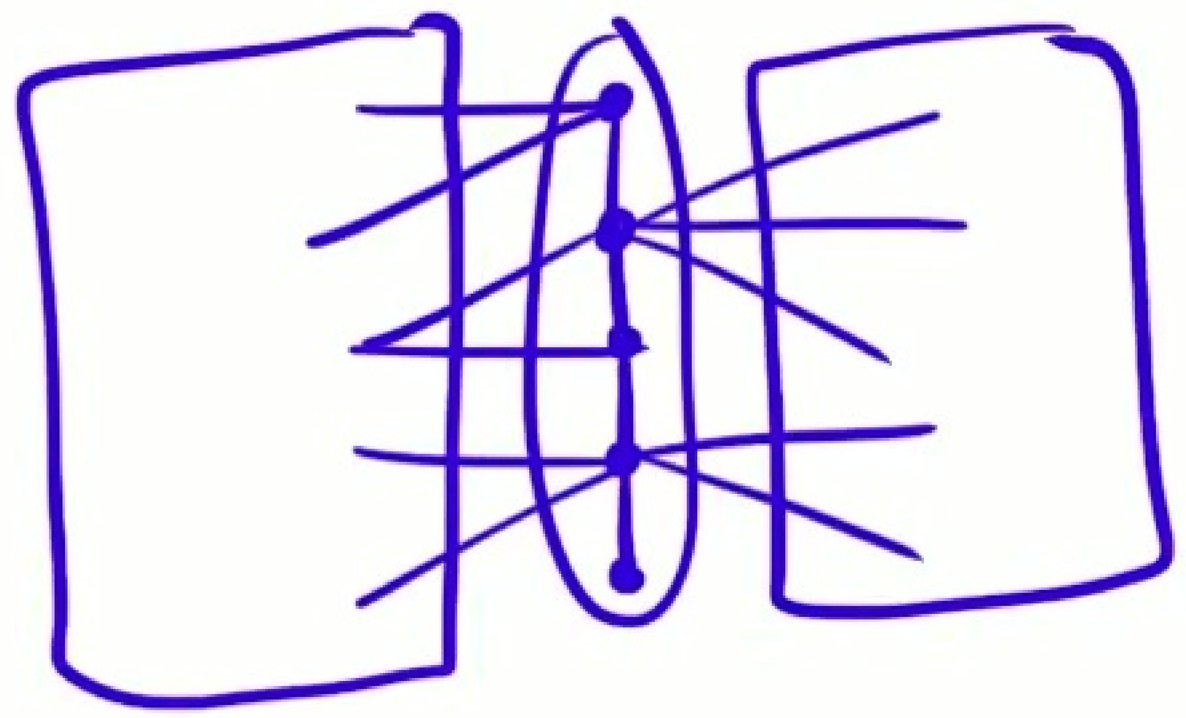


Thm: Every planar graph diameter-D has len-$O(D)$ balanced path separator.

Divide + Conquer

Diameter computation?



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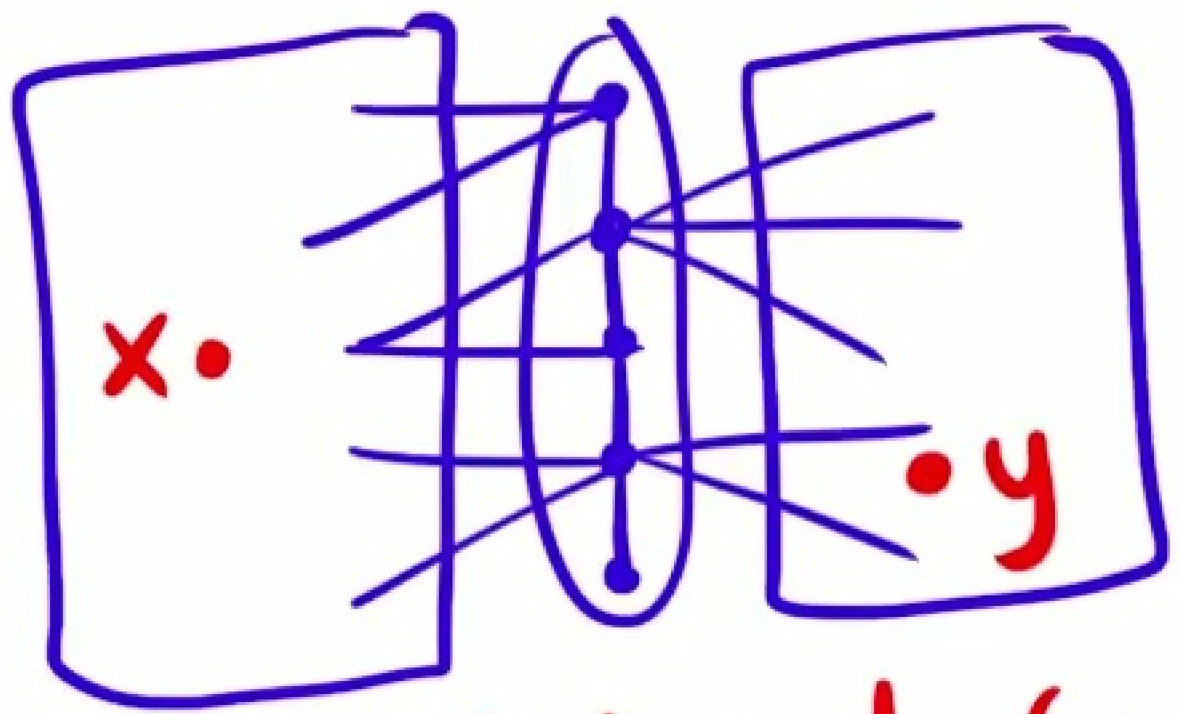

$\mathrm{diam}(G) = d_G(x, y)$

Divide + Conquer

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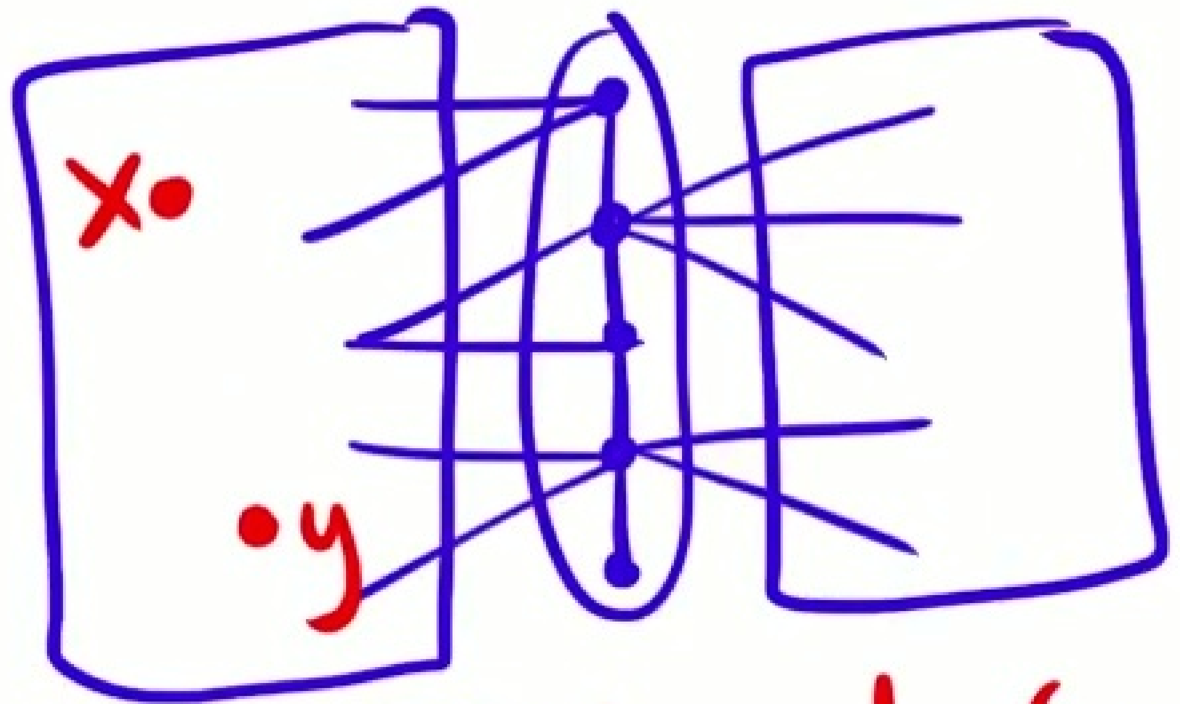

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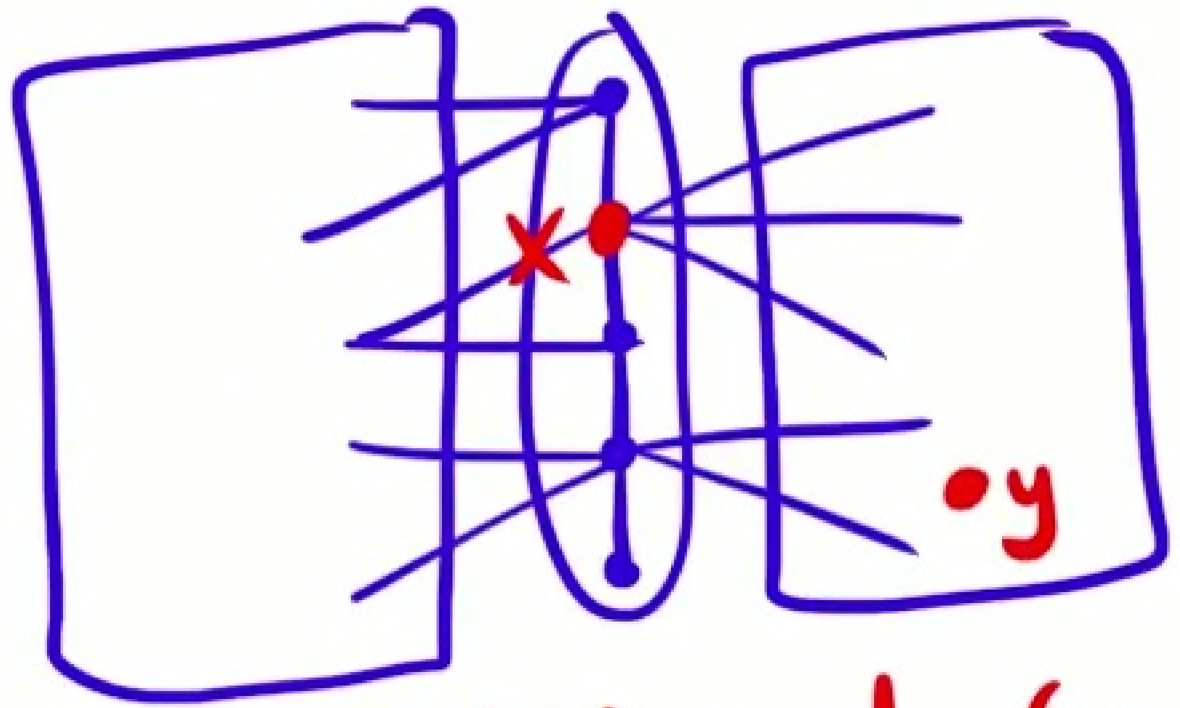

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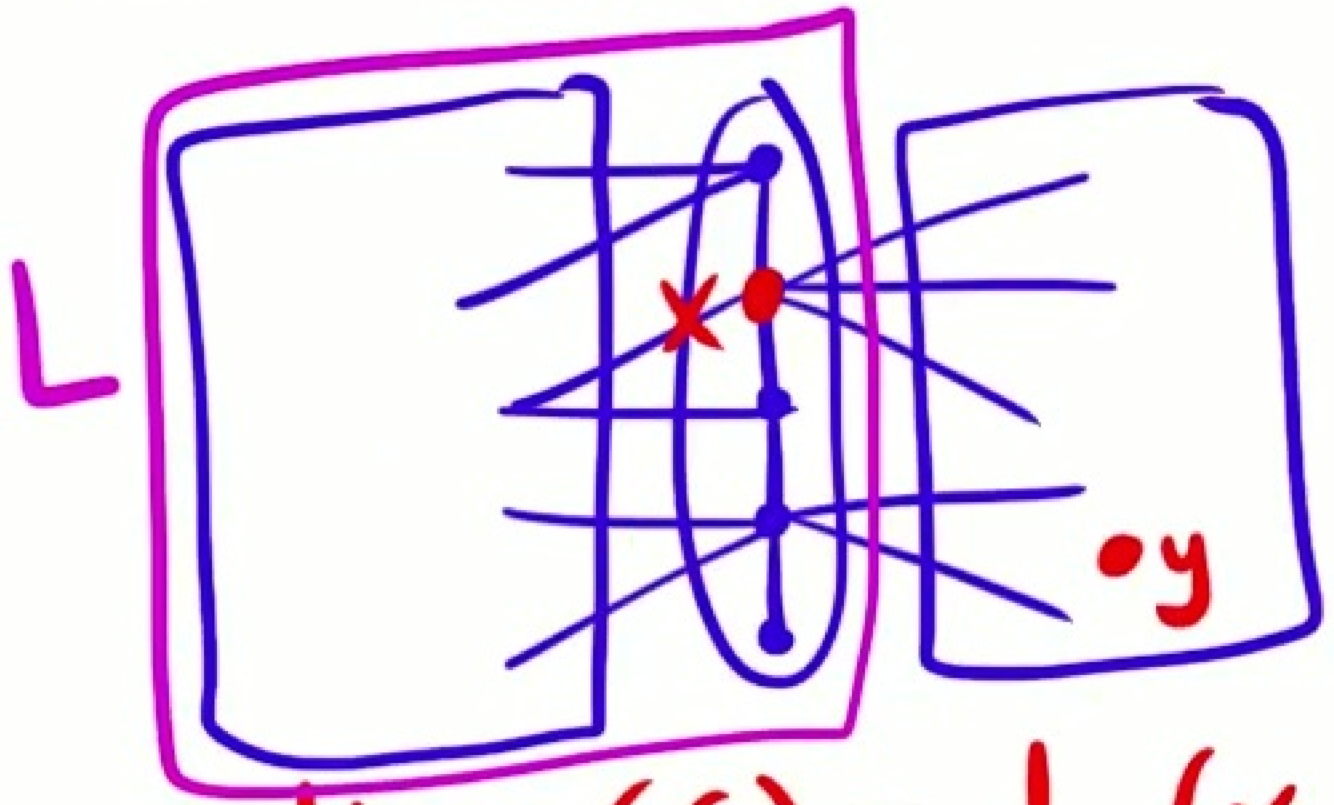

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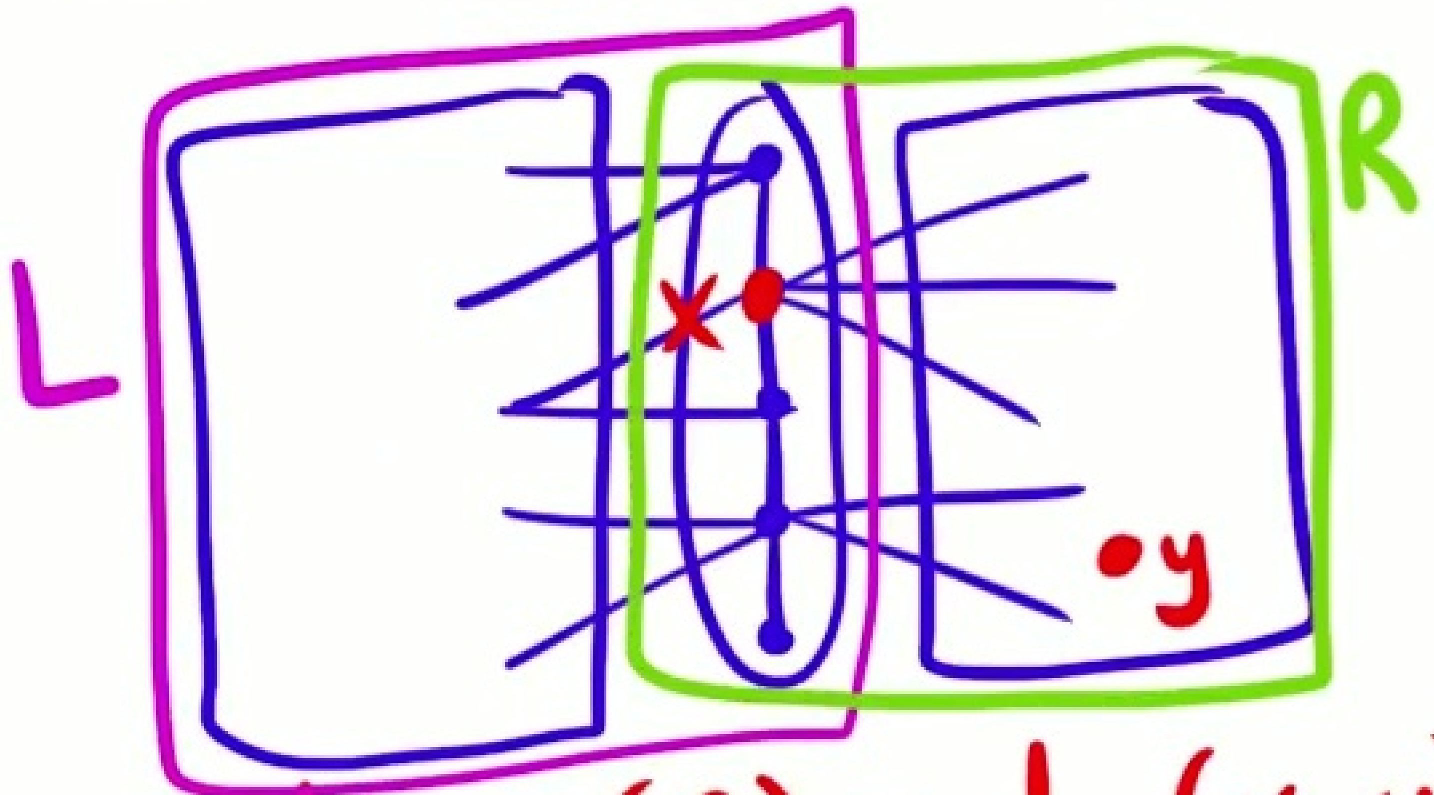

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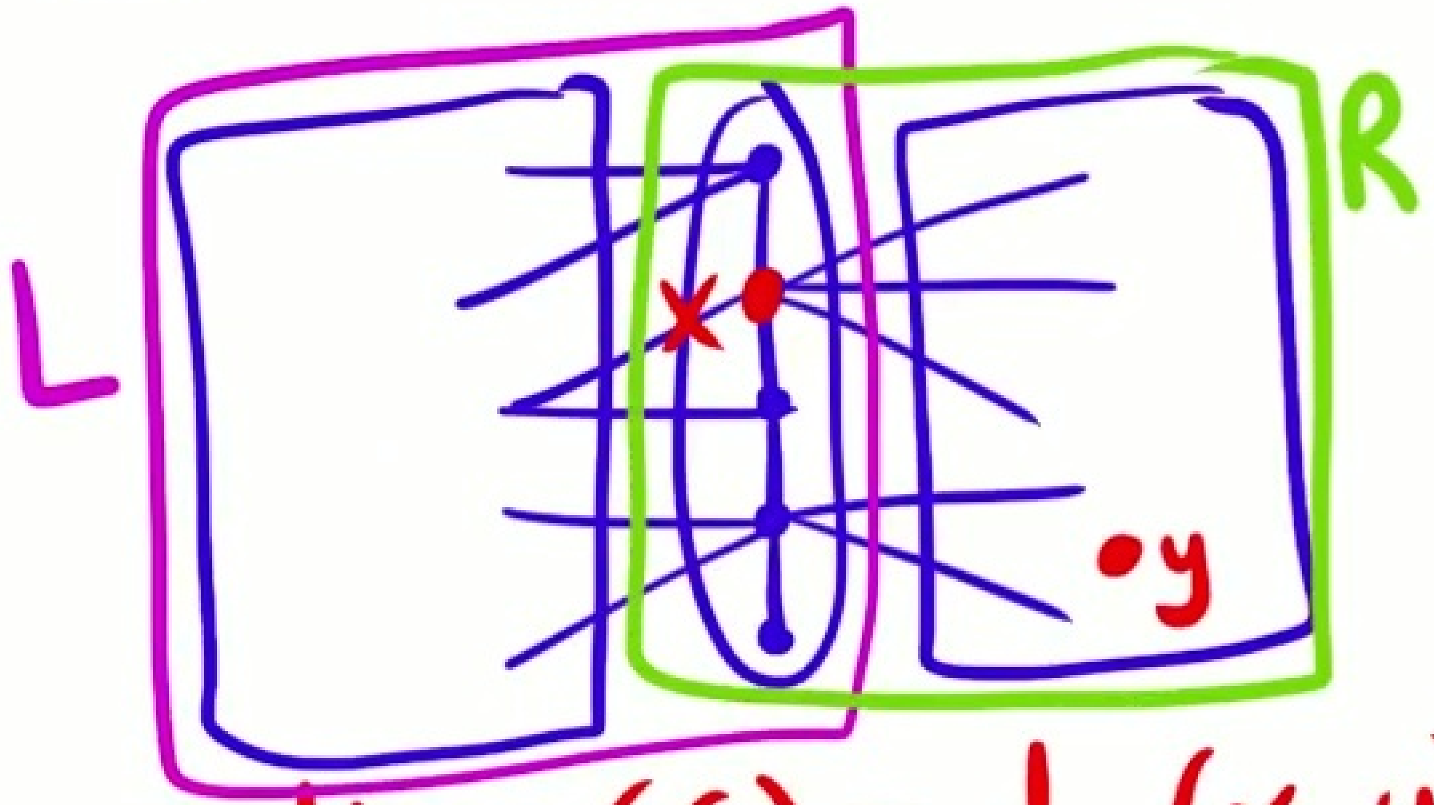

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Divide + Conquer

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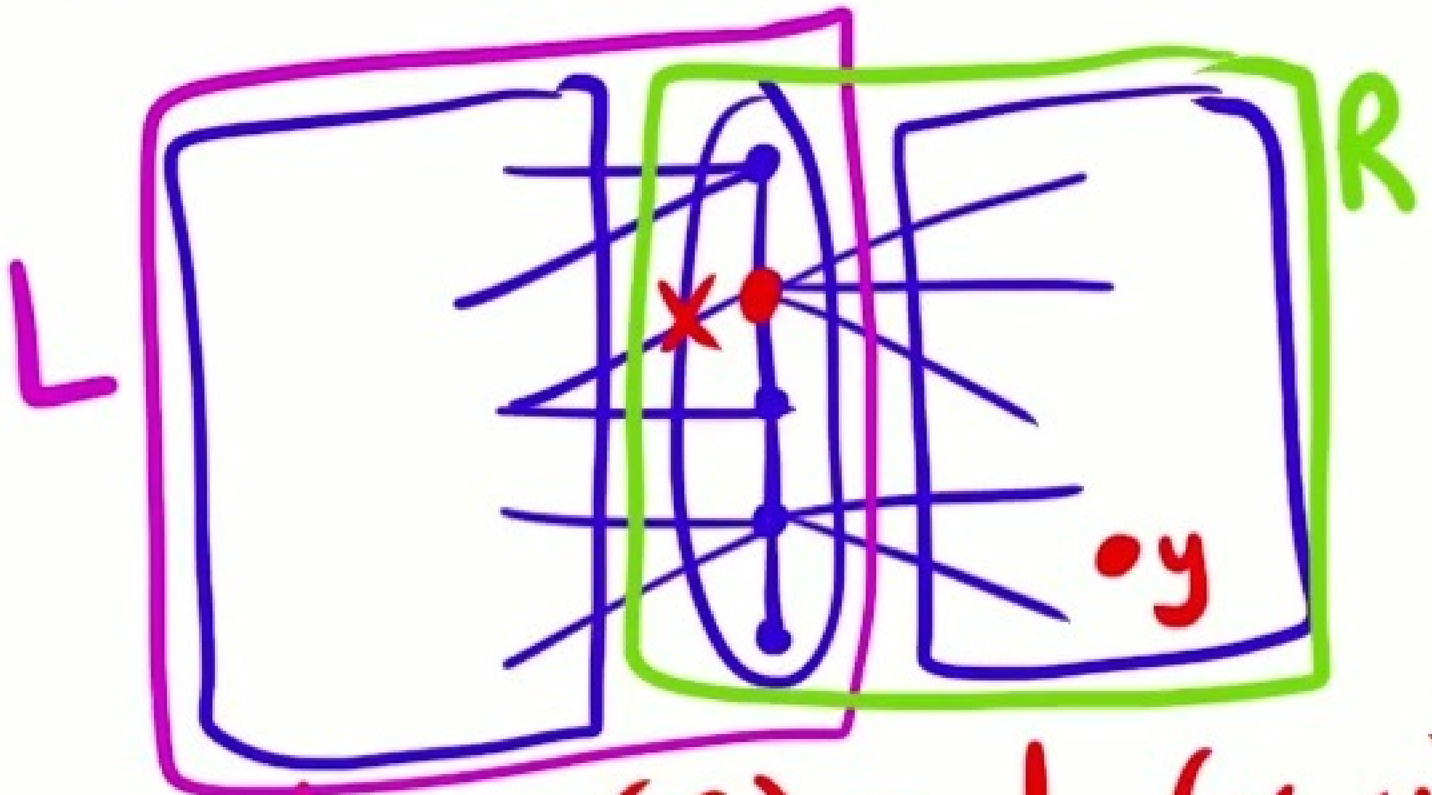

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$$\max_{x,y} d_G(x,y) = \max\left\{ \overset{T(|L|+D^{O(1)})}{\max_{x,y\in L} d_G(x,y)},\ \overset{T(|R|+D^{O(1)})}{\max_{x,y\in R} d_G(x,y)},\ \underbrace{\max_{x\in L,\, y\in R} d_G(x,y)}_{T_{conquer}(n)} \right\}$$

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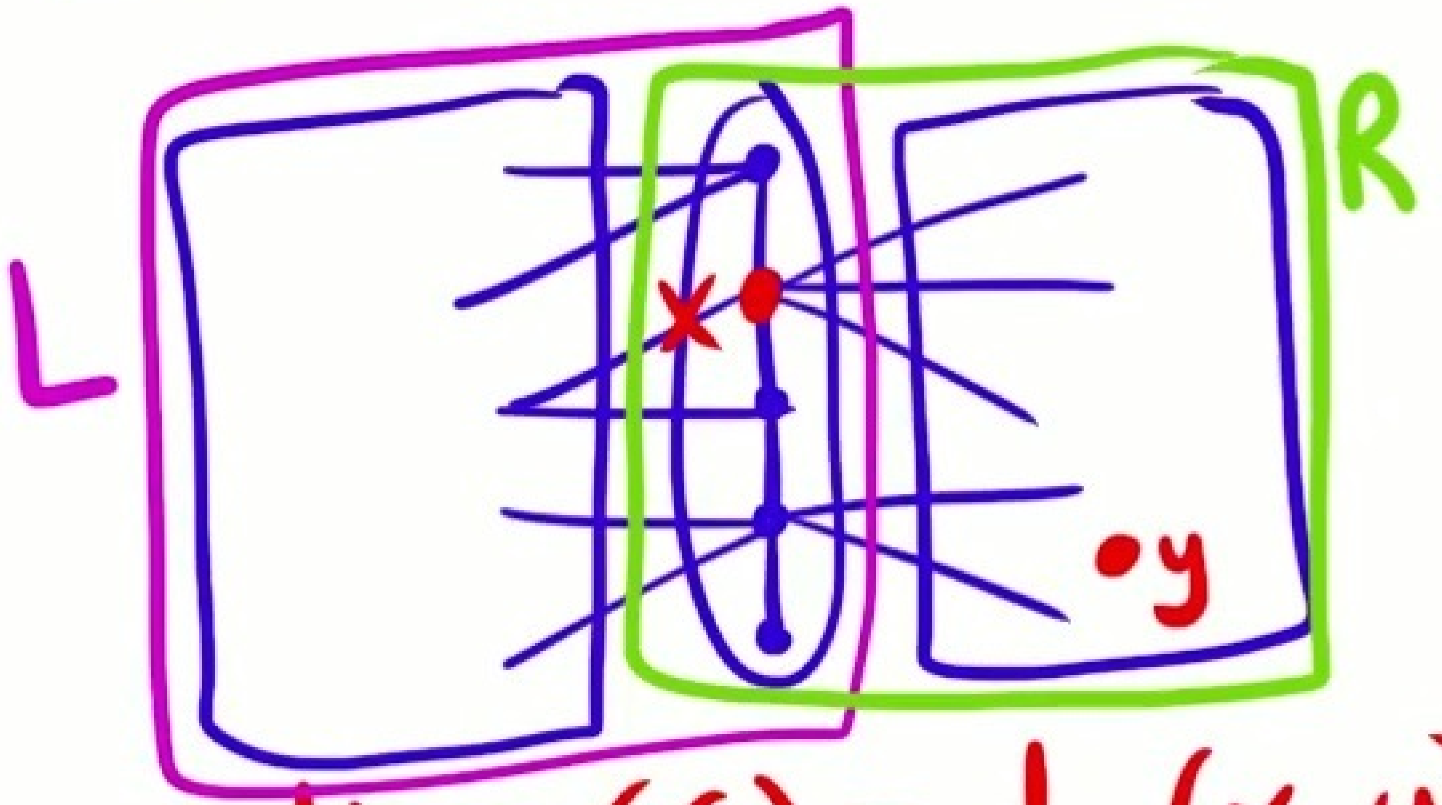

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$$T(n) = T(|L|+D^{O(1)}) + T(|R|+D^{O(1)}) + T_{conquer}(n)$$

Divide + Conquer

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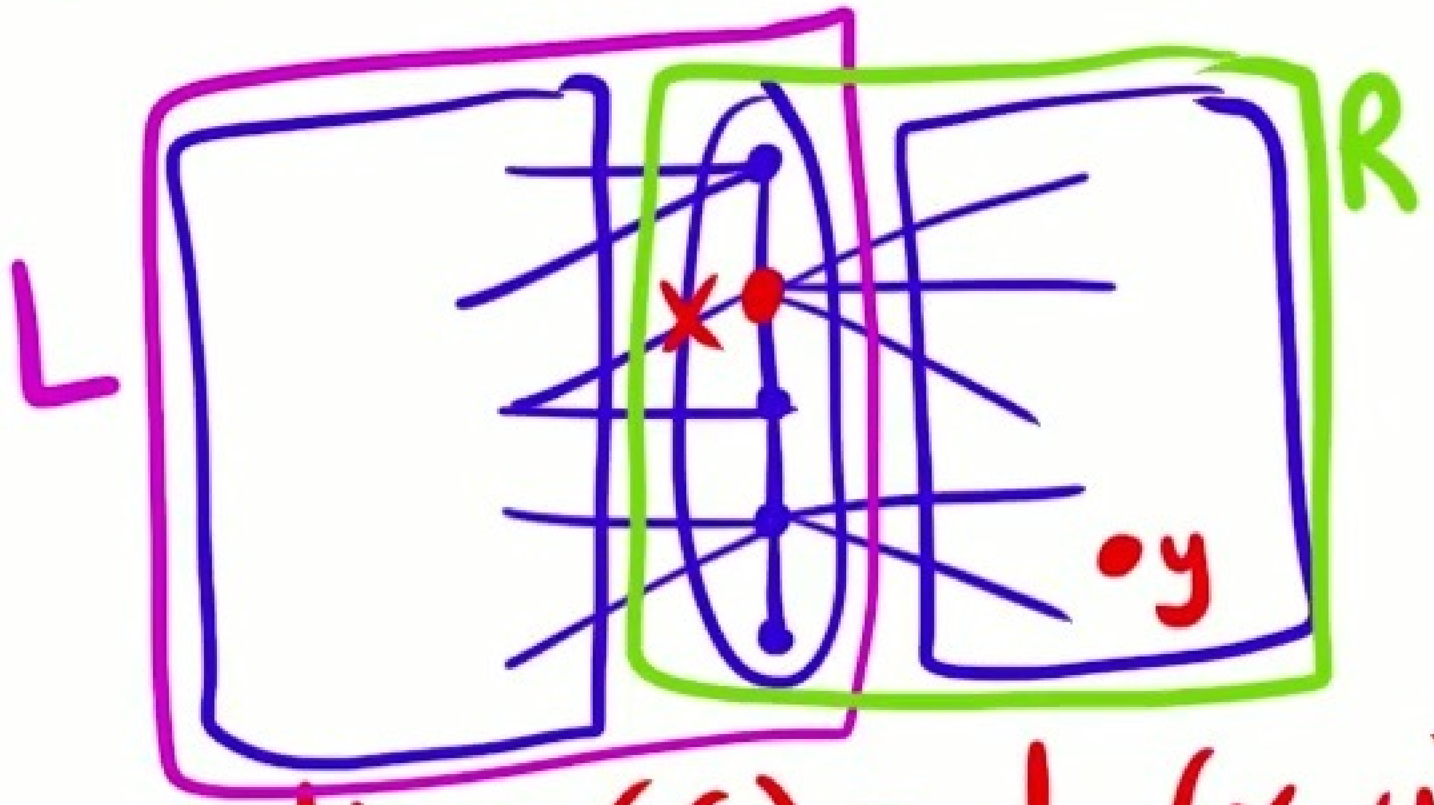

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$$|L|, |R| \leq 0.9n + O(D)$$

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Divide + Conquer

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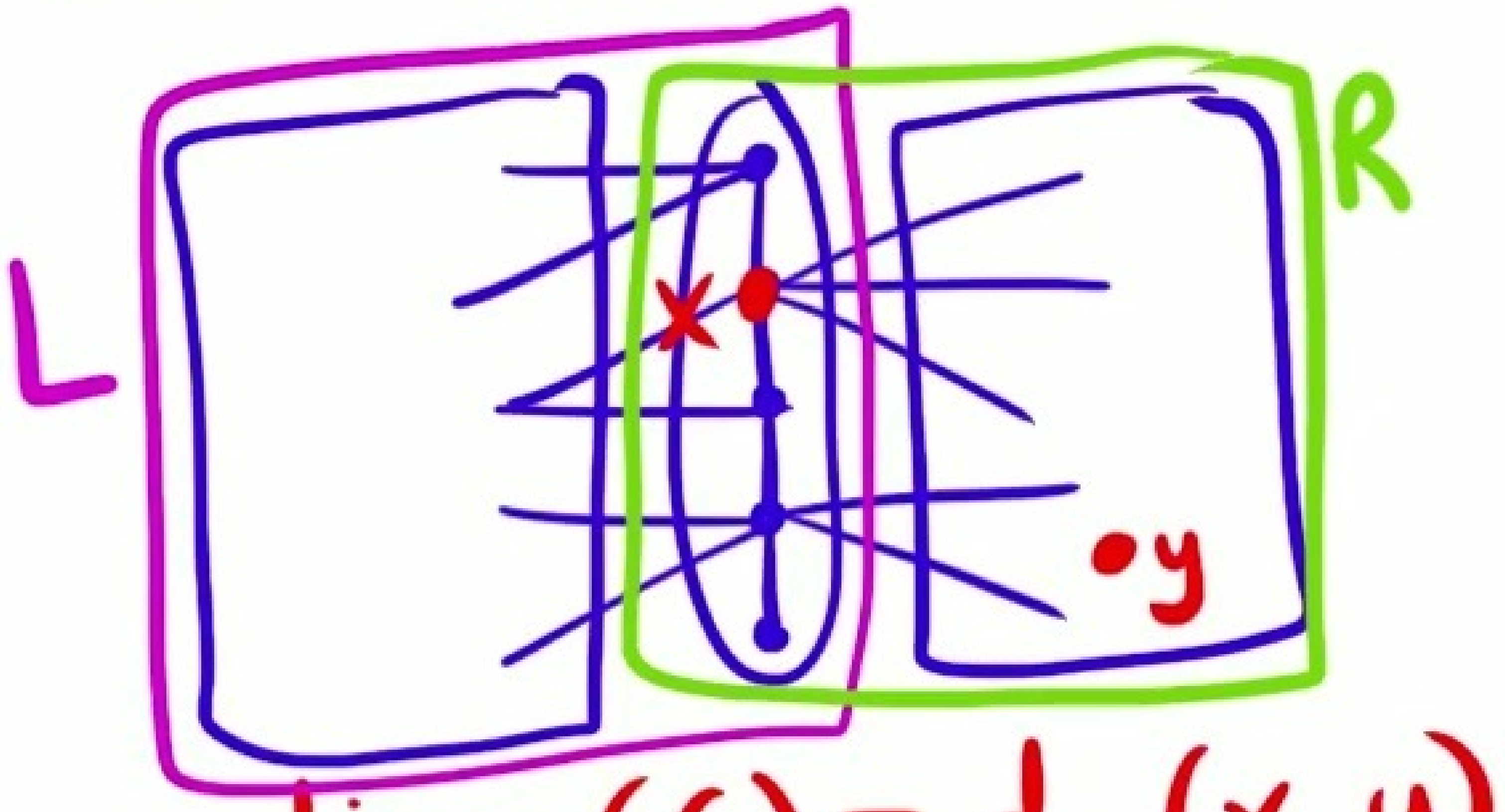

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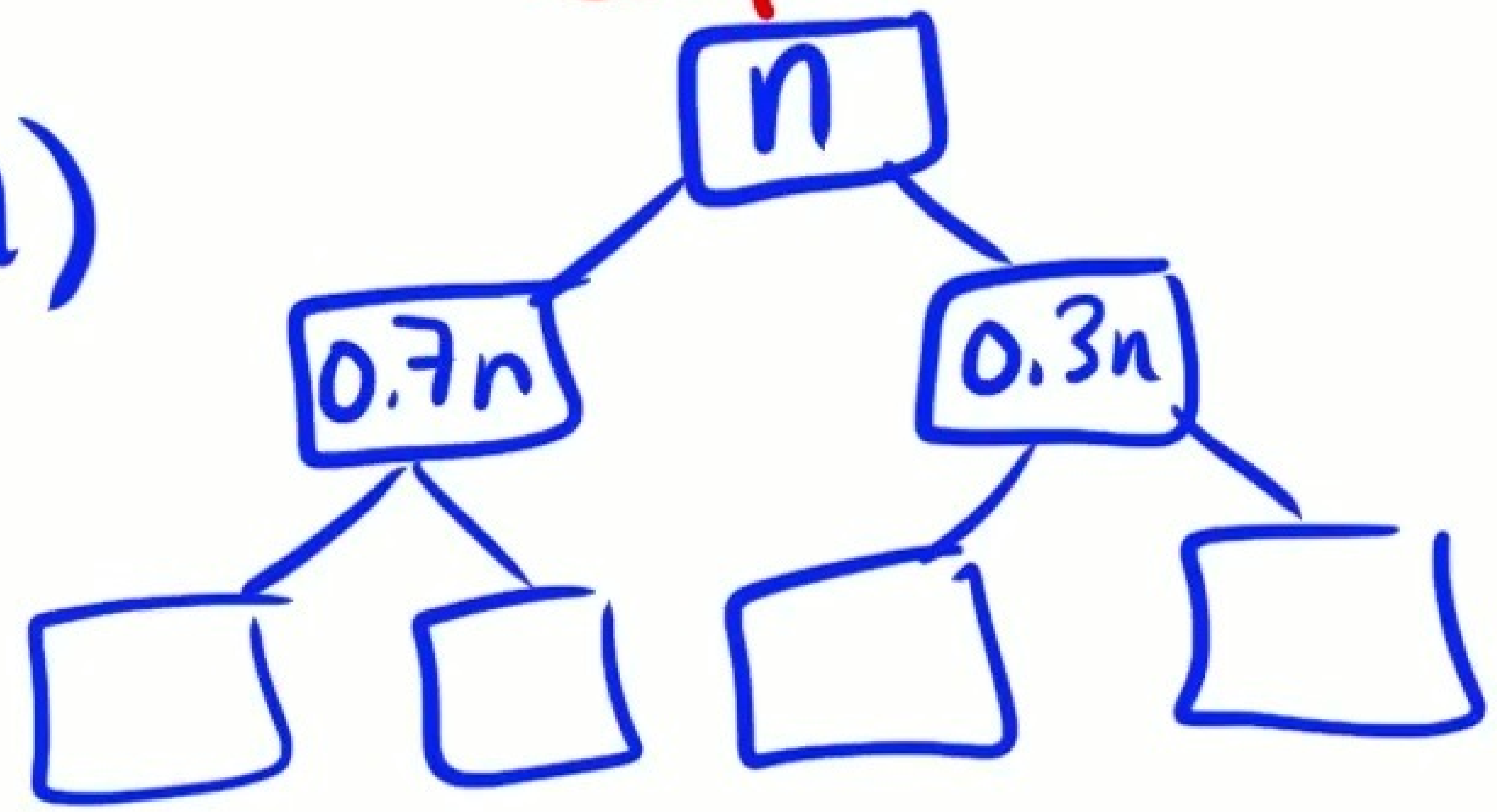

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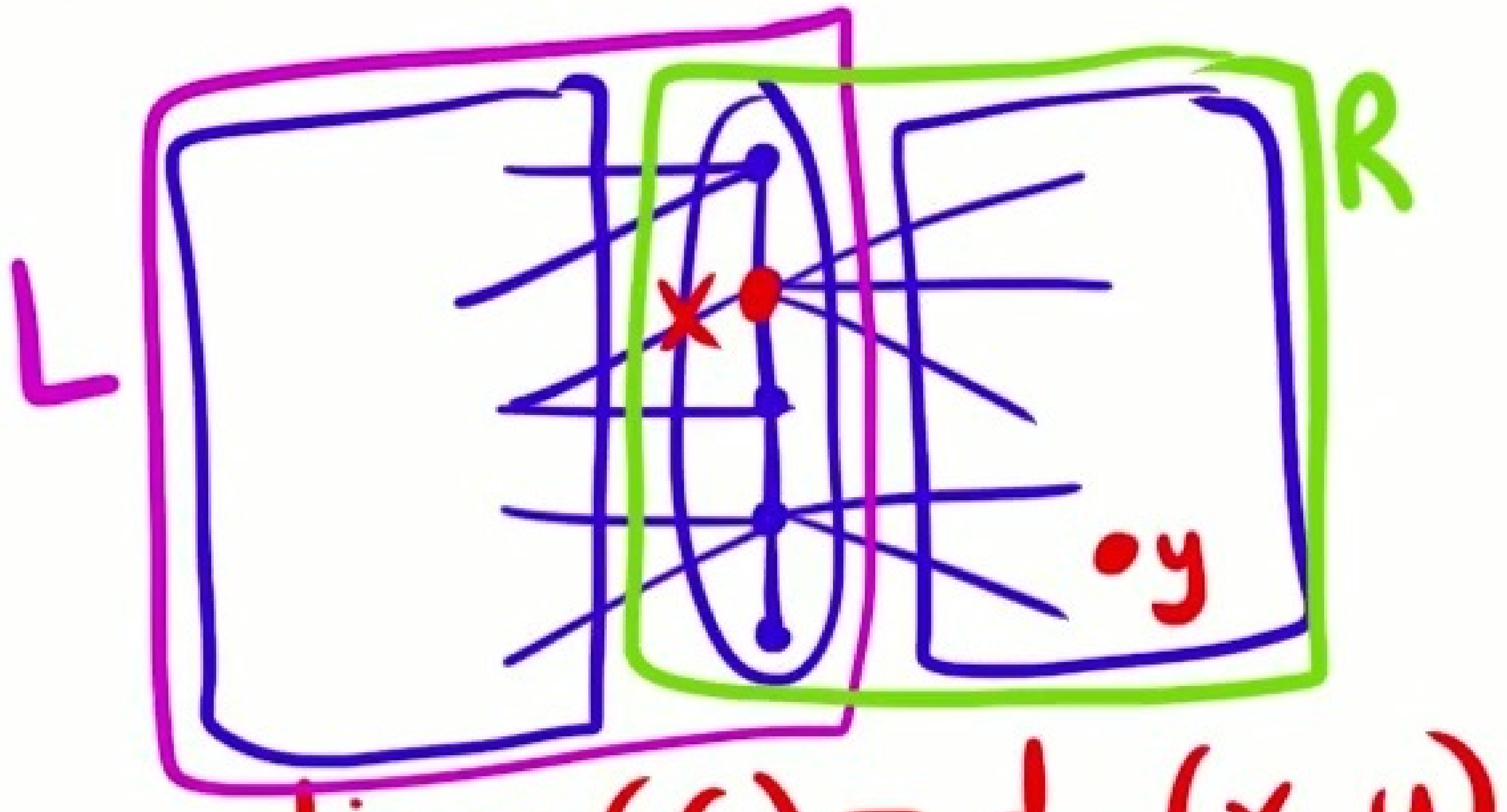

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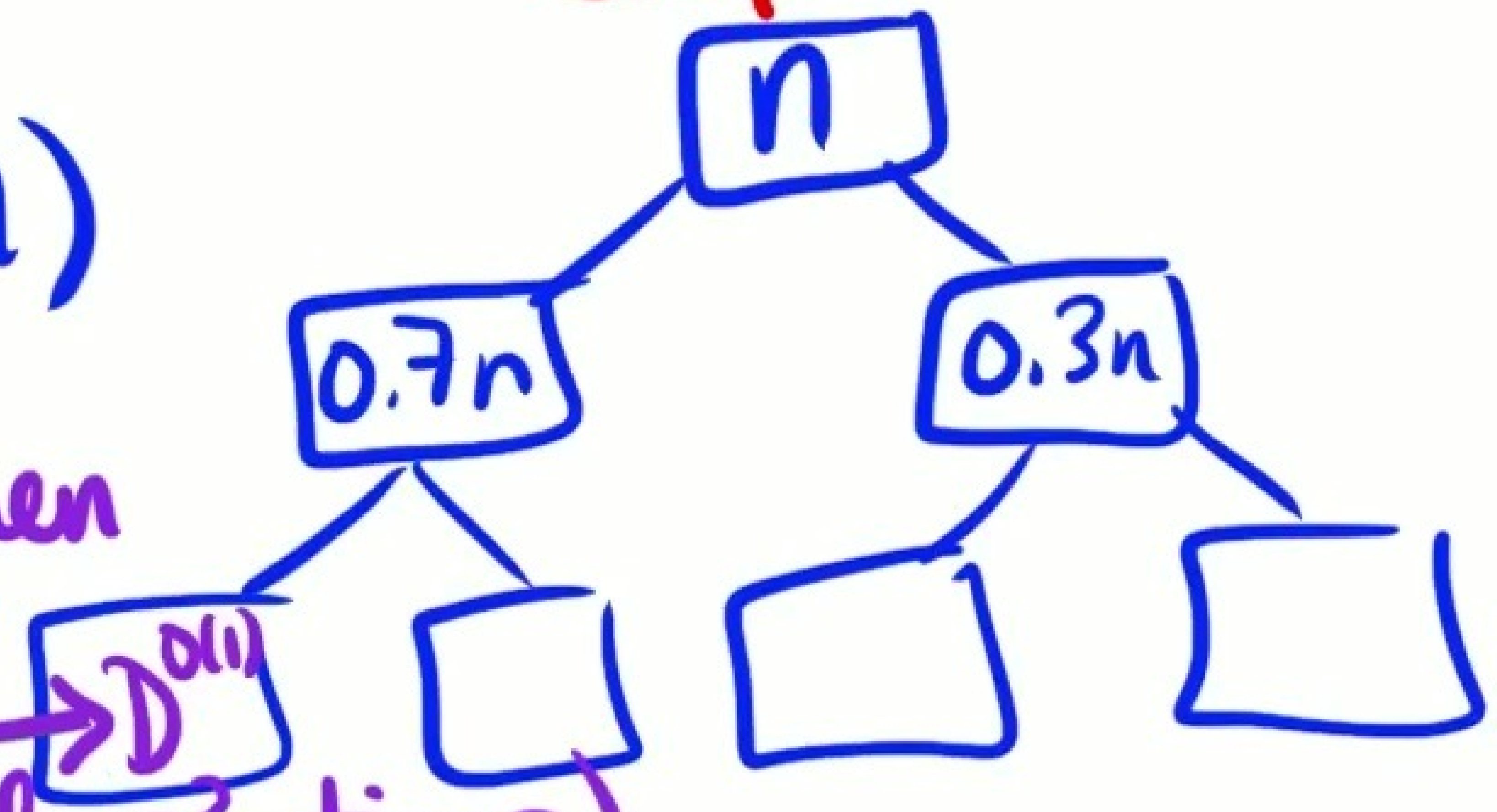

stop when size $D^{O(1)}$ (run trivial n^2 time)

Divide + Conquer

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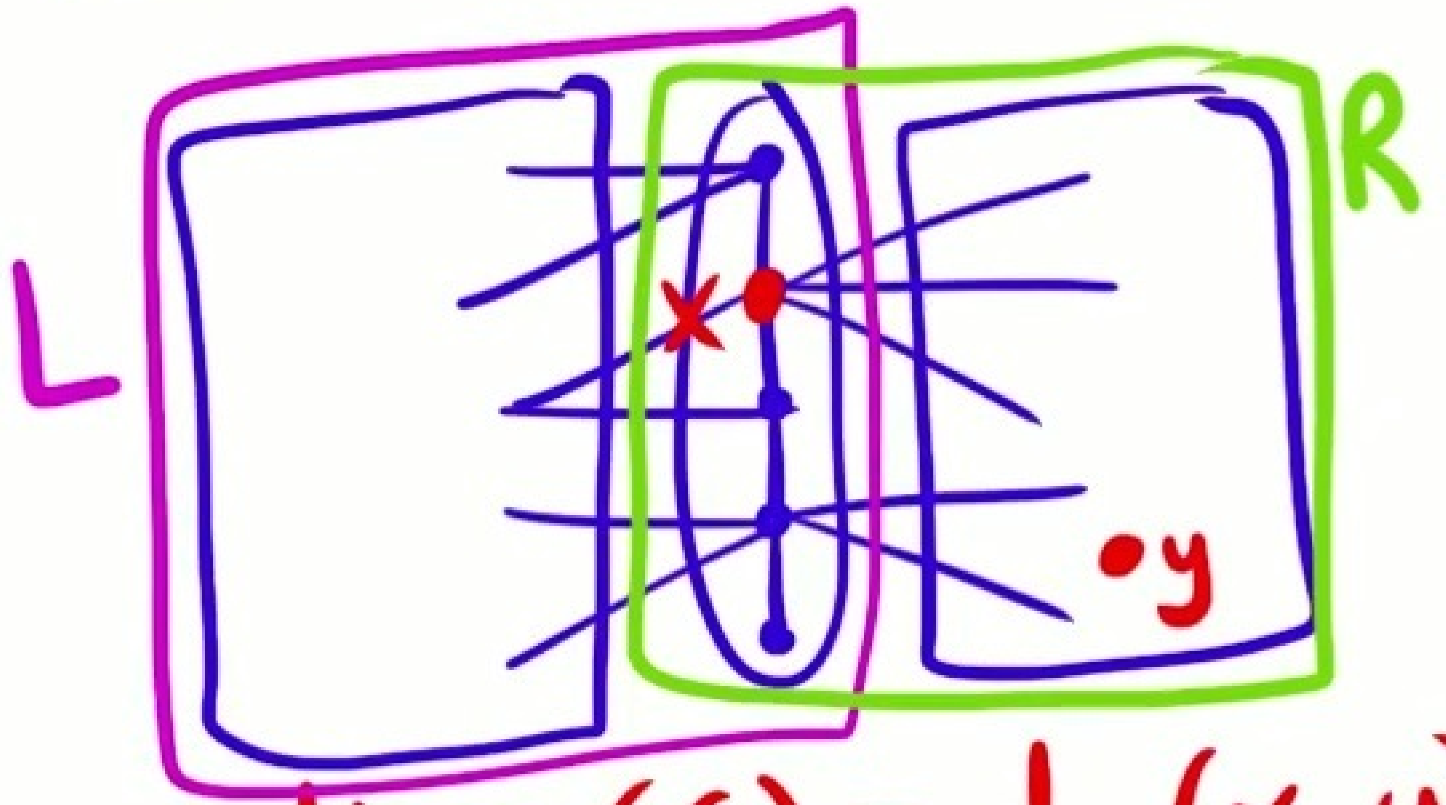

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$$\boxed{\text{If } T_{conquer}(n) = nD^{O(1)}, \text{ then } T(n) = nD^{O(1)}.}$$

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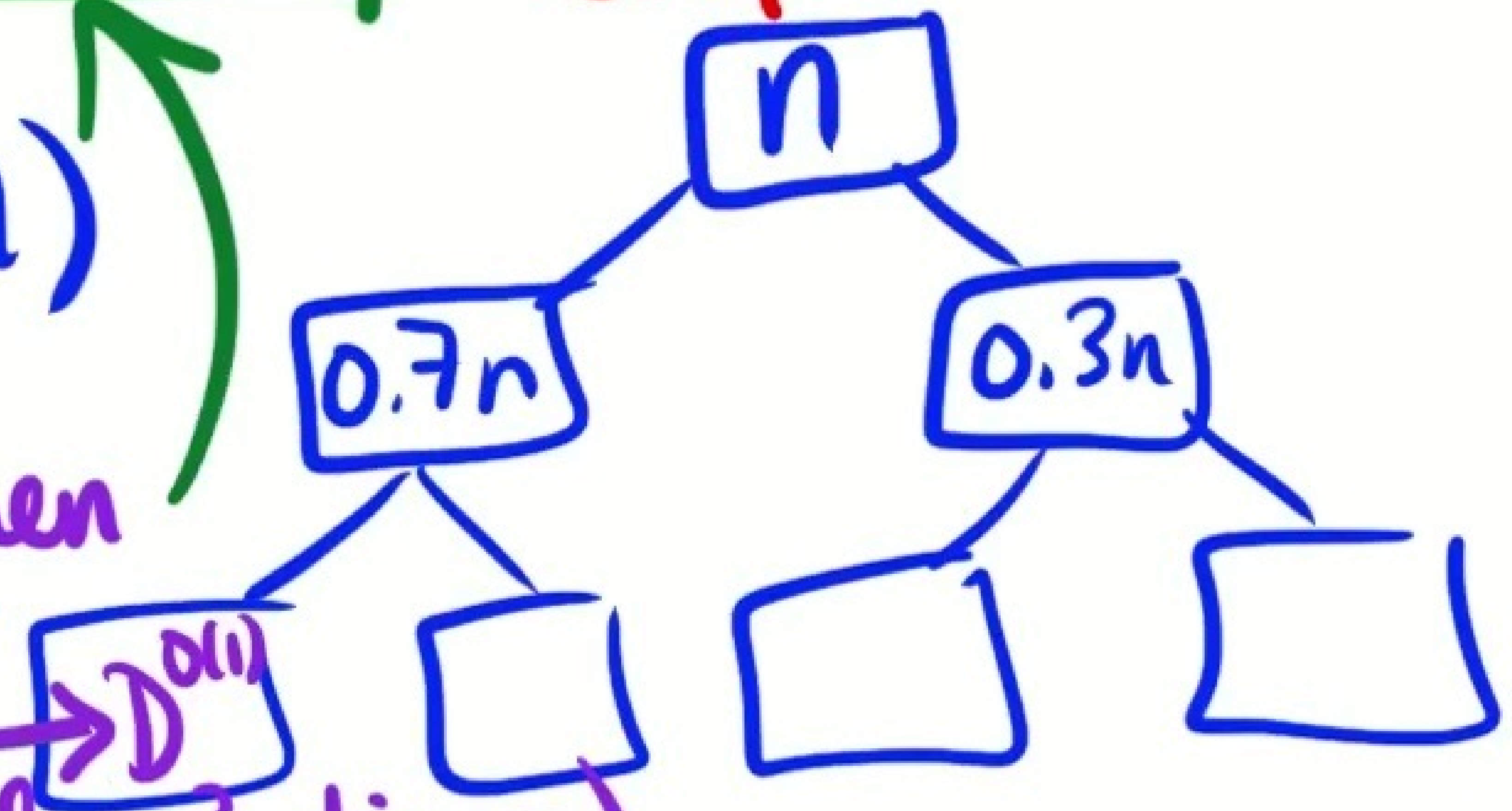

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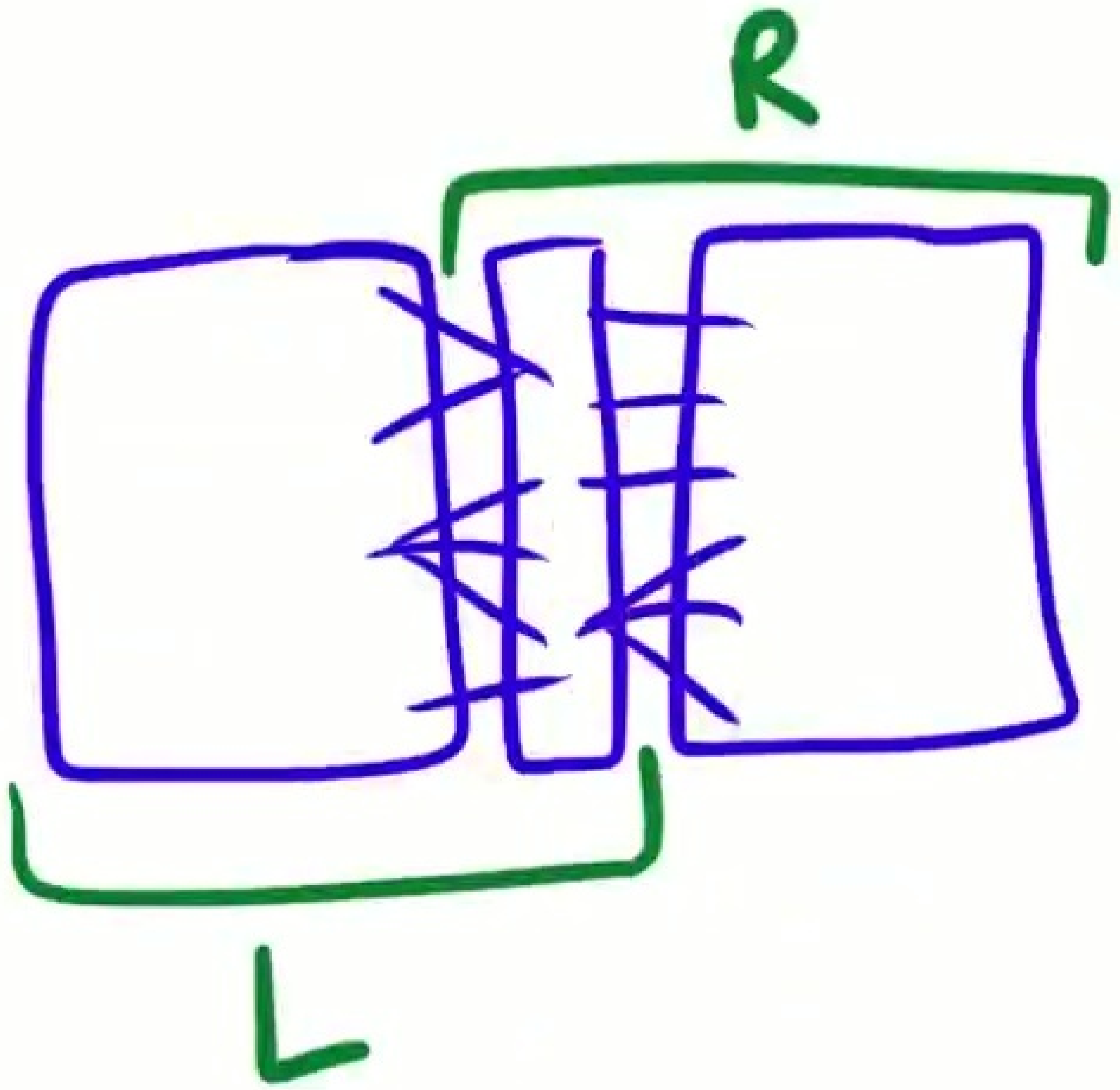
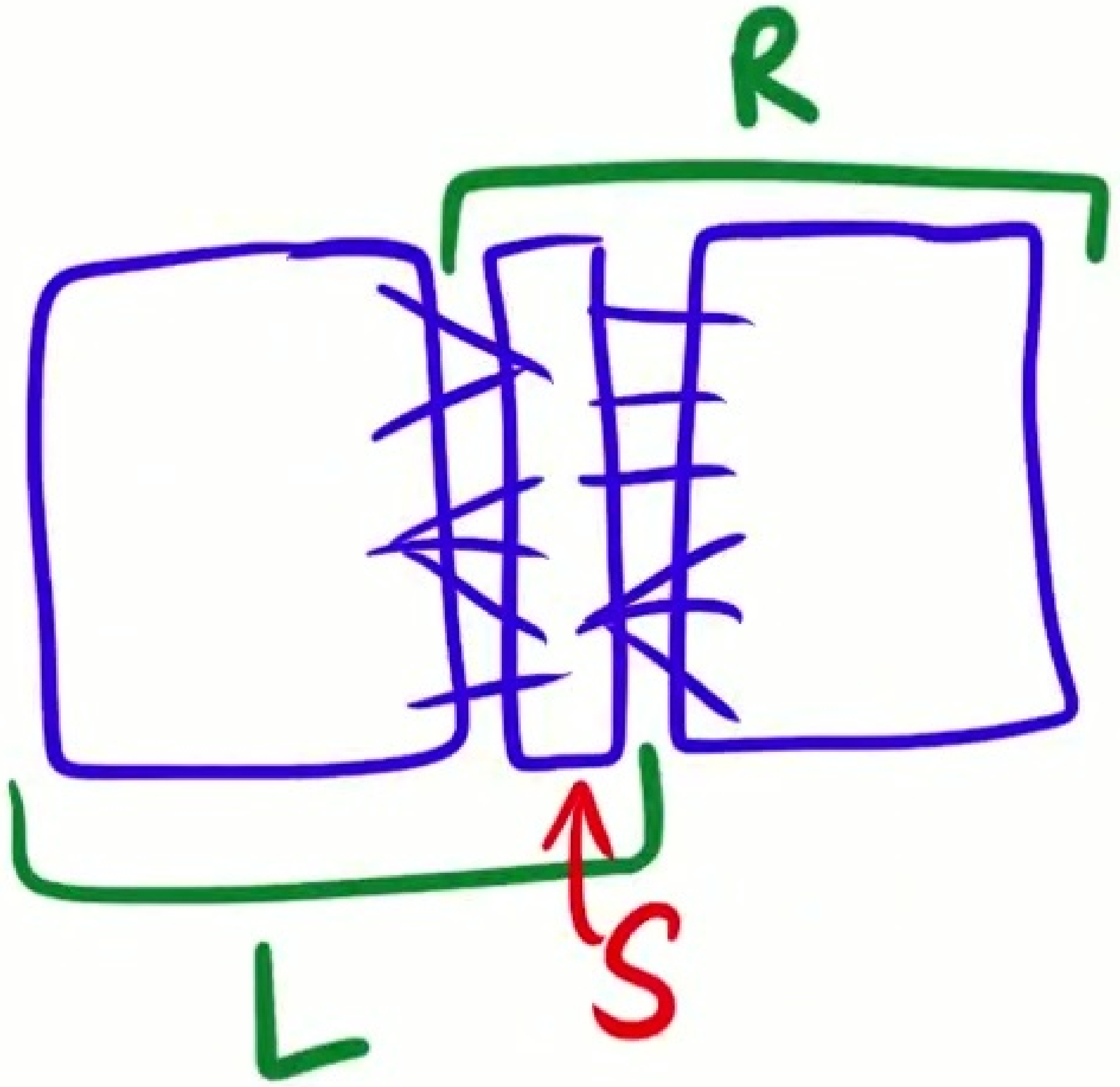
Conquer Step



$$\boxed{\text{compute } \max_{x \in L,\, y \in R} d(x,y)}$$

Conquer Step



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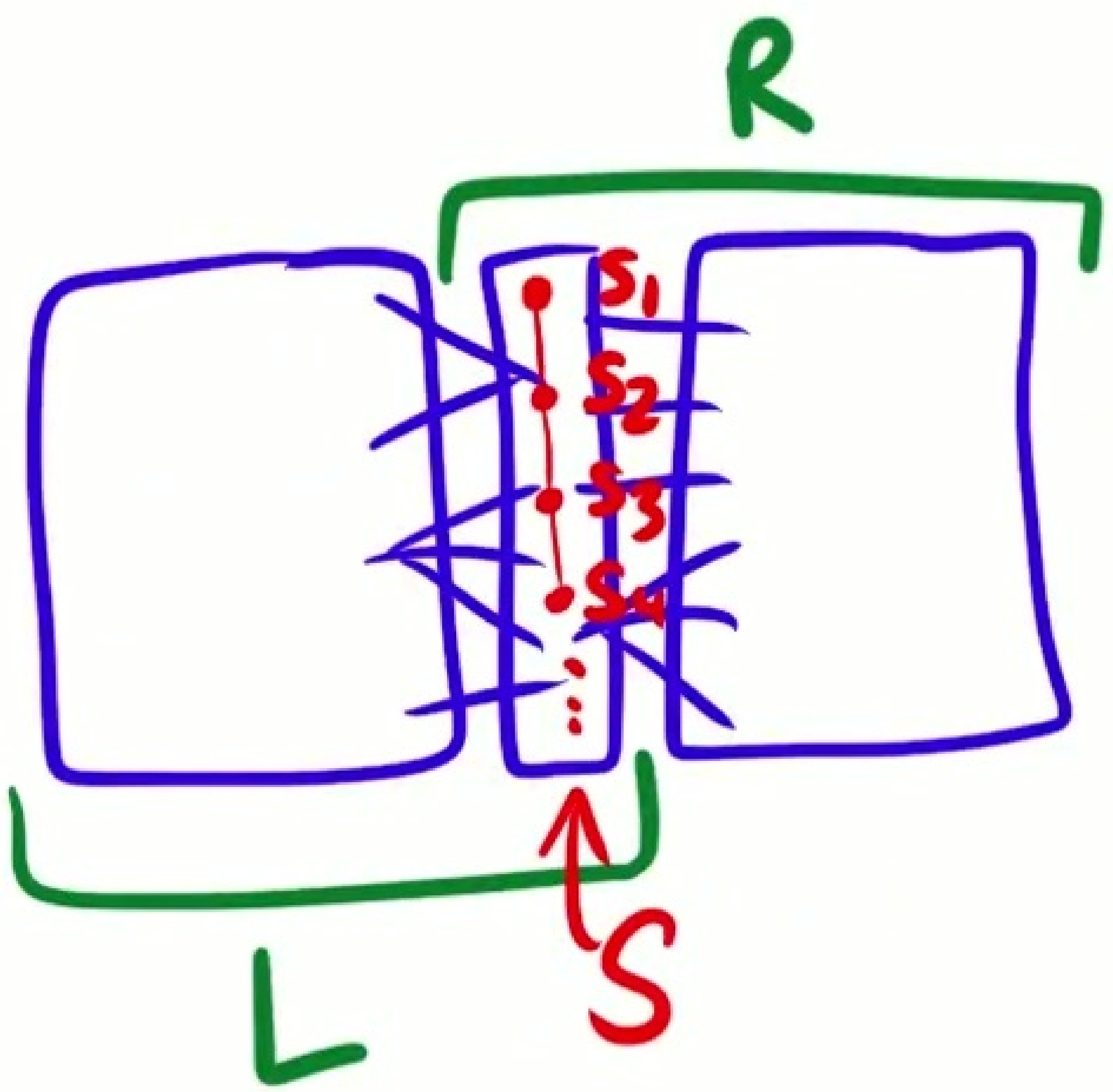
S is separator

$\Rightarrow \forall x \in L, y \in R:$

$$d(x,y) = \min_{s \in S} \big(d(x,s) + d(s,y) \big)$$

Conquer Step



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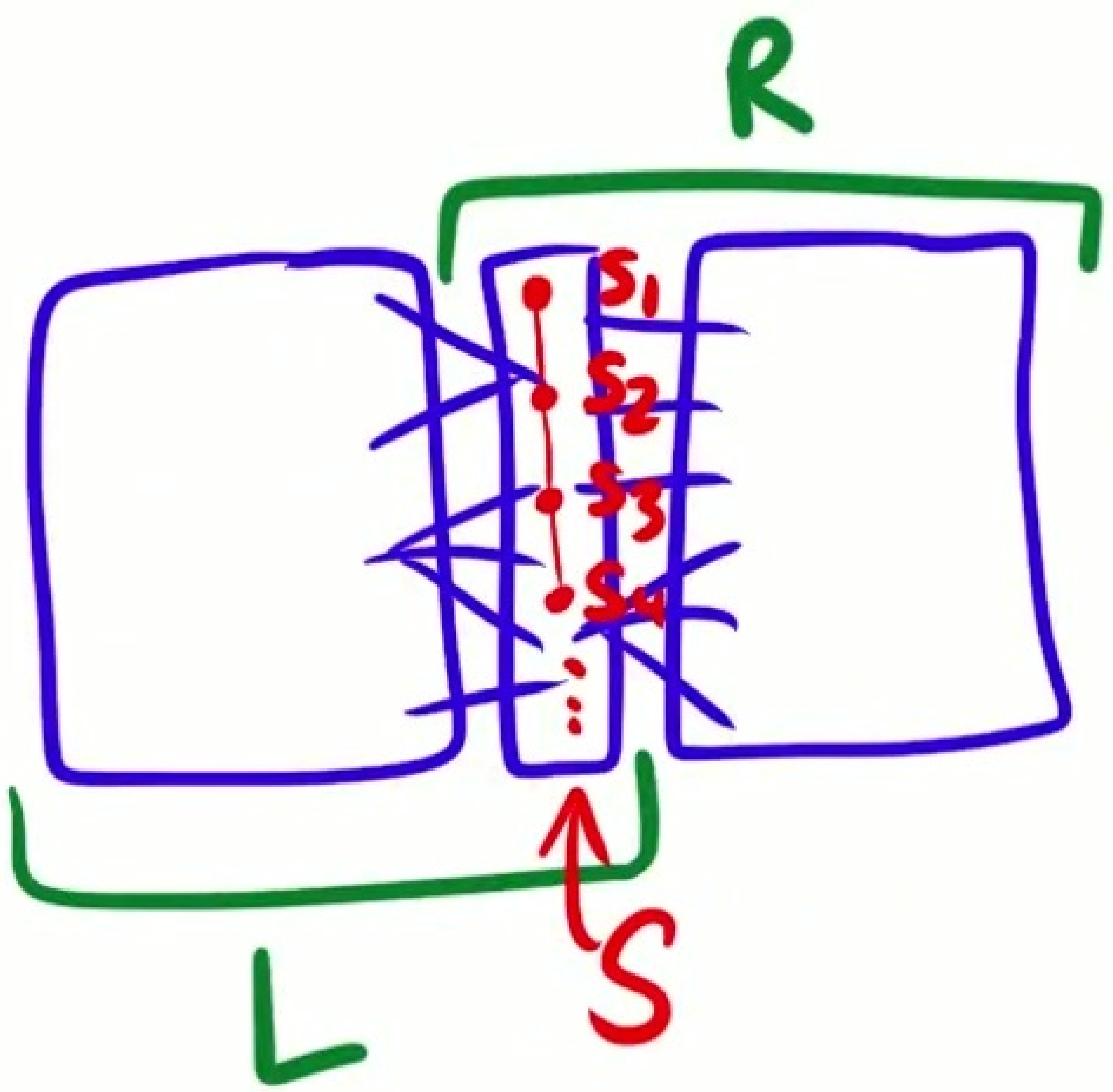

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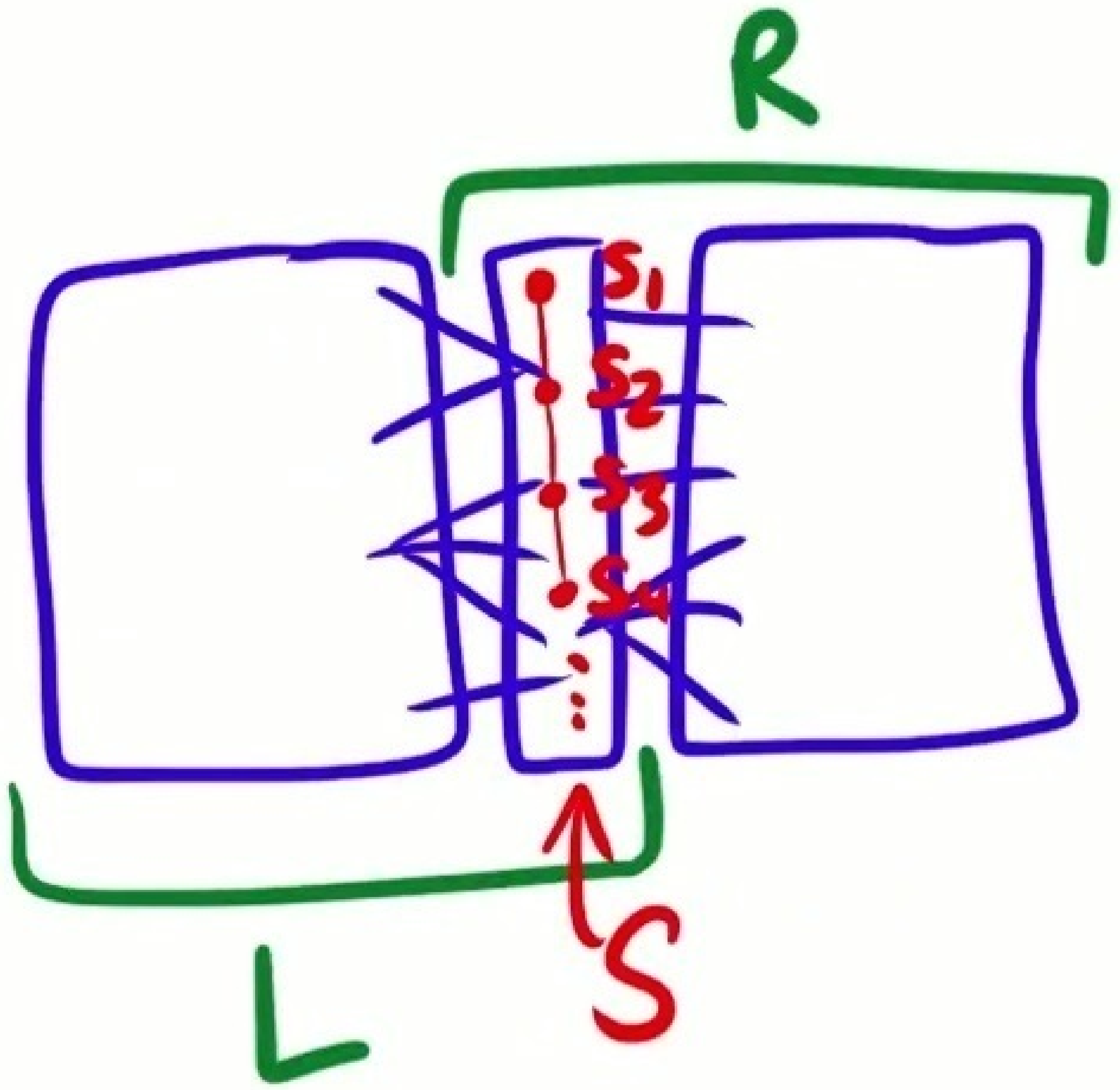
$\mathcal{R} := \{ \langle d(y,s_1), \ldots, d(y,s_k) \rangle : y \in R \}$

Conquer Step



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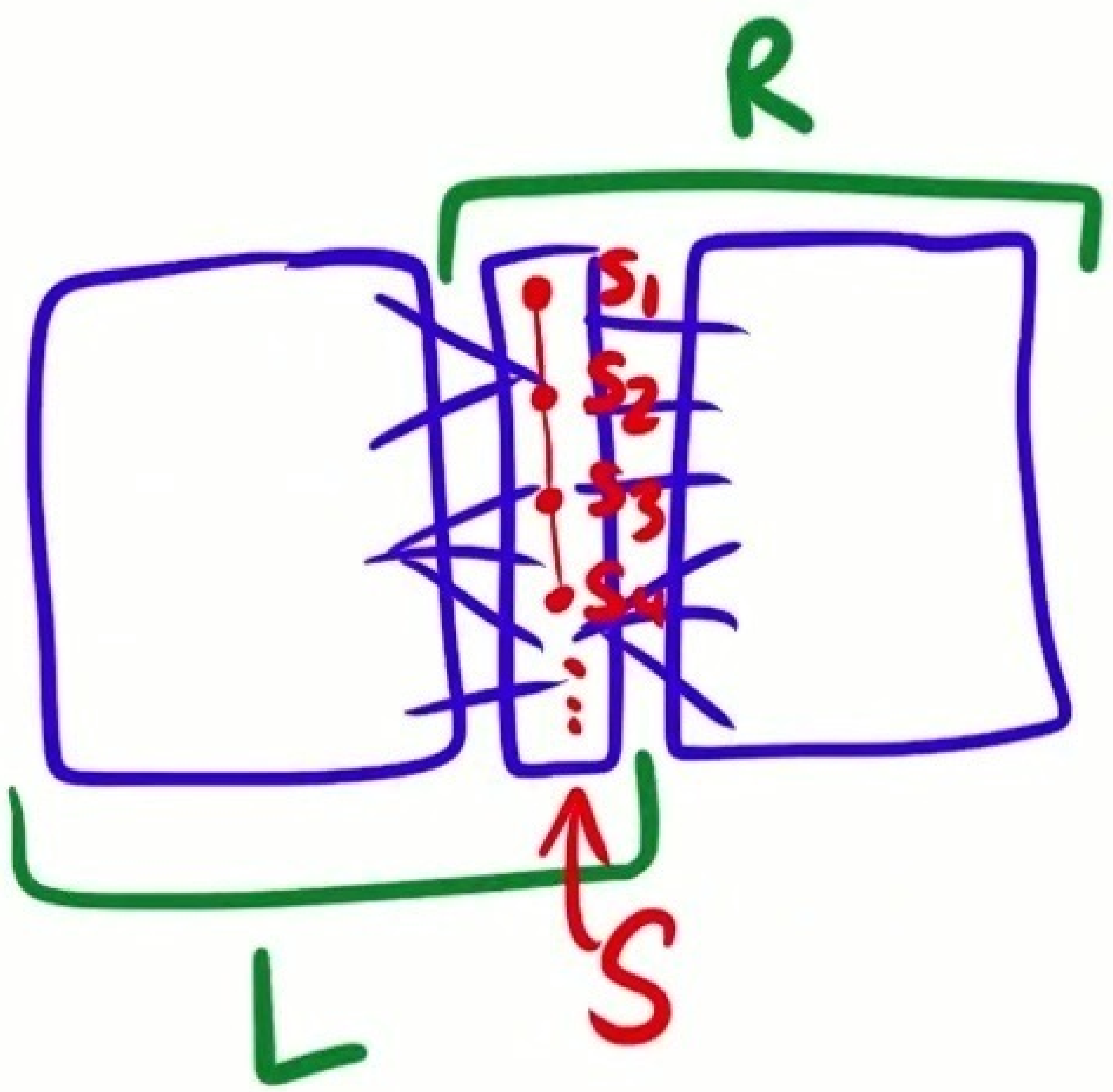
Then, want $\max_{u \in \mathcal{L}, v \in \mathcal{R}} \text{function}(u,v)$.

Conquer Step



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Then, want $\max_{u \in \mathcal{L}, v \in \mathcal{R}} \text{function}(u,v)$.

Runtime: $\tilde{O}(nD) + |\mathcal{L}| \cdot |\mathcal{R}| \cdot O(D)$

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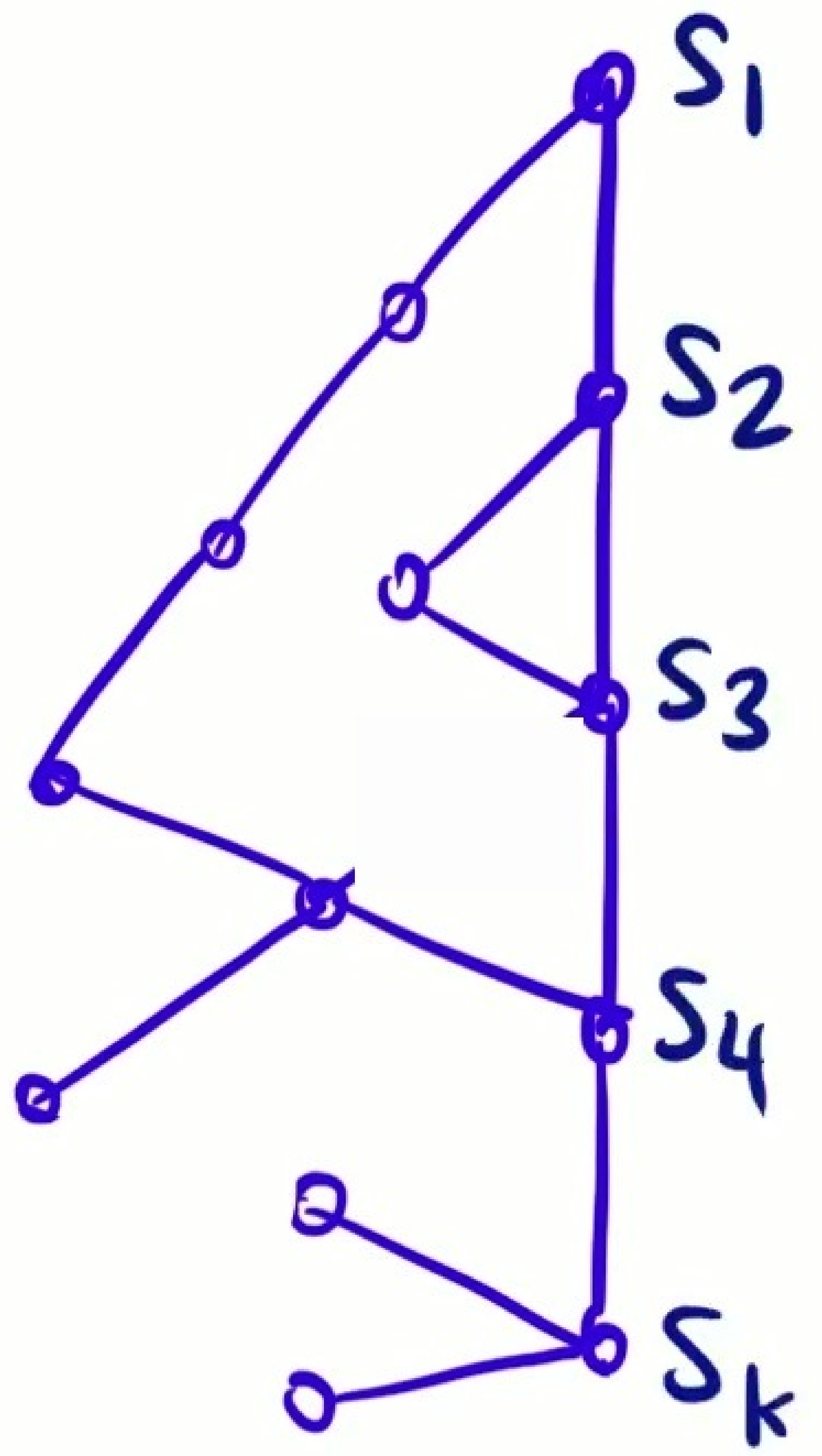

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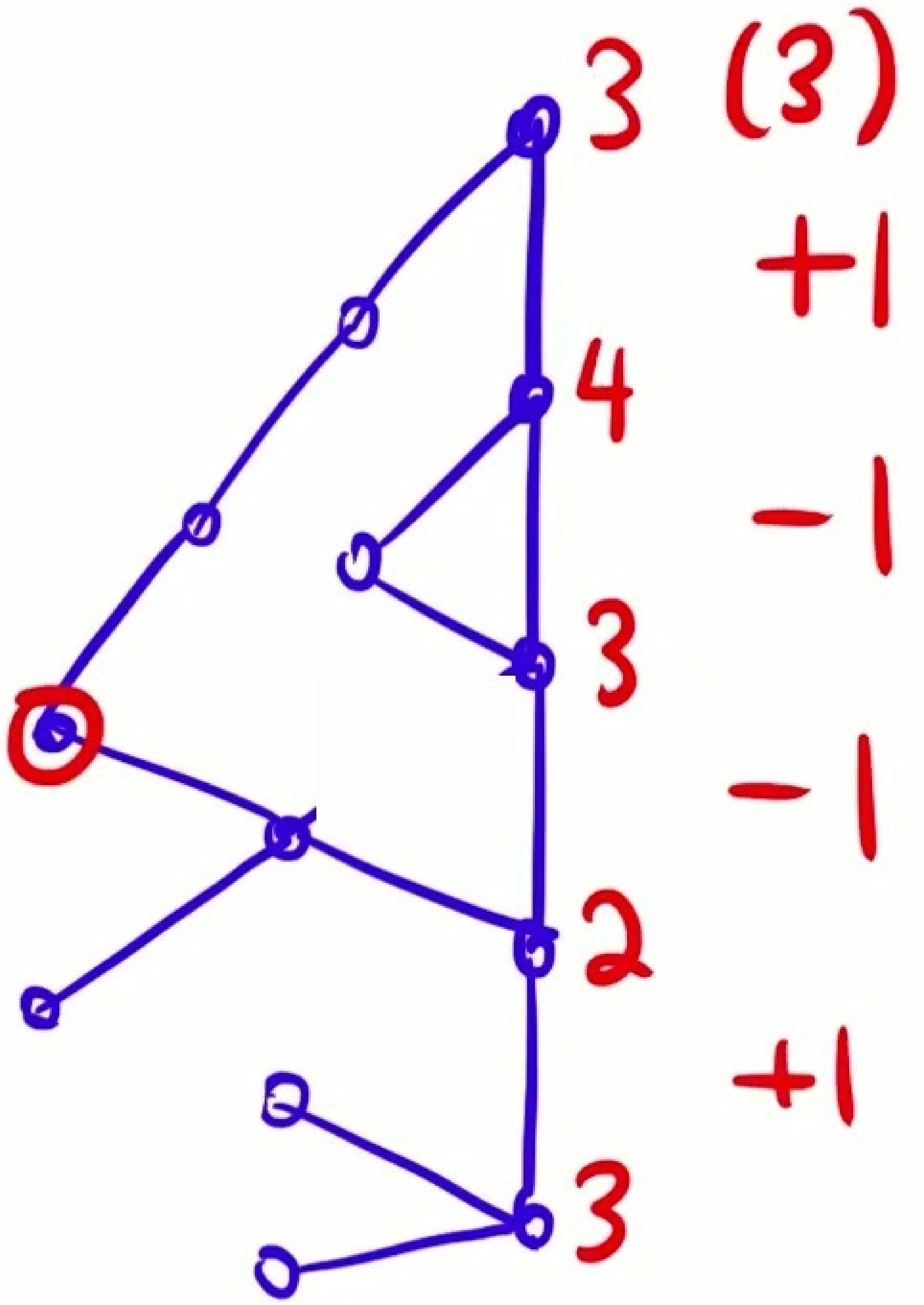

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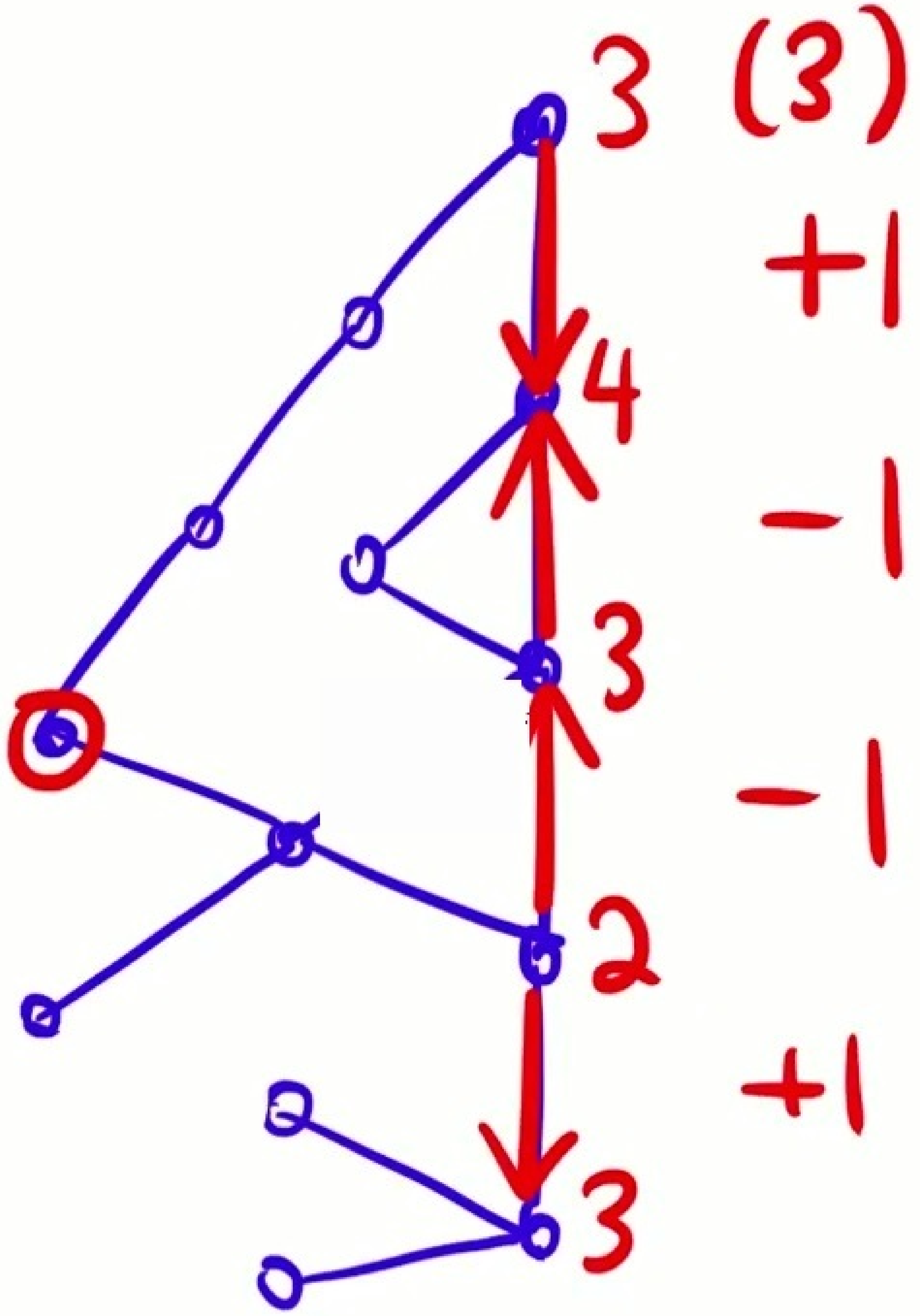

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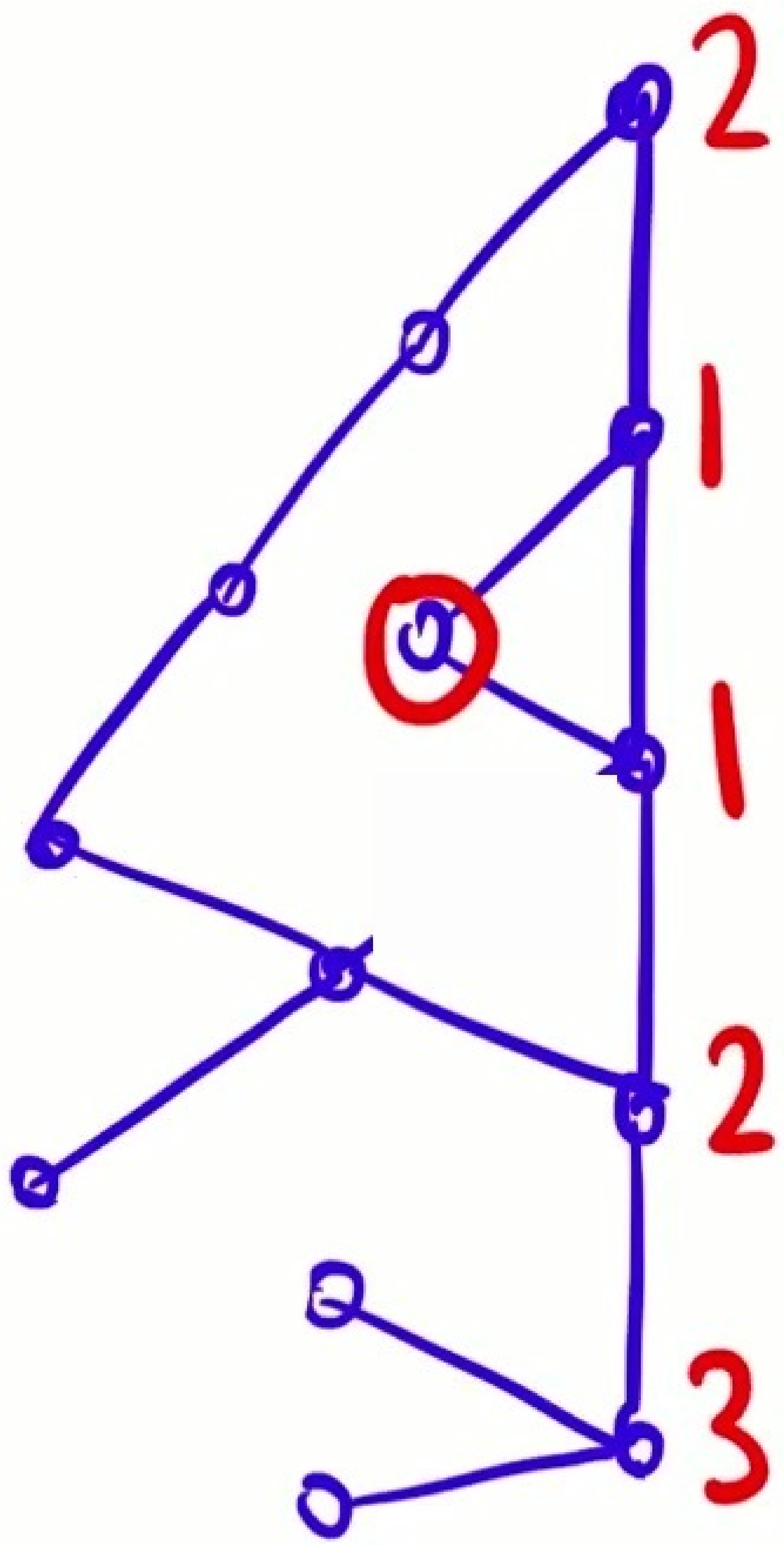

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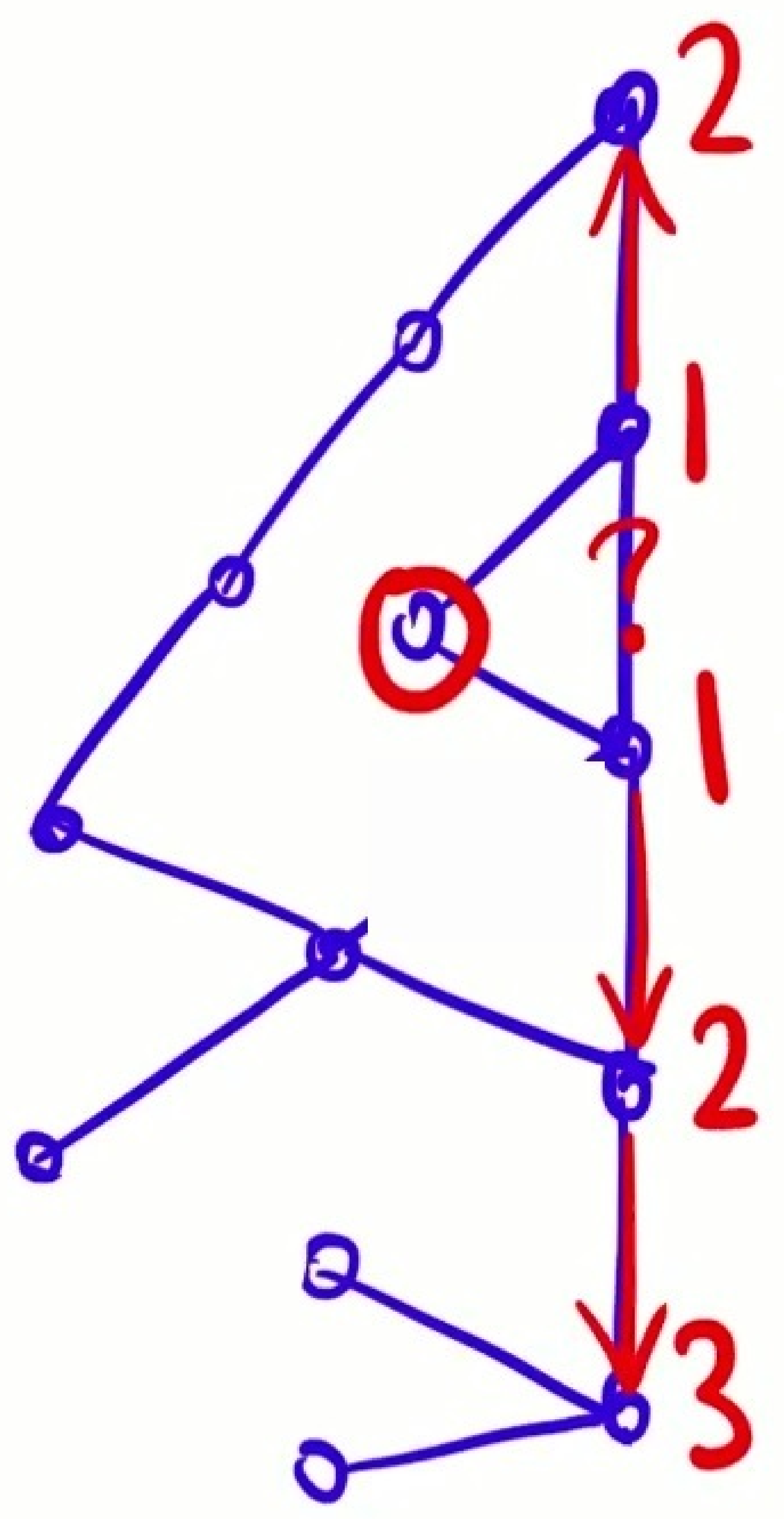

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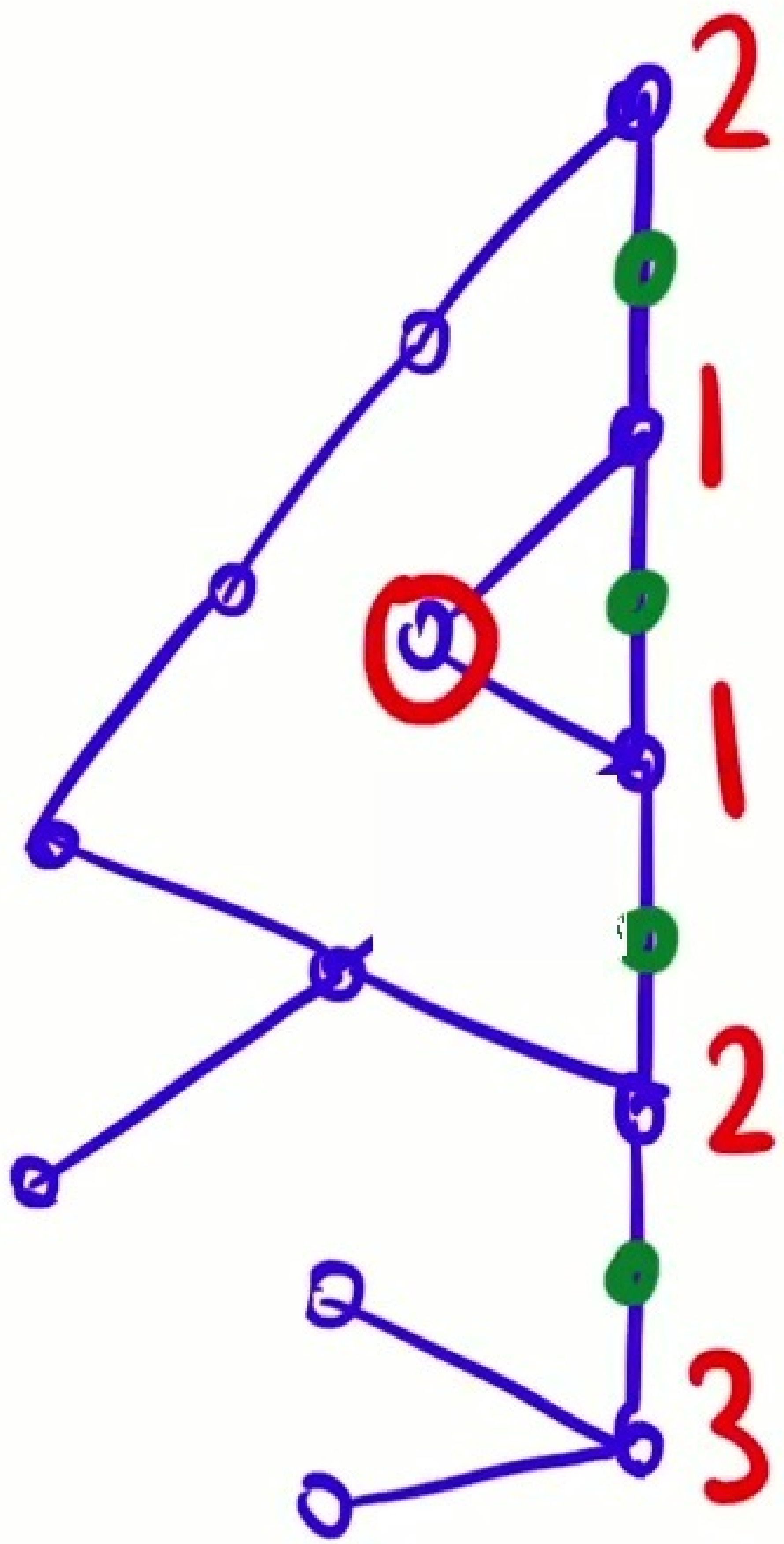

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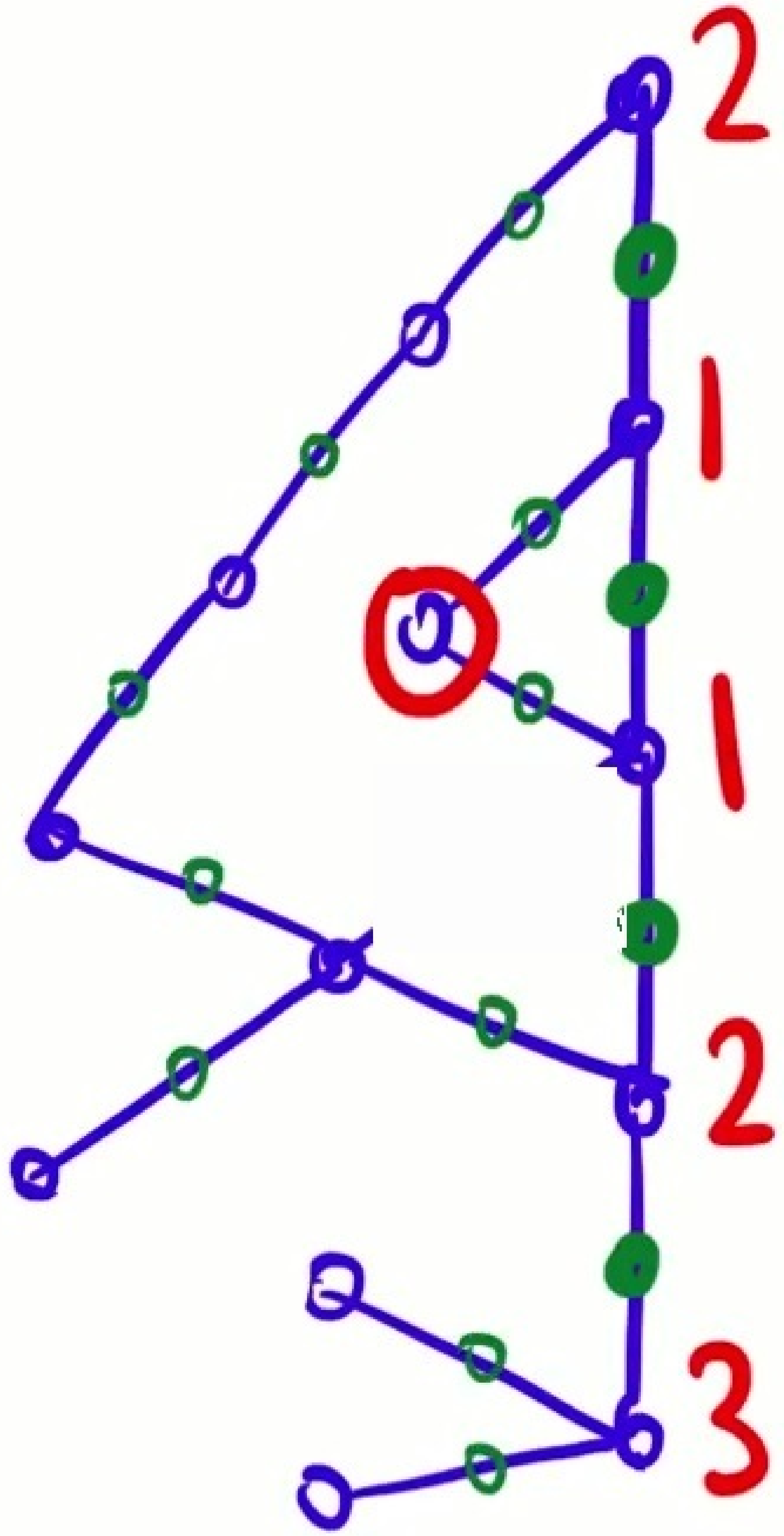
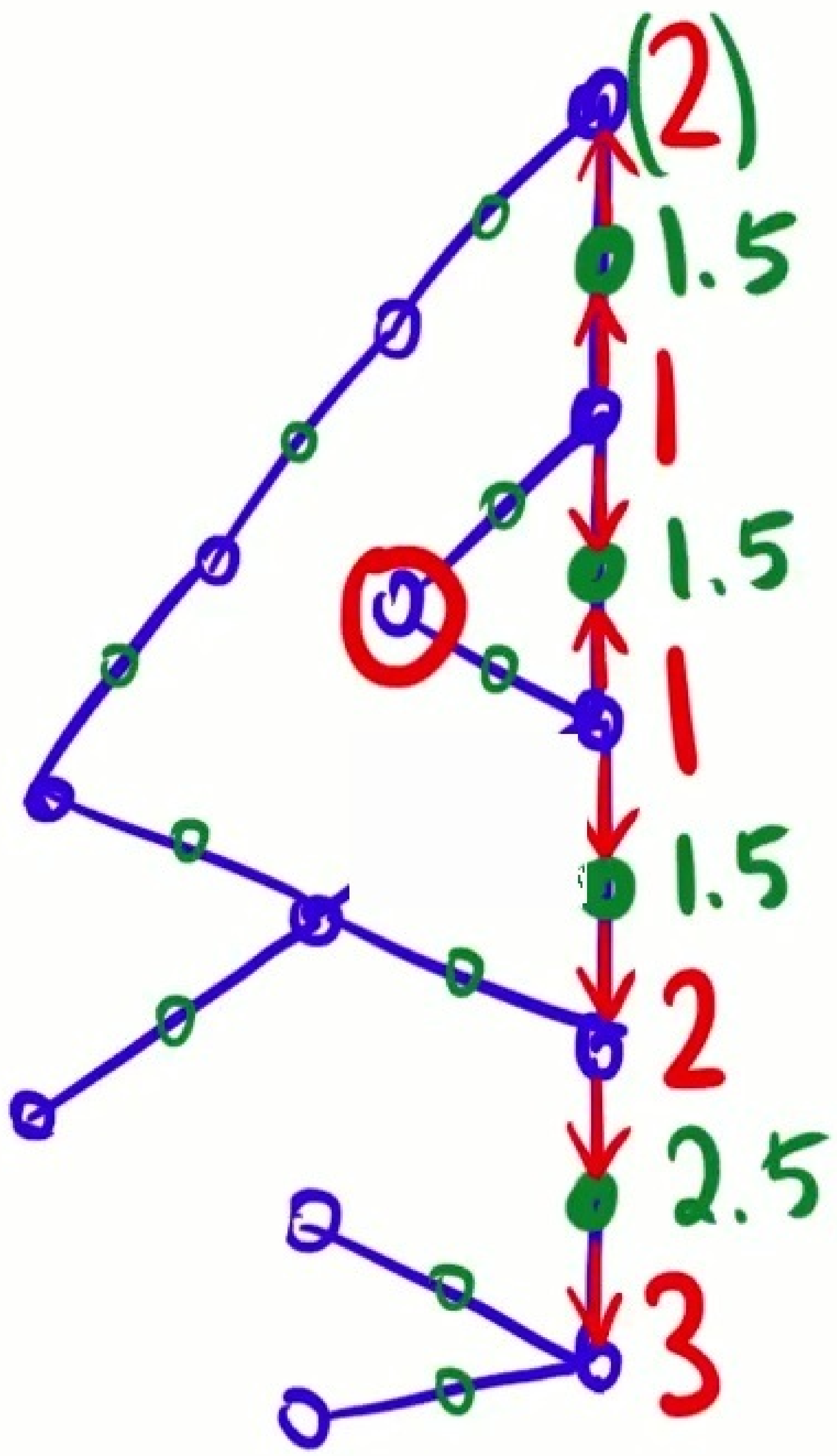
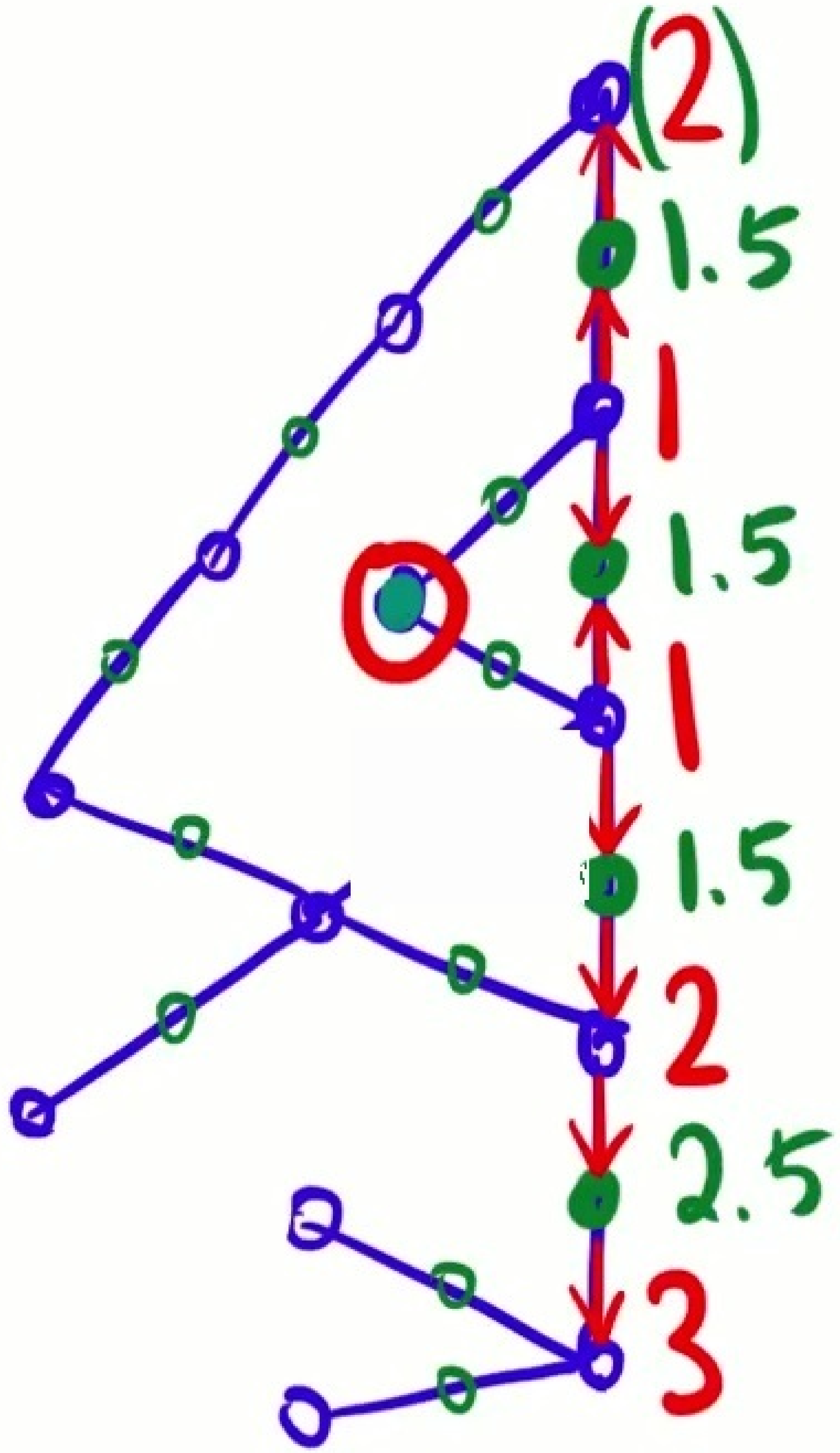


$$|\mathcal{L}| = O(Dk^3)$$



$|\mathcal{L}| = O(Dk^3)$



(2) ↑ ↑ ↓ ↑ ↓ ↓ ↓ ↓

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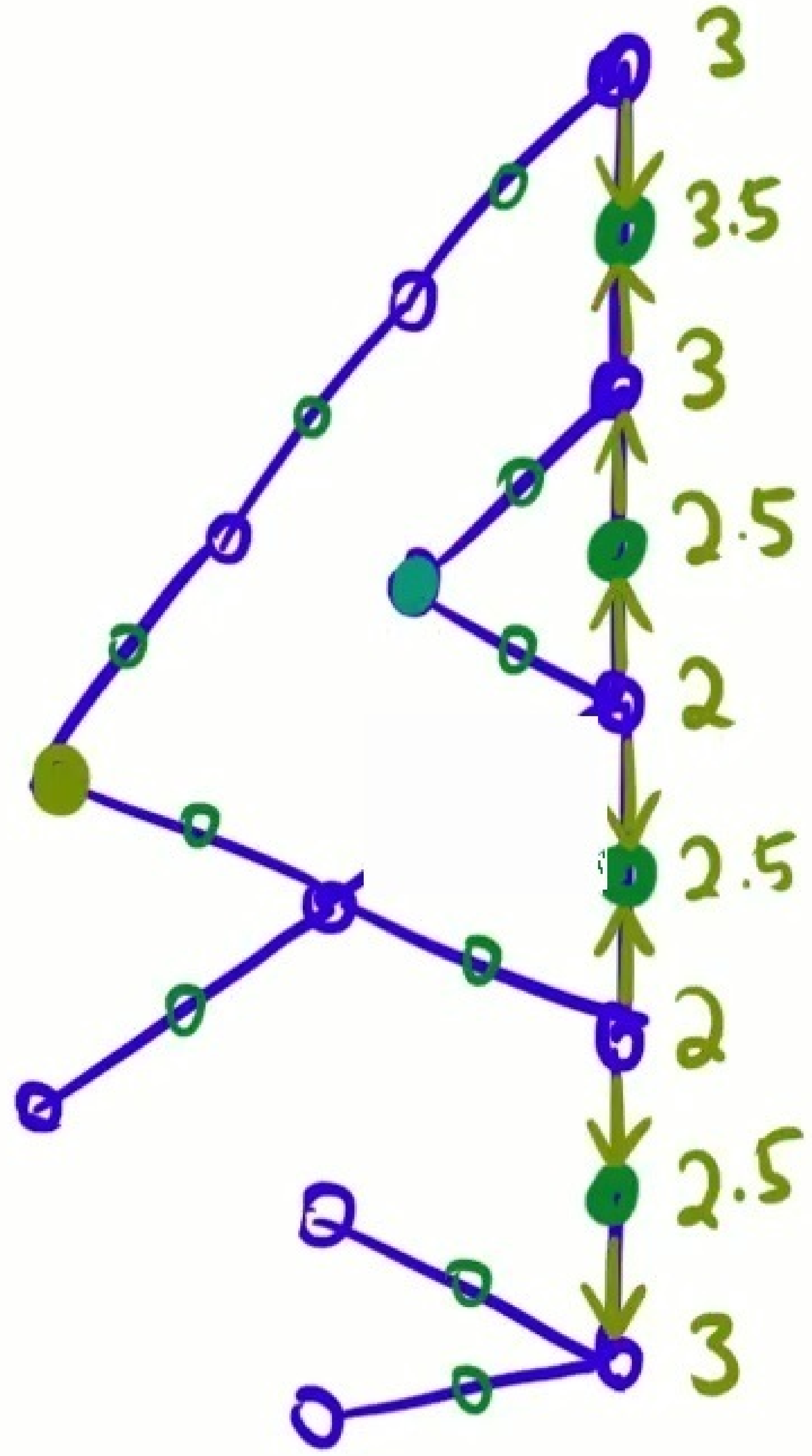
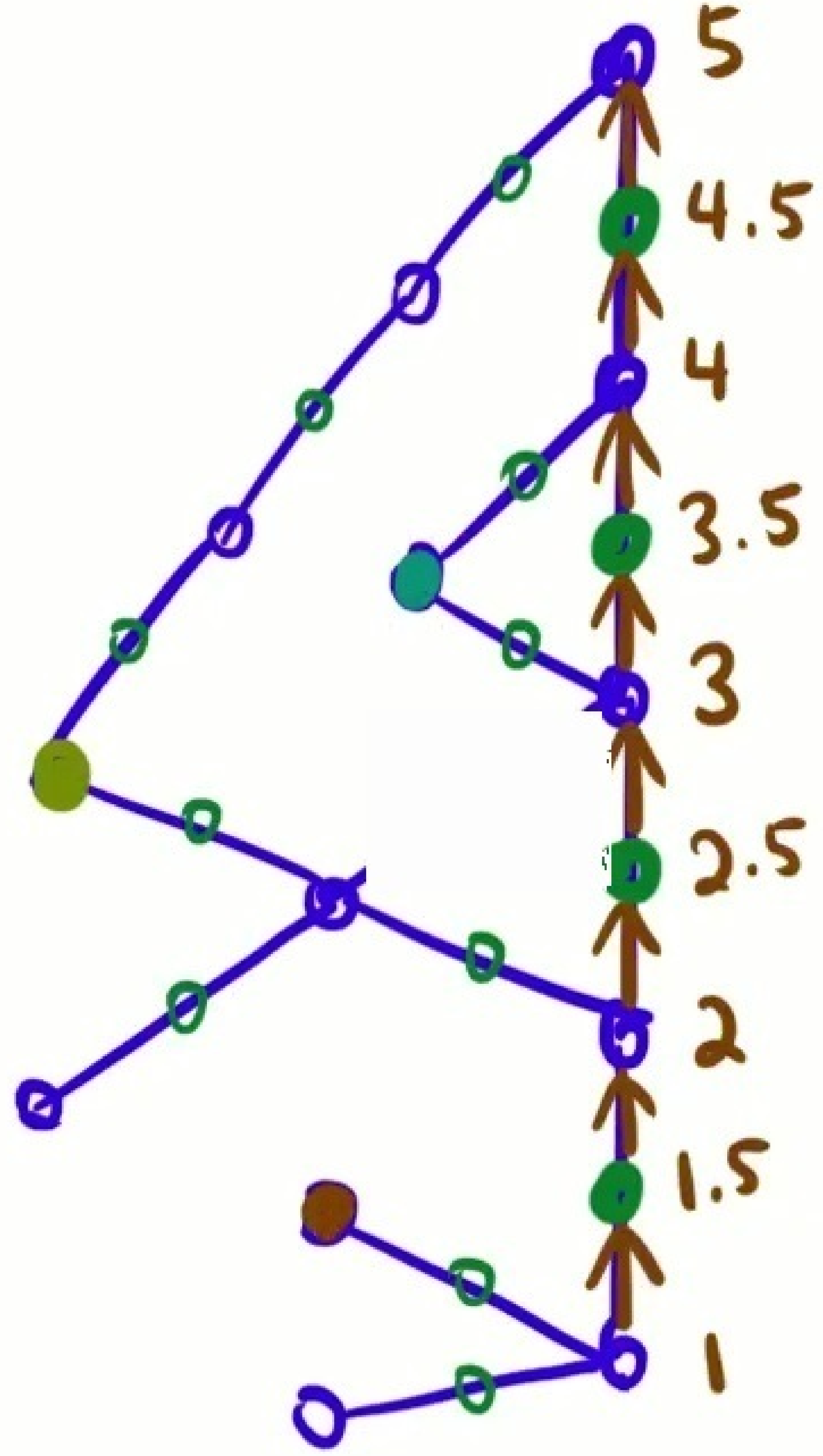


(2) ↑ ↑ ↓ ↑ ↓ ↓ ↓ ↓

(3) ↓ ↑ ↑ ↑ ↓ ↑ ↓ ↓

$|\mathcal{L}| = O(Dk^3)$

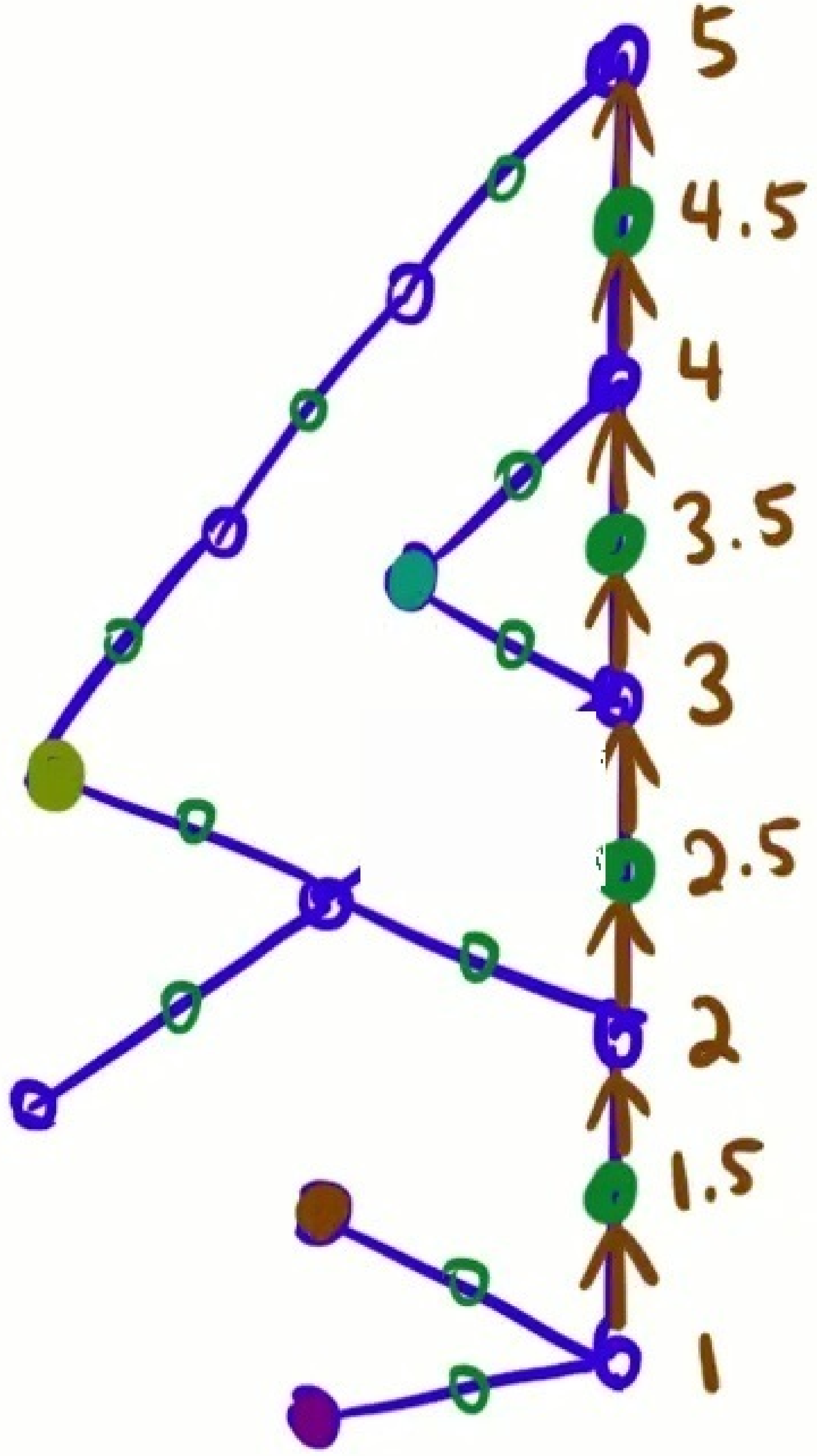


(2) $\uparrow \uparrow \downarrow \uparrow \downarrow \downarrow \downarrow \downarrow$

(3) $\downarrow \uparrow \uparrow \uparrow \downarrow \uparrow \downarrow \downarrow$

(5) $\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$

$$|\mathcal{L}| = O(Dk^3)$$



(2) $\uparrow \uparrow \downarrow \uparrow \downarrow \downarrow \downarrow \downarrow$

(3) $\downarrow \uparrow \uparrow \uparrow \downarrow \uparrow \downarrow \downarrow$

(5) $\uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow$

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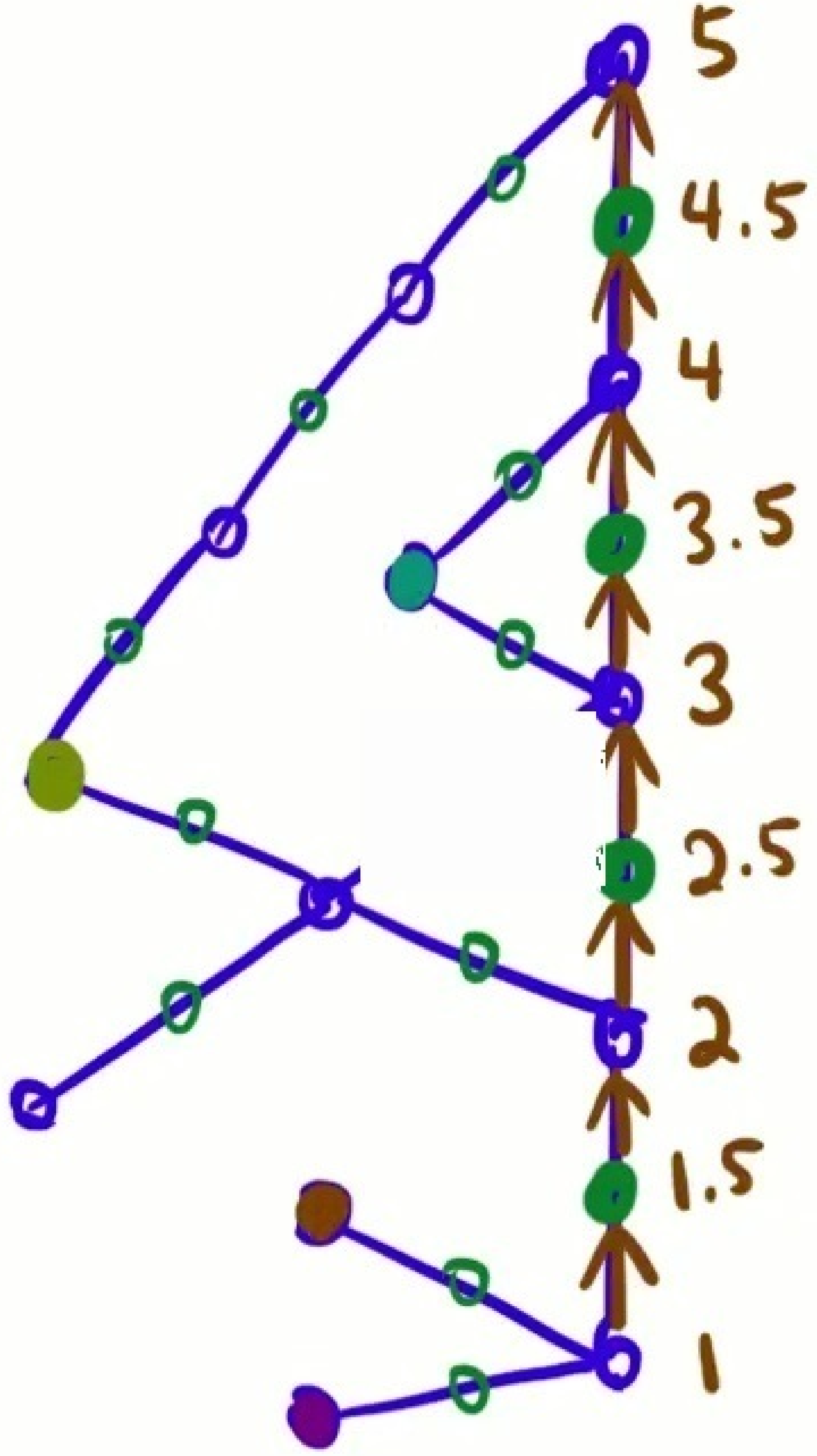

$(2)\ \uparrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow$

$(3)\ \downarrow\ \uparrow\ \uparrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow\ \downarrow$

$(5)\ \uparrow\ \uparrow\ \uparrow\ \uparrow\ \uparrow\ \uparrow\ \uparrow\ \uparrow$

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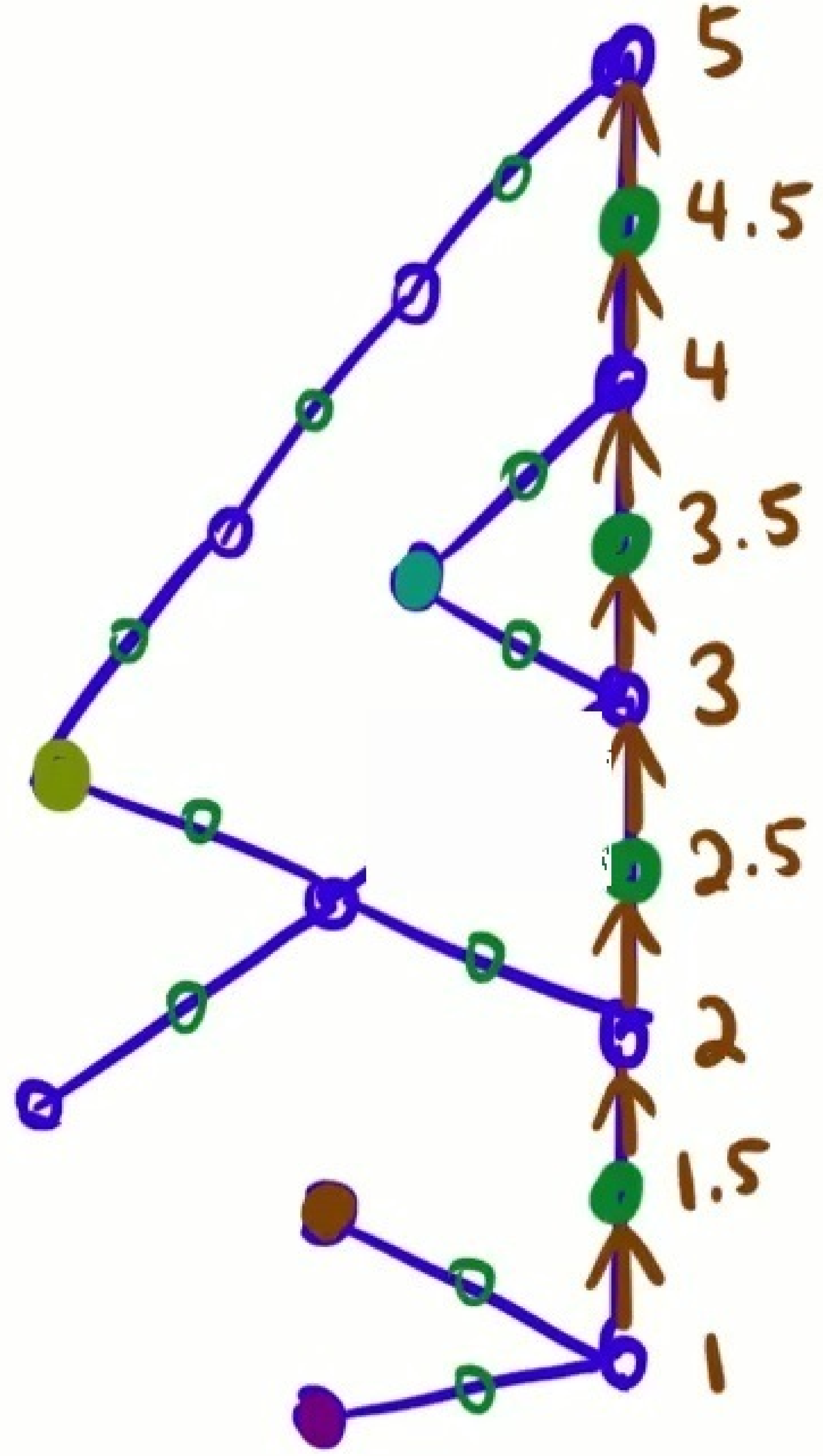
$\le O(D)$
choices

$|\mathcal{L}| = O(Dk^3)$



(2) ↑ ↑ ↓ ↑ ↓ ↓ ↓ ↓

(3) ↓ ↑ ↑ ↑ ↓ ↑ ↓ ↓

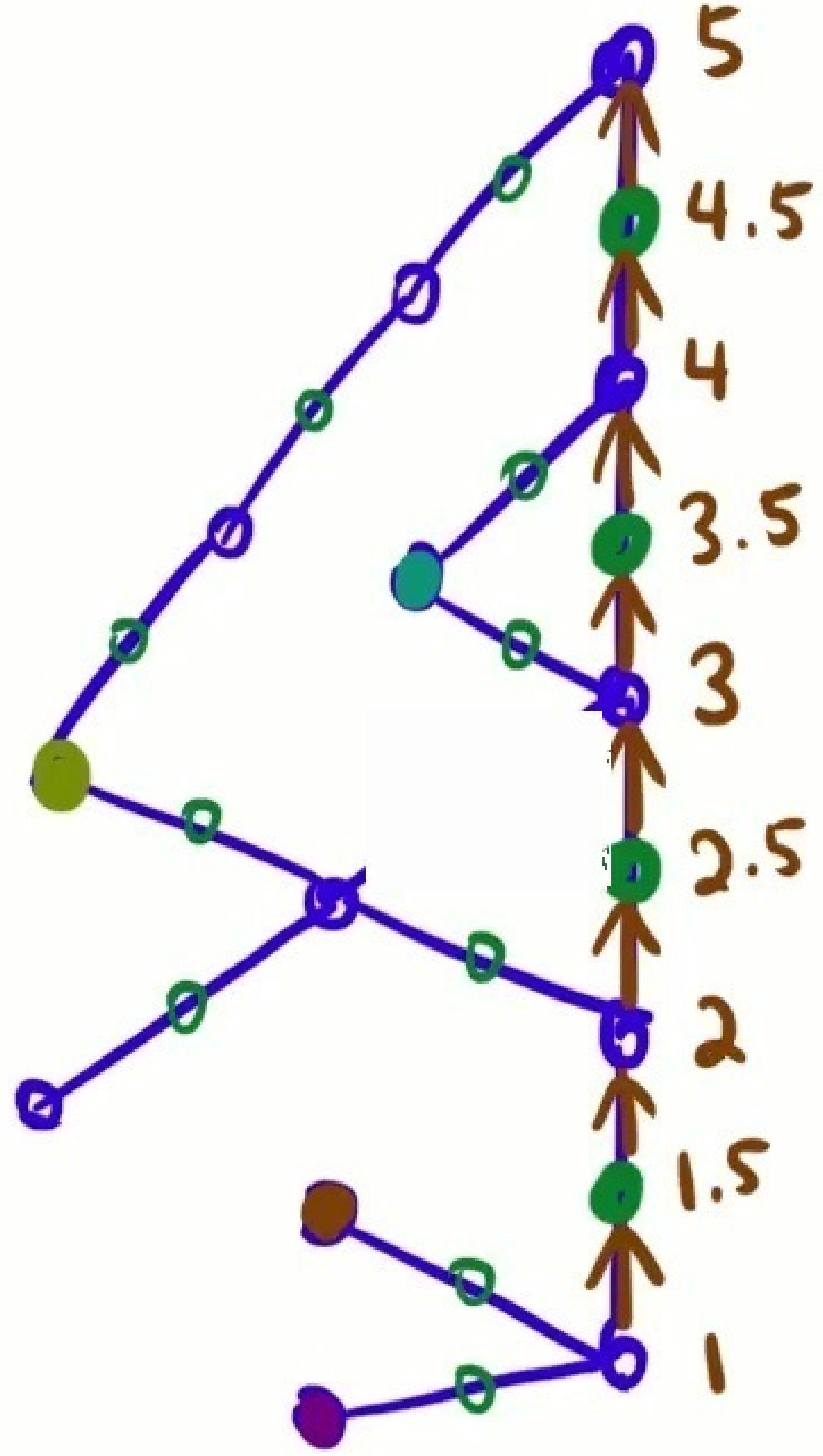
(5) ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

(5) ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

$\leq O(D)$ choices

How many distinct strings?

$|\mathcal{L}| = O(Dk^3)$



(2)	↑	↑	↓	↑	↓	↓	↓	↓
(3)	↓	↑	↑	↑	↓	↑	↓	↓
(5)	↑	↑	↑	↑	↑	↑	↑	↑
(5)	↑	↑	↑	↑	↑	↑	↑	↑

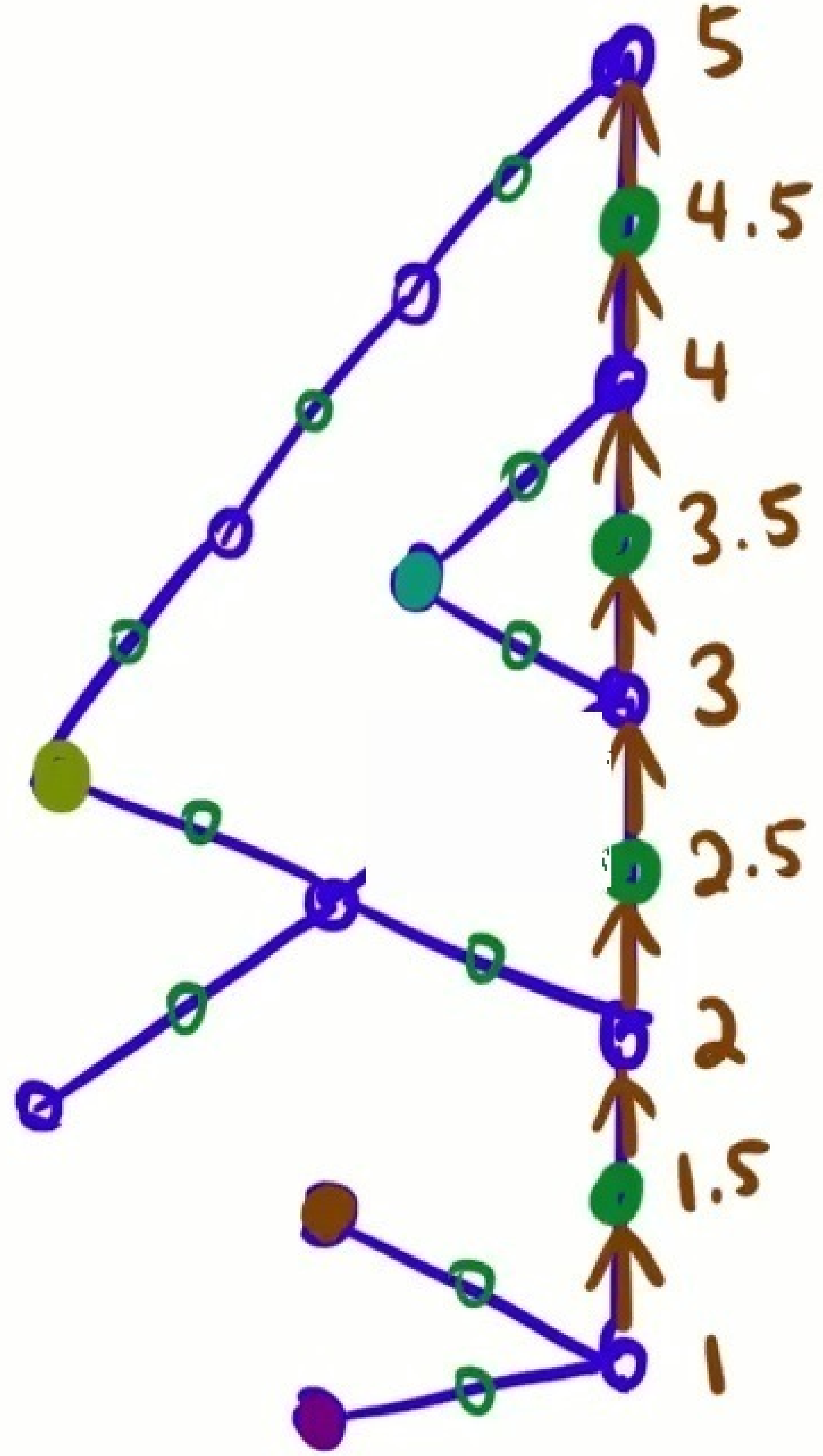
$\le O(D)$ choices

How many <u>distinct</u> strings?

Def [VC dimension]

A matrix of ↑/↓'s has <u>VC dim</u> $< d$ if it contains no 2^d-by-d submatrix whose rows span all length-d strings.

$|\mathcal{L}| = O(Dk^3)$



$$\begin{array}{c} (2) \\ (3) \\ (5) \\ (5) \end{array} \underbrace{\begin{array}{cccccccc} \uparrow & \uparrow & \downarrow & \uparrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \downarrow & \uparrow & \uparrow & \uparrow & \downarrow & \uparrow & \downarrow & \downarrow \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{array}}_{\text{How many distinct strings?}}$$

$\le O(D)$ choices

Def [VC dimension]

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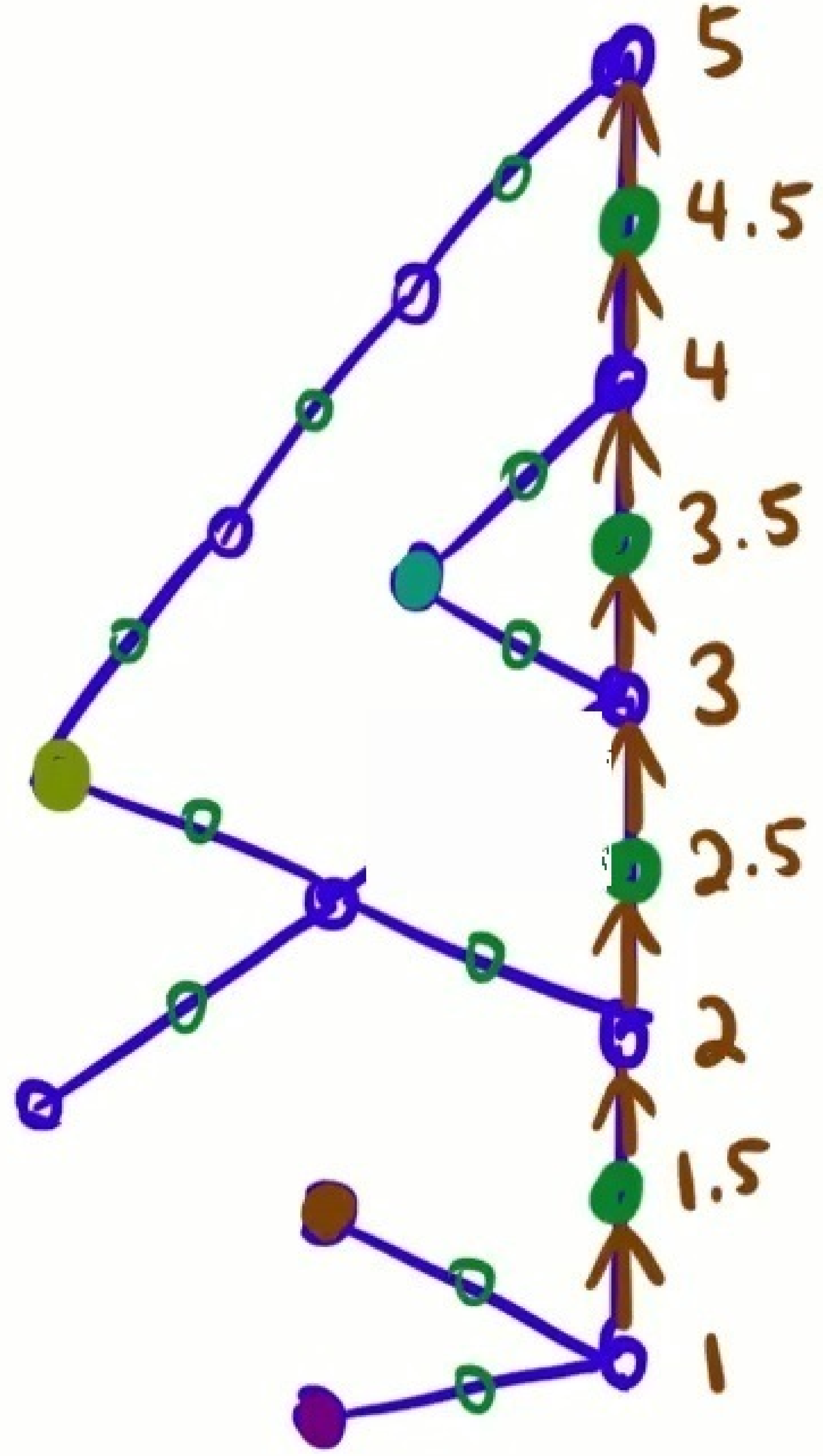
$d=3$: no

$$\begin{array}{ccc} \uparrow & \downarrow & \uparrow \\ \downarrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \downarrow \\ \uparrow & \uparrow & \uparrow \\ \downarrow & \downarrow & \downarrow \\ \uparrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \uparrow \\ \downarrow & \uparrow & \downarrow \end{array}$$

$|\mathcal{L}| = O(Dk^3)$



(2)	↑	↑	↓	↑	↓	↓	↓	↓
(3)	↓	↑	↑	↑	↓	↑	↓	↓
(5)	↑	↑	↑	↑	↑	↑	↑	↑
(5)	↑	↑	↑	↑	↑	↑	↑	↑

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↓	↓	↑
↑	↓	↓
↑	↑	↑
↓	↓	↓
↑	↑	↓
↓	↑	↑
↓	↑	↓

Sauer's Lemma:
If k columns and VC dim $< d$, then $O(k^{d-1})$ distinct rows.

$|\mathcal{L}| = O(Dk^3)$

Lemma: If planar graph, then

VC dim < 4.

$\Longrightarrow O(k^3)$ distinct rows

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Pf: suppose not: $\exists\ 2^4 \times 4$ submtx.

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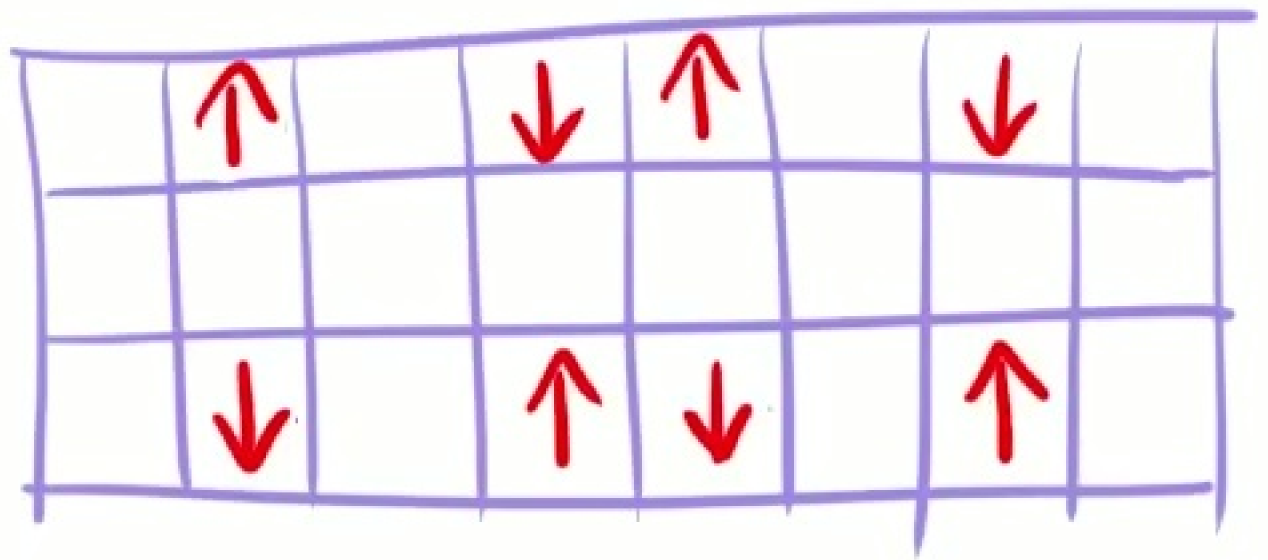
Pf: suppose not: $\exists\ 2^4 \times 4$ submtx.

Then, there is 2x4 submtx $\begin{matrix}\uparrow & \downarrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \downarrow & \uparrow\end{matrix}$ somewhere in matrix.



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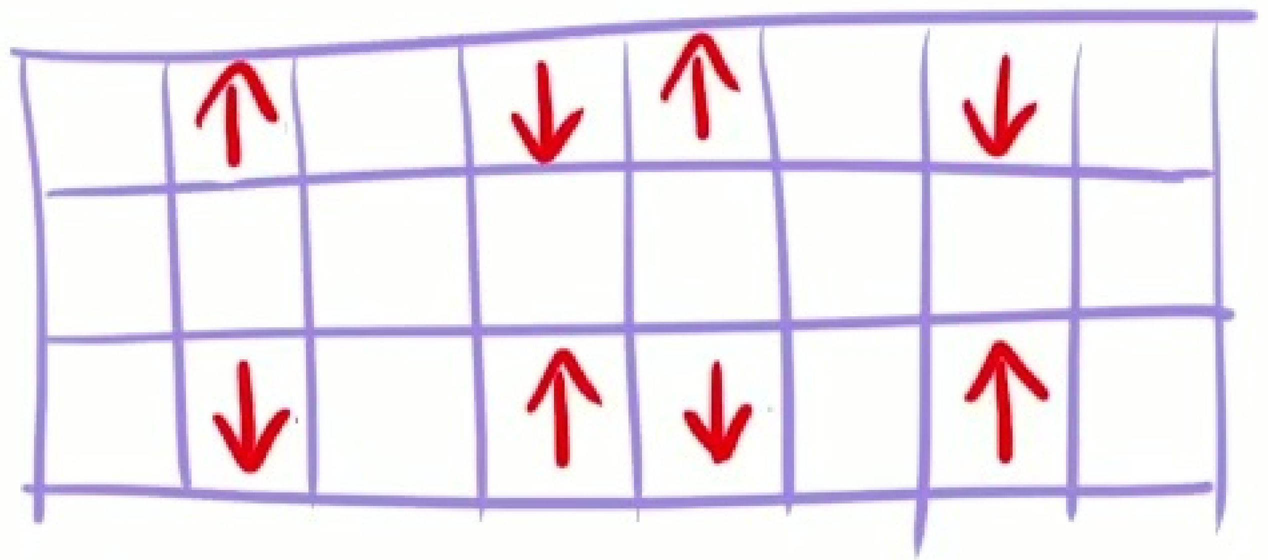

To show: violates planarity!

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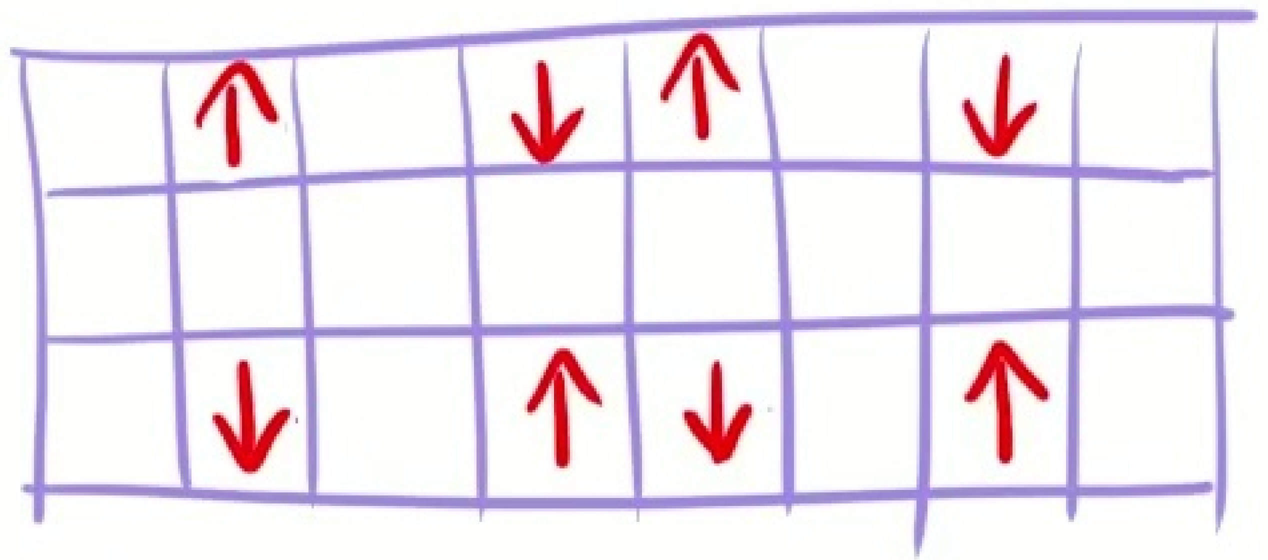

To show: violates planarity!
violates <u>Monge property</u>

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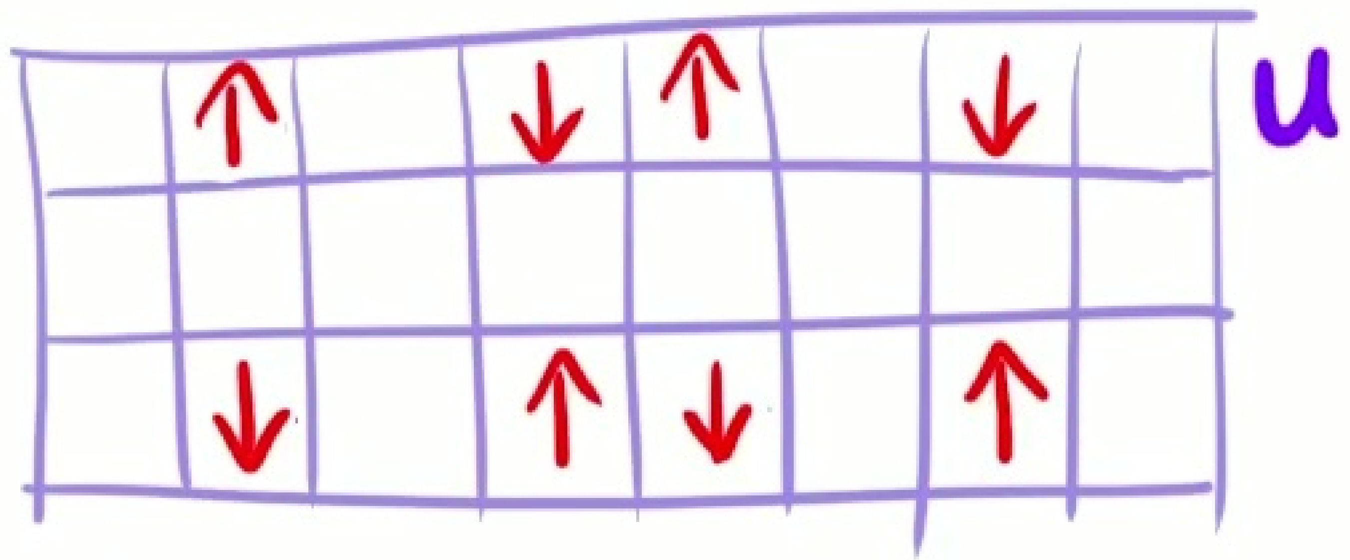

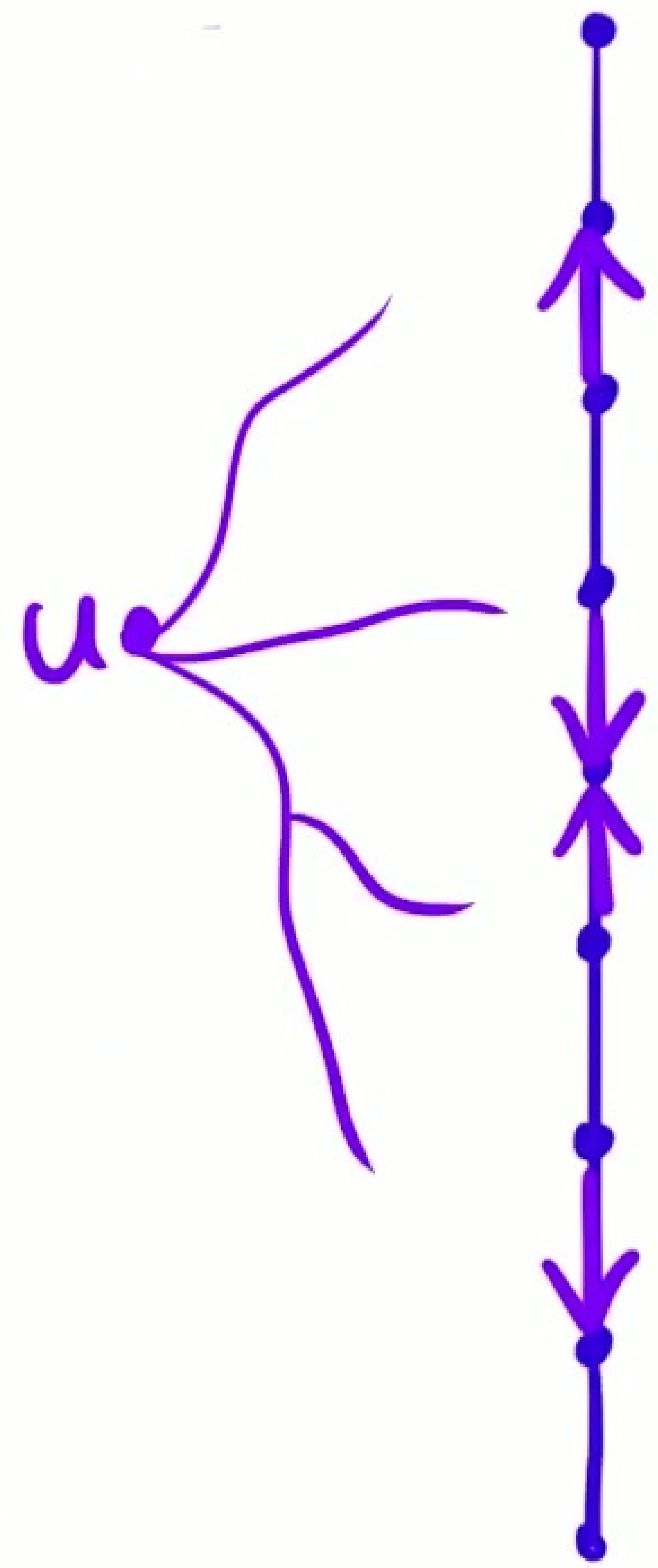

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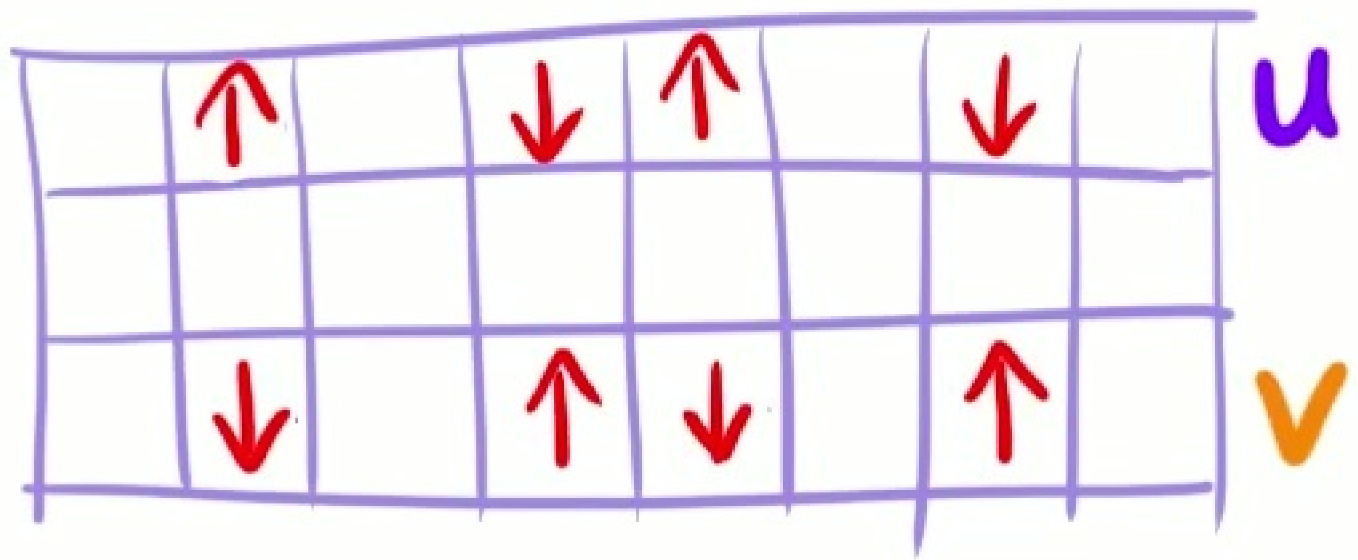

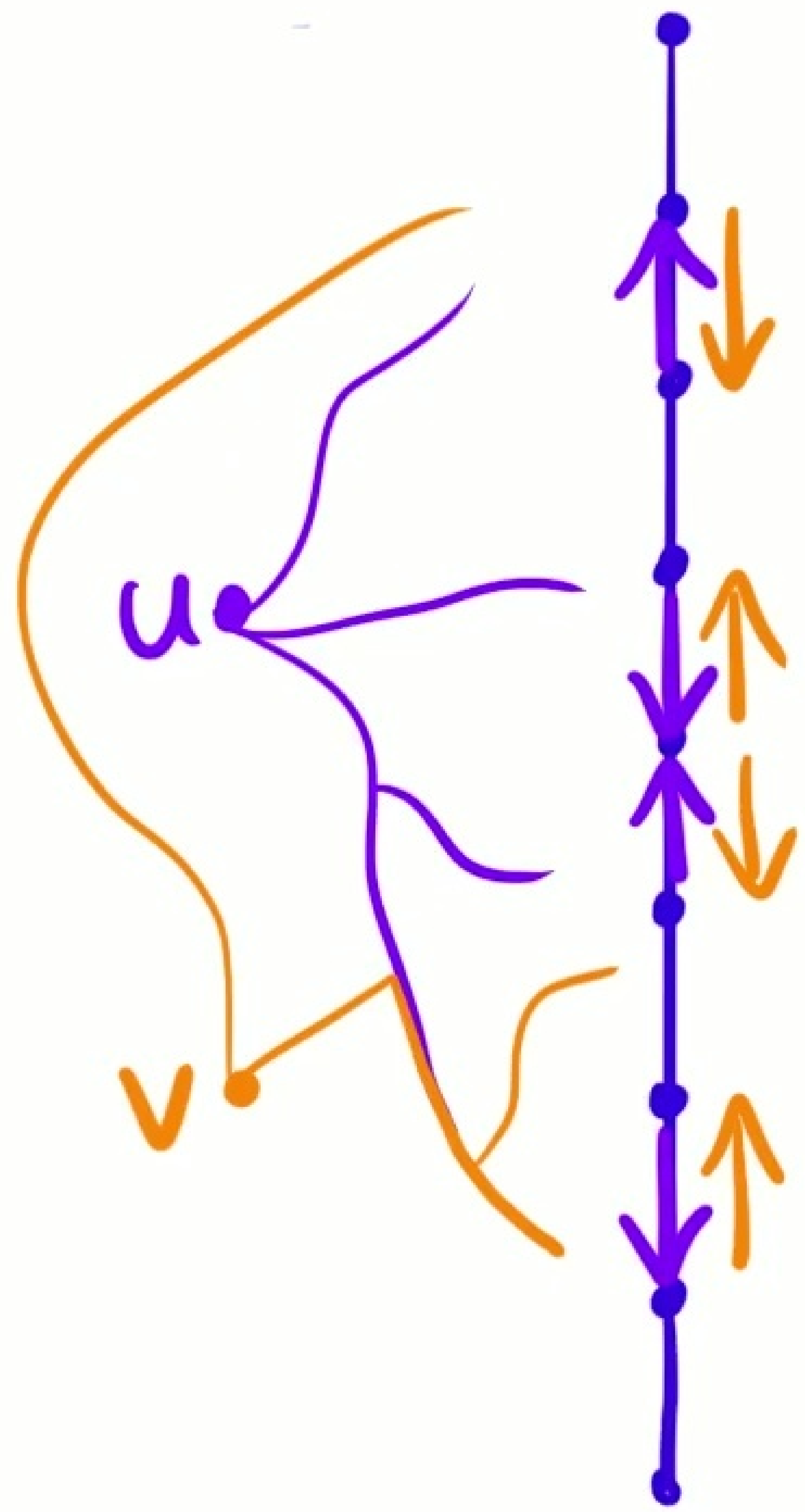

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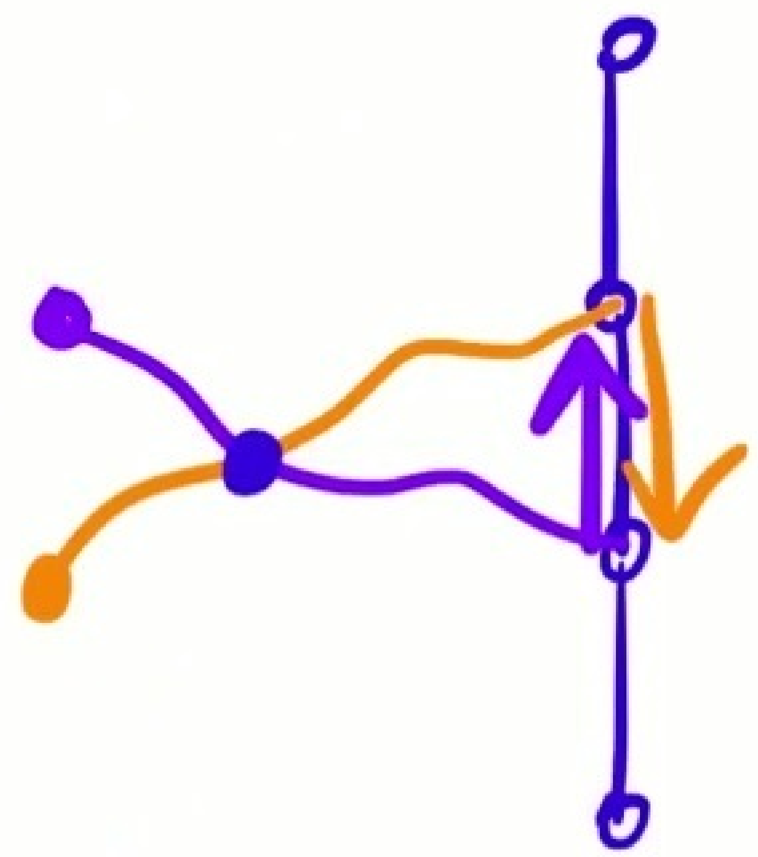
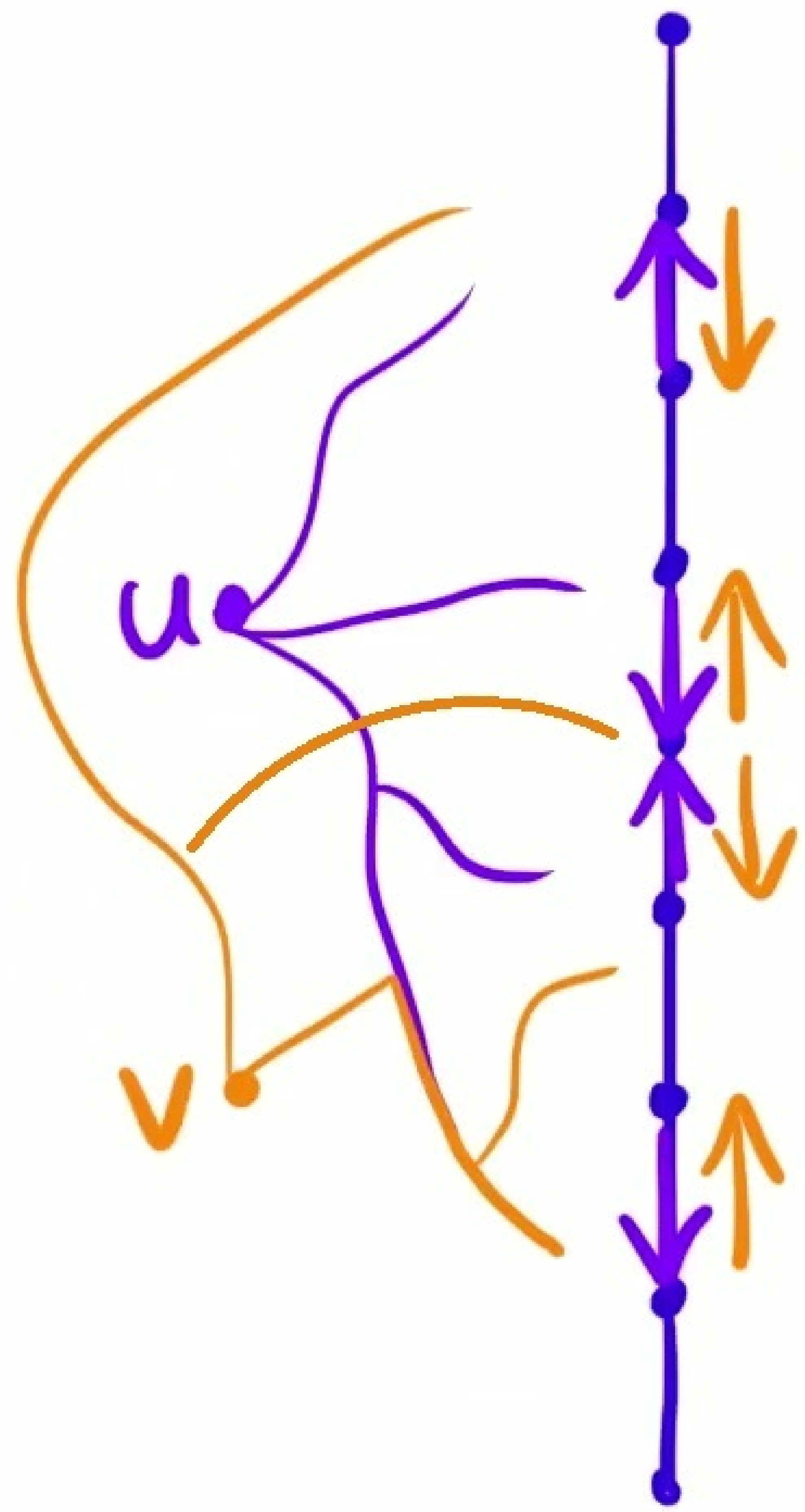
Monge property:

Cannot have:





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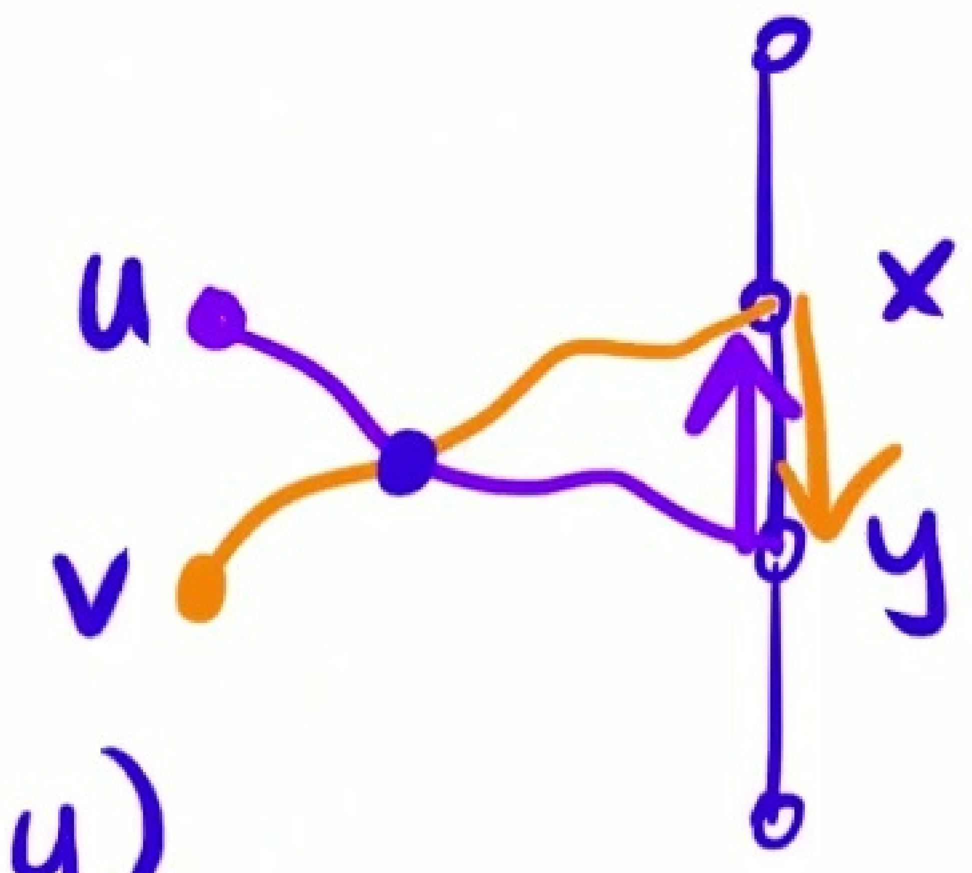
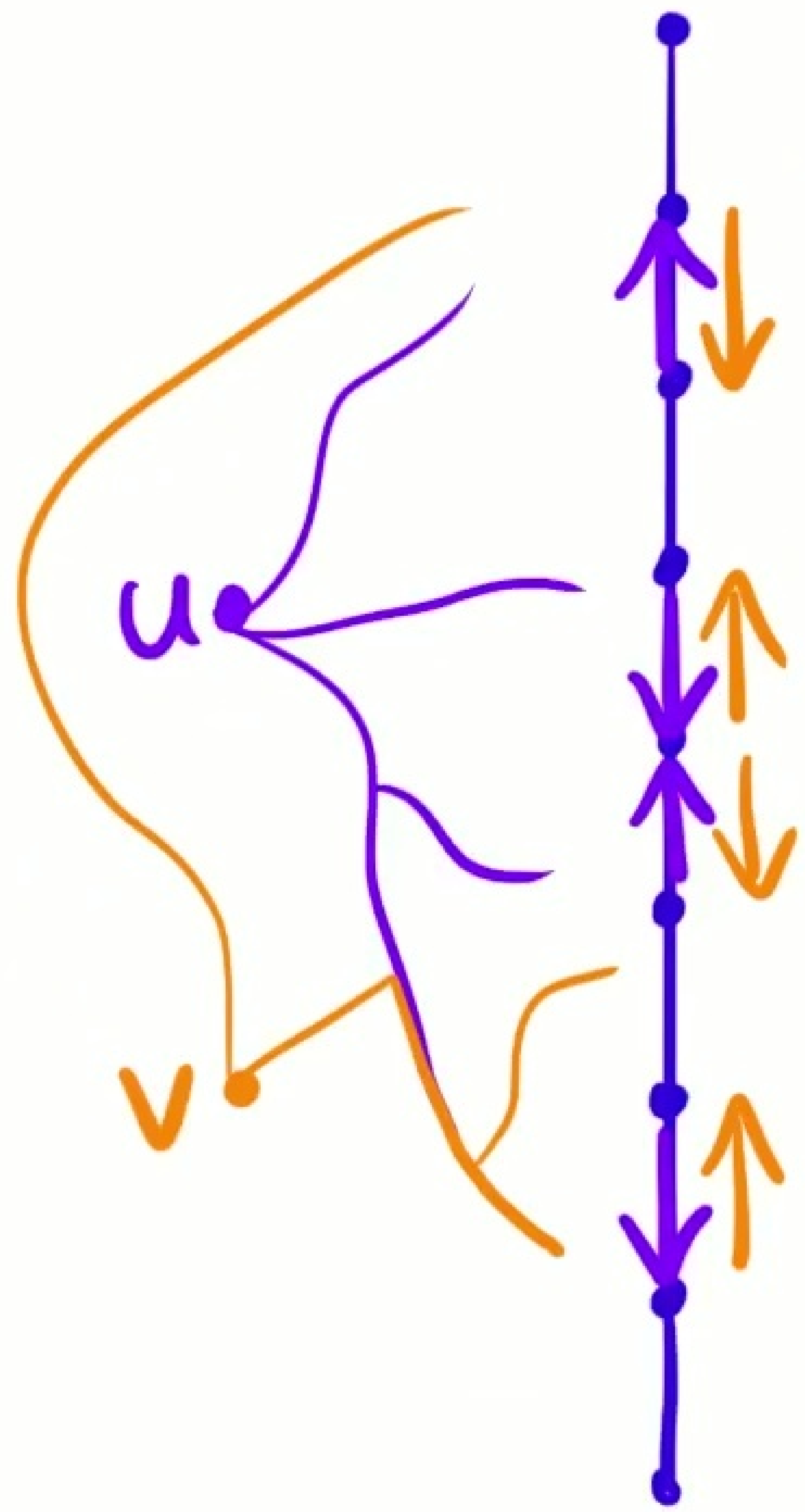


Pf: $d(u,x) > d(u,y)$

$d(v,y) > d(v,x)$



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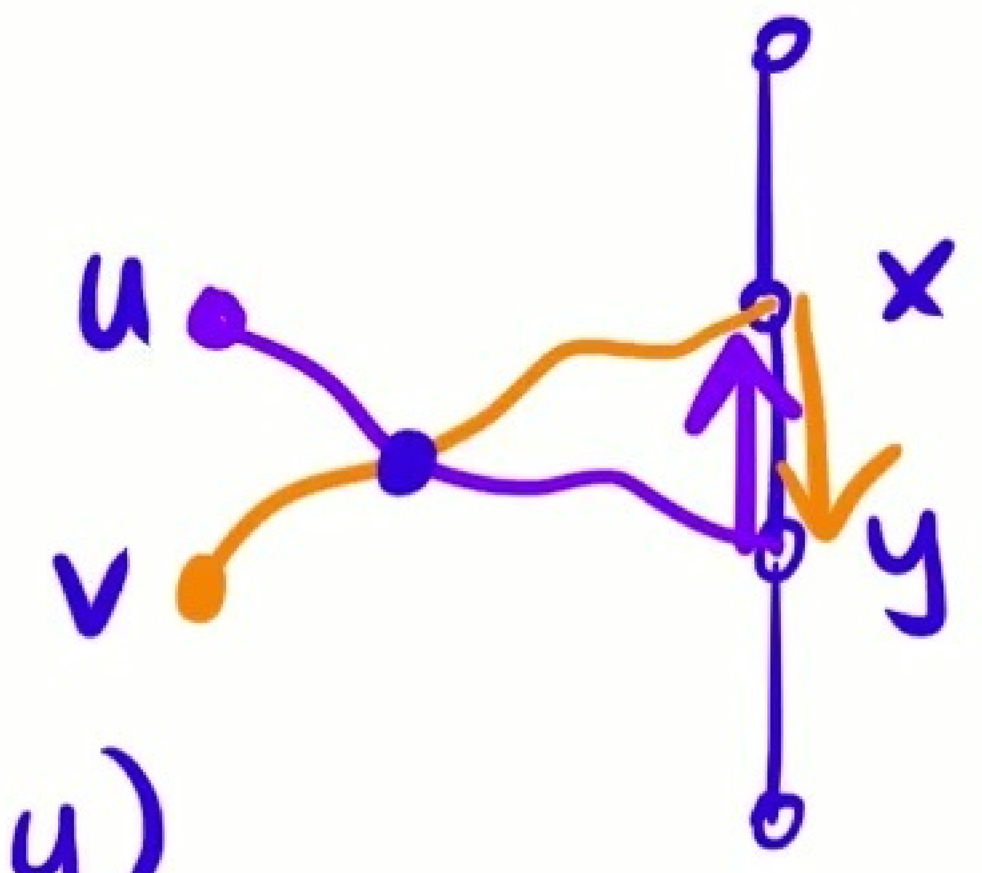

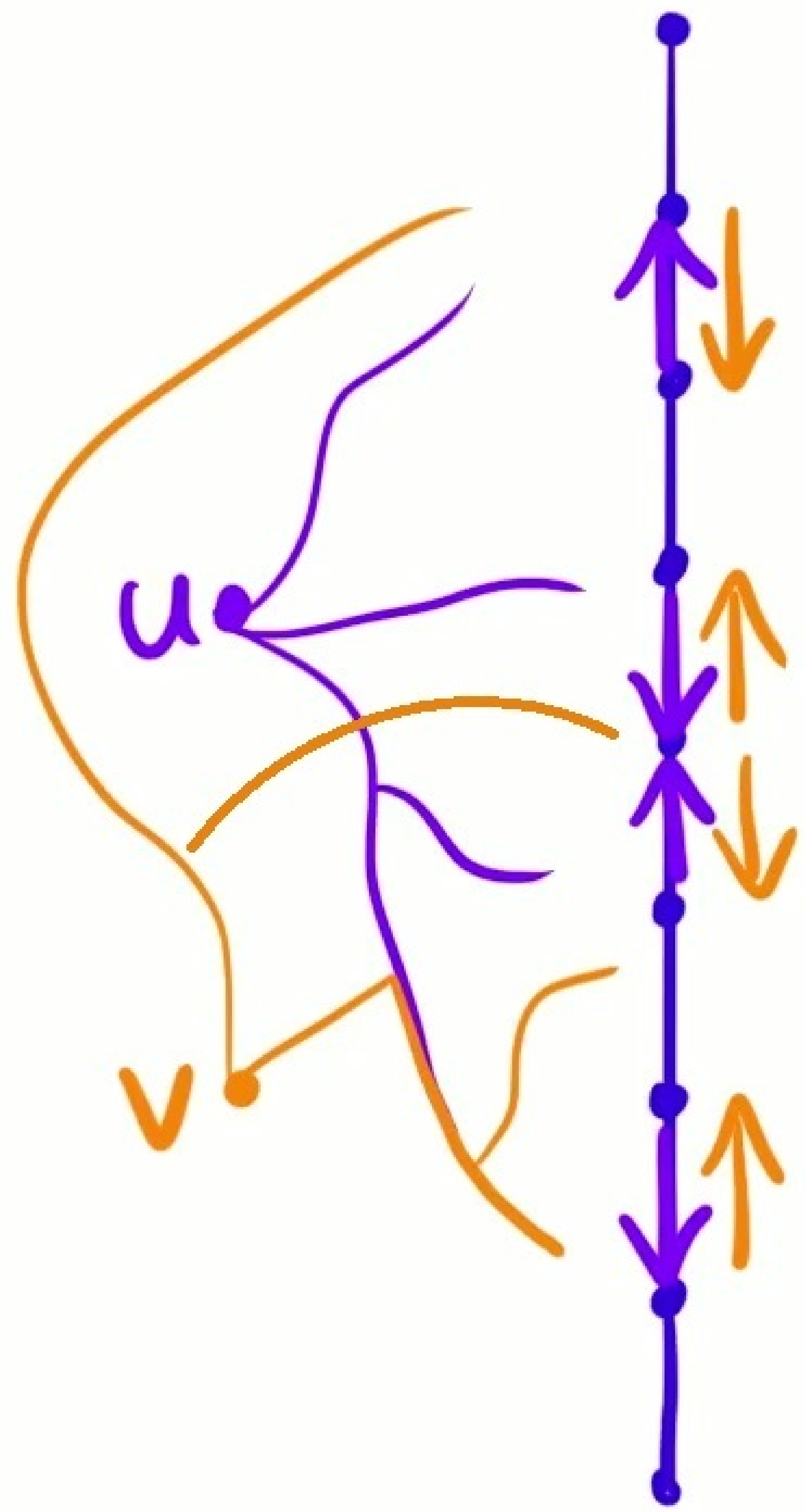

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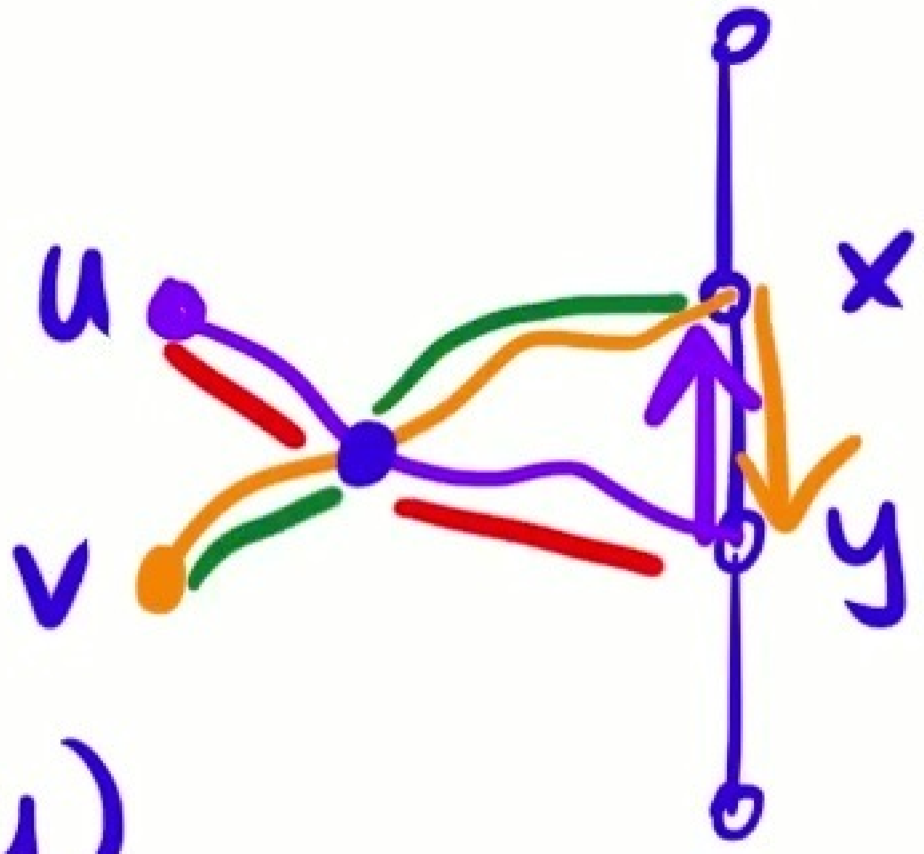

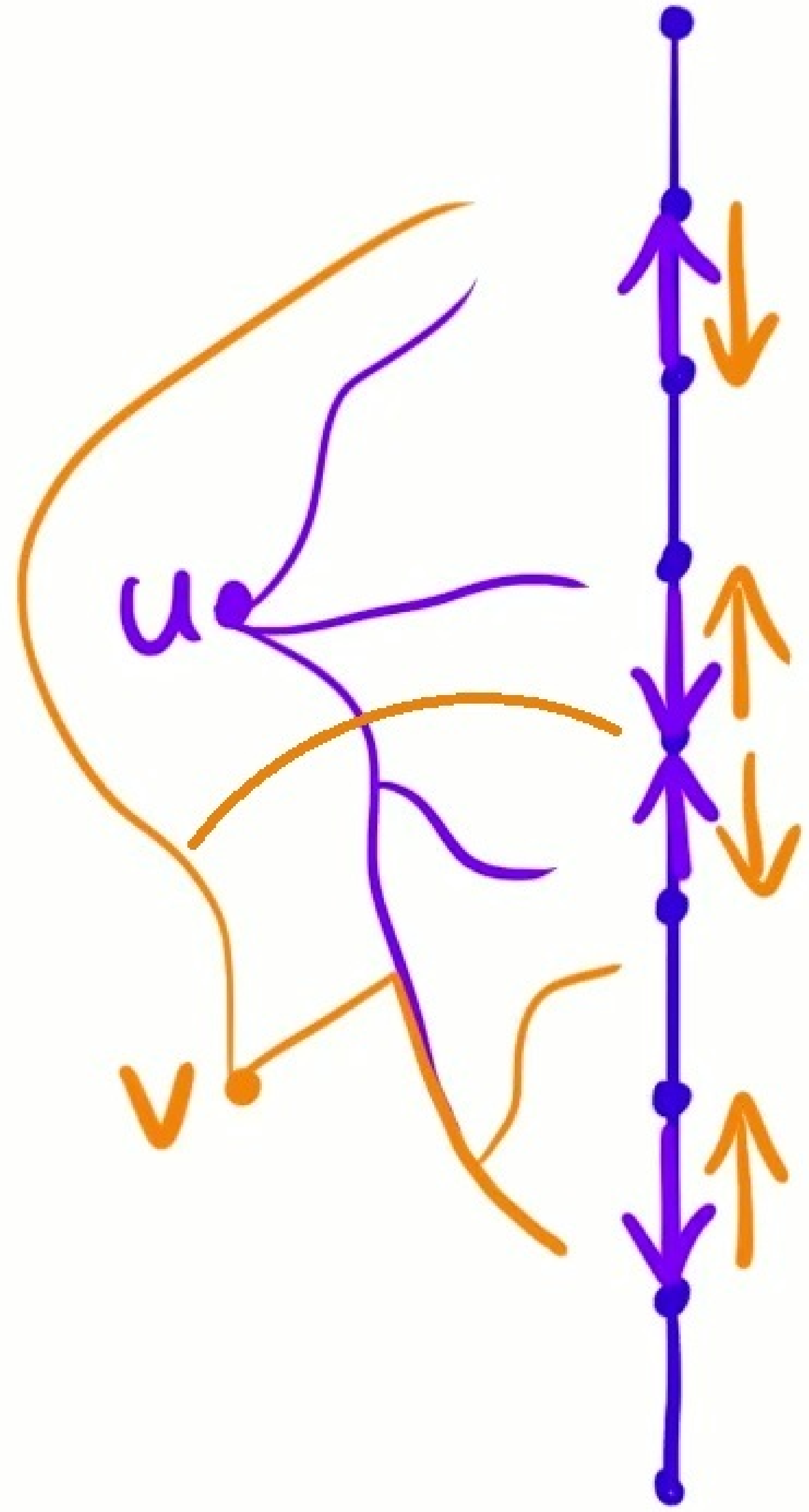

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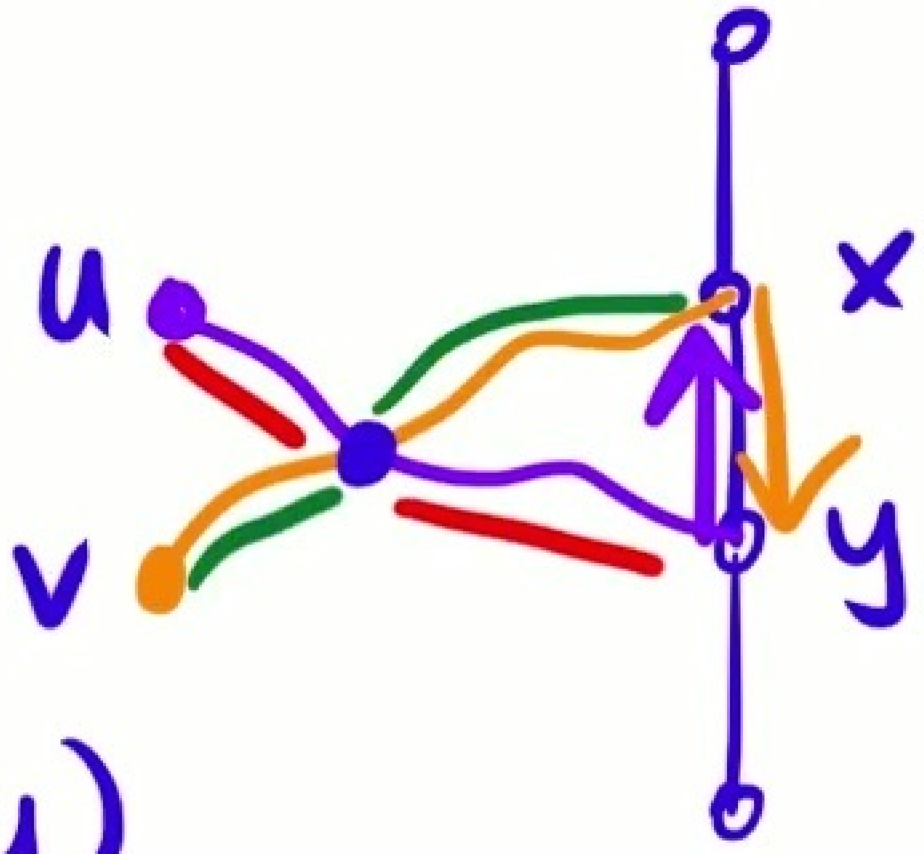

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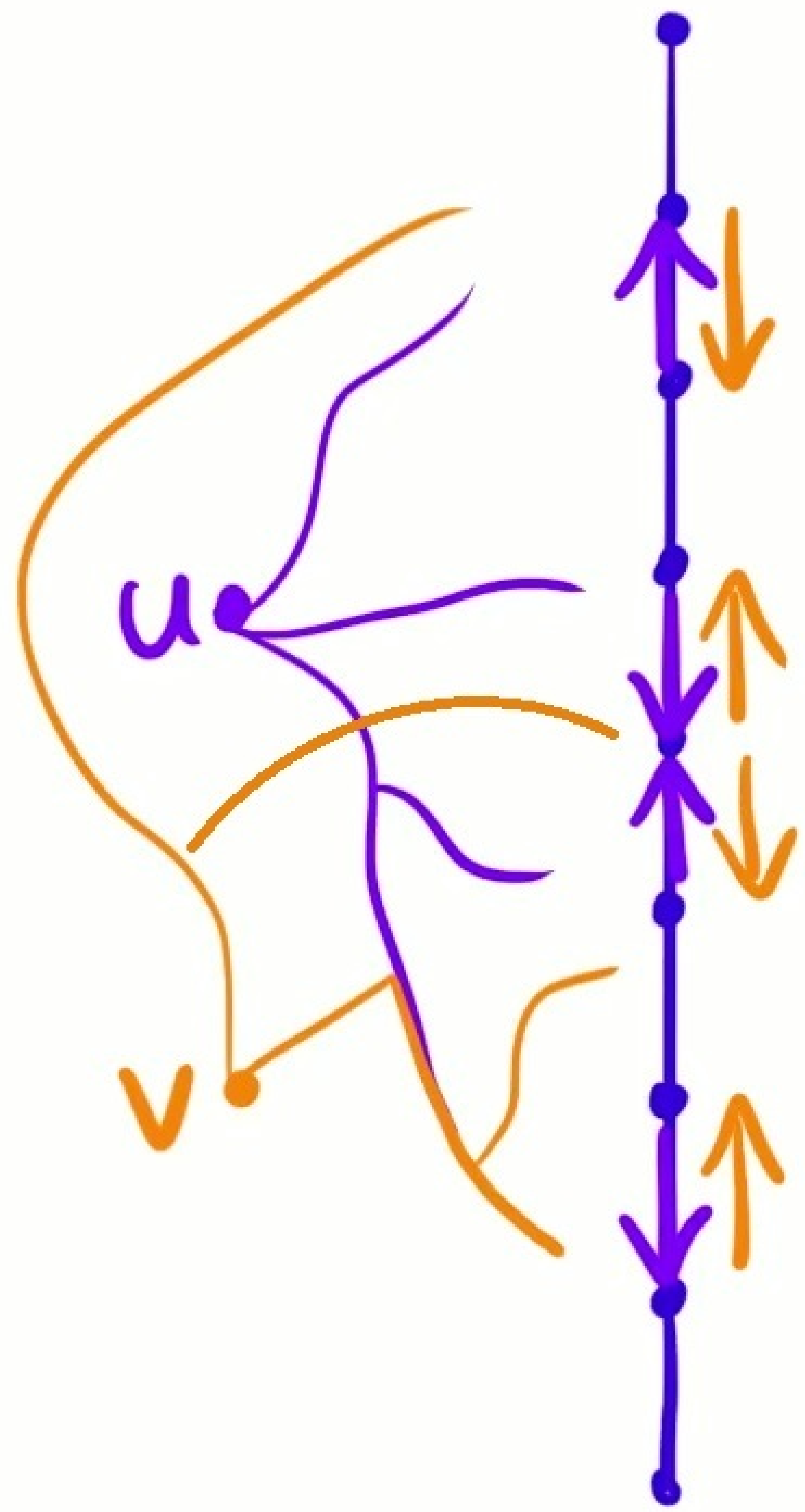
Contradiction.



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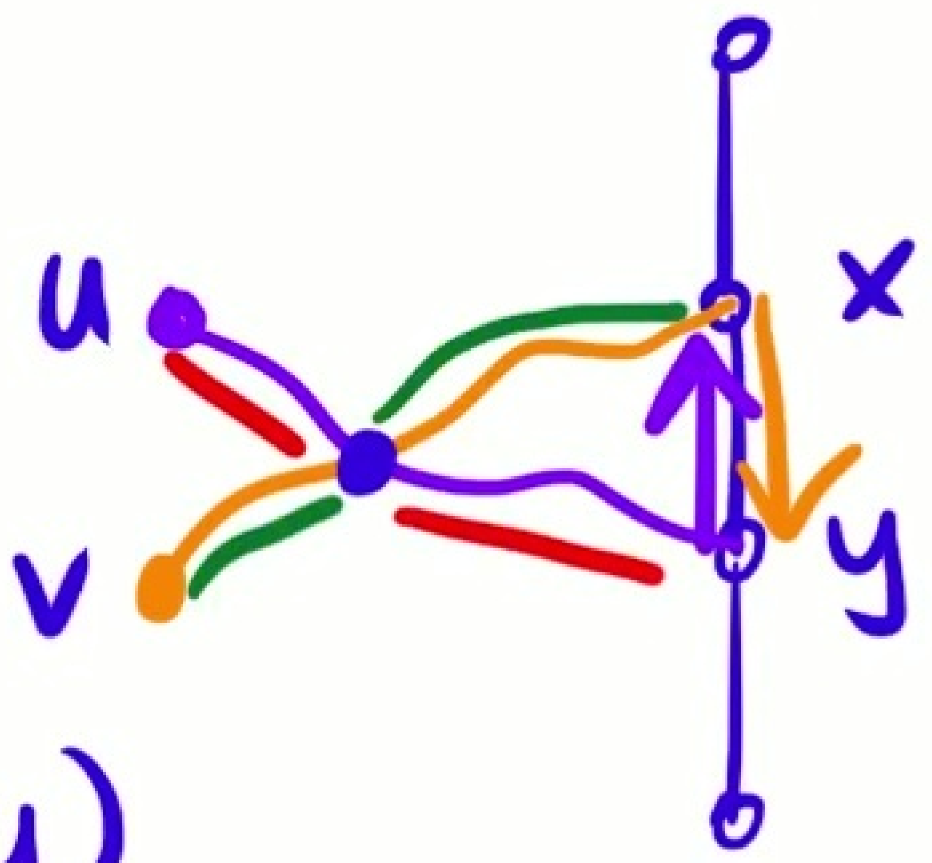

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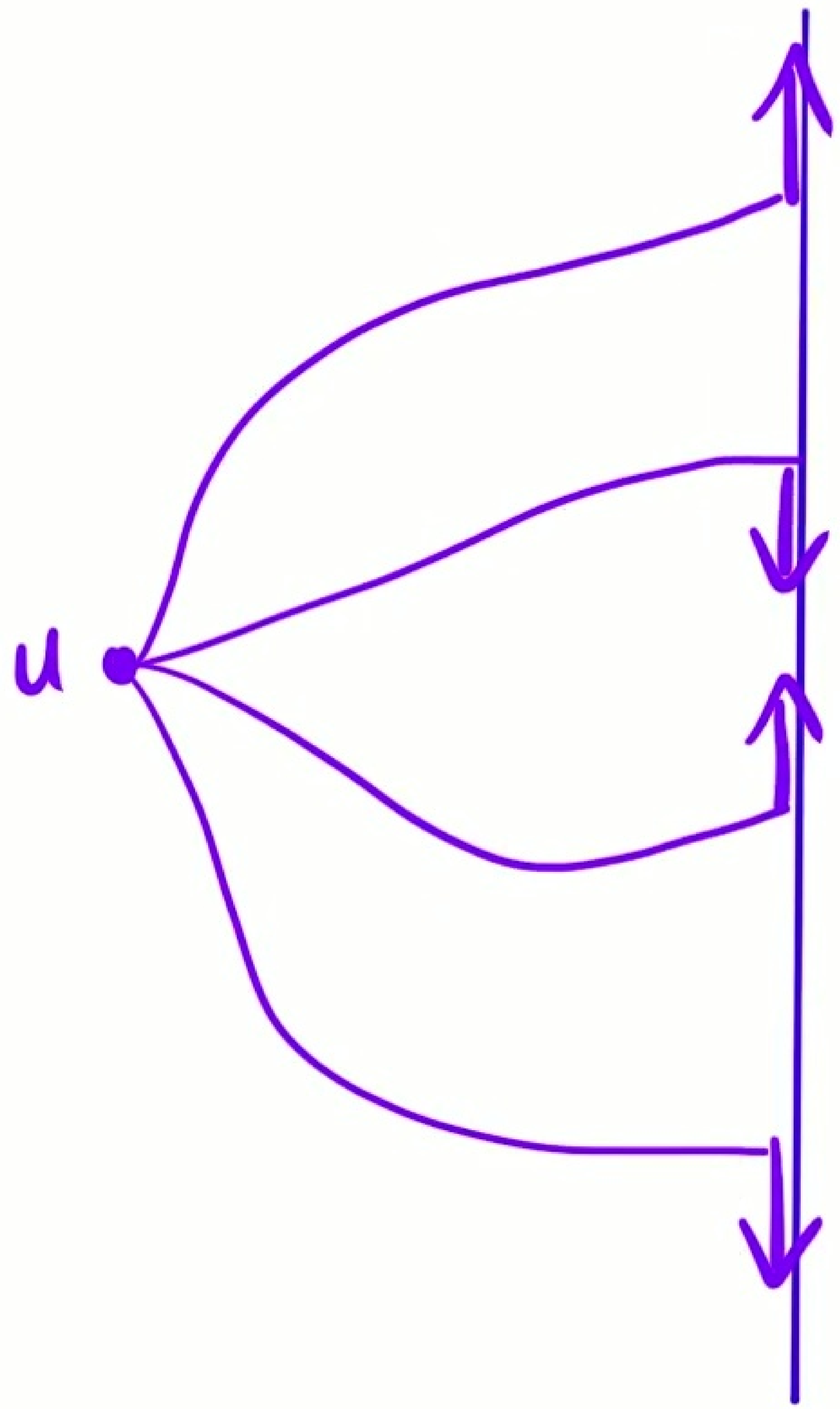

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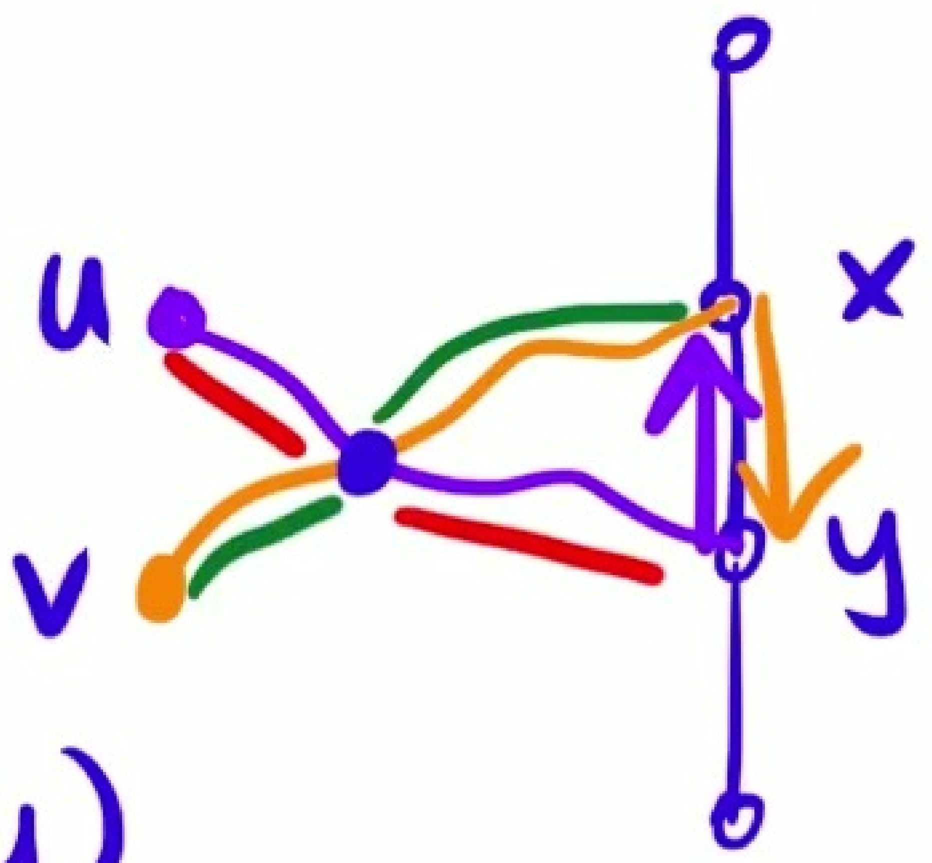

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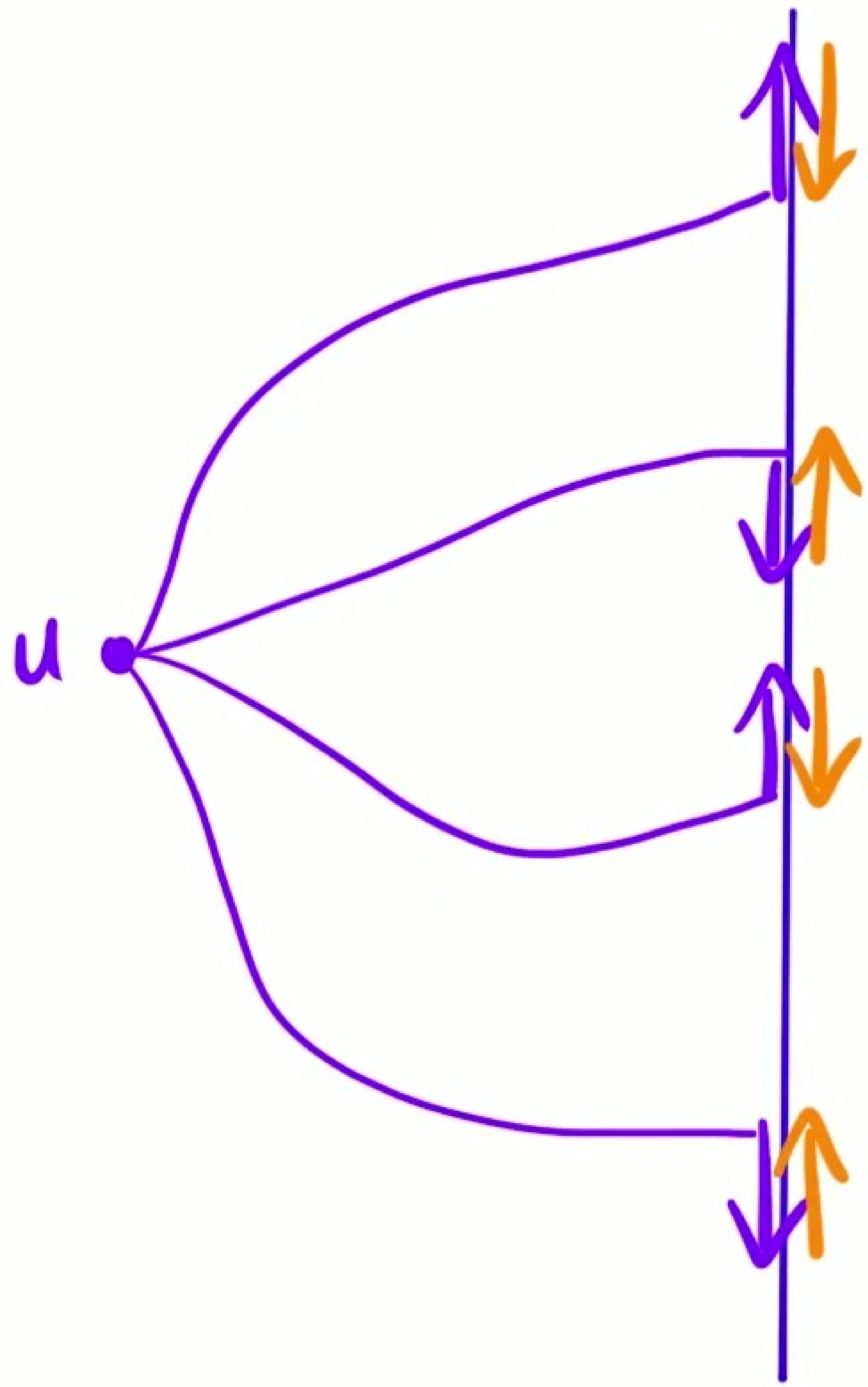

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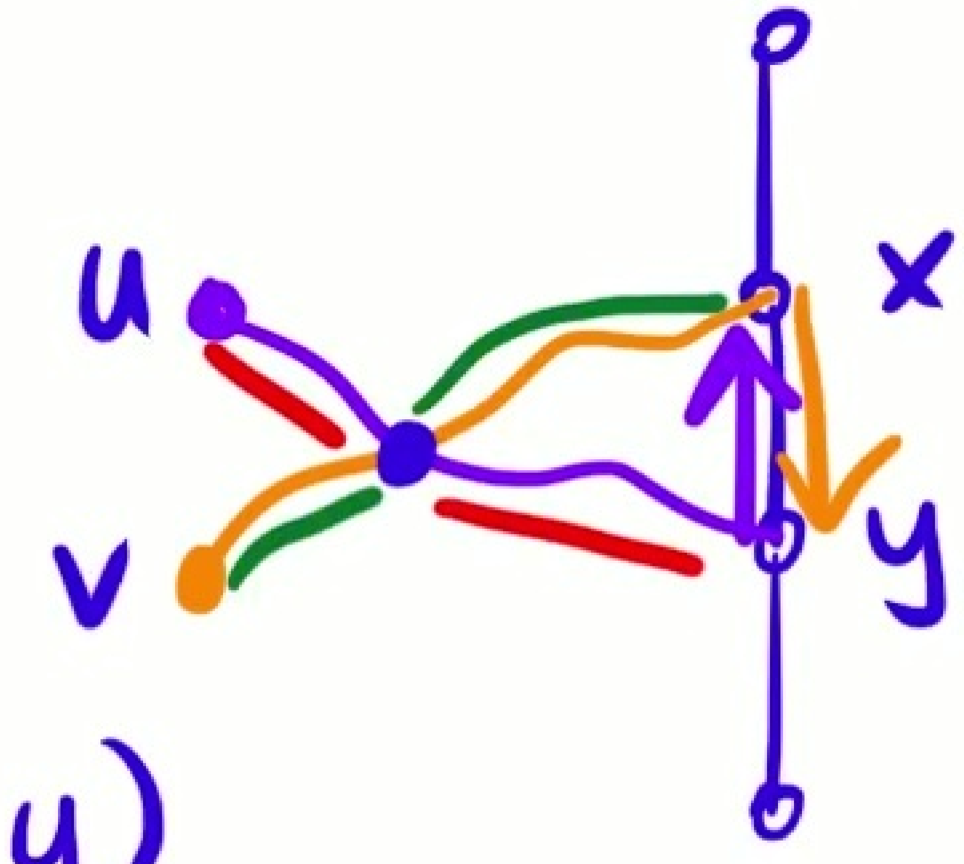

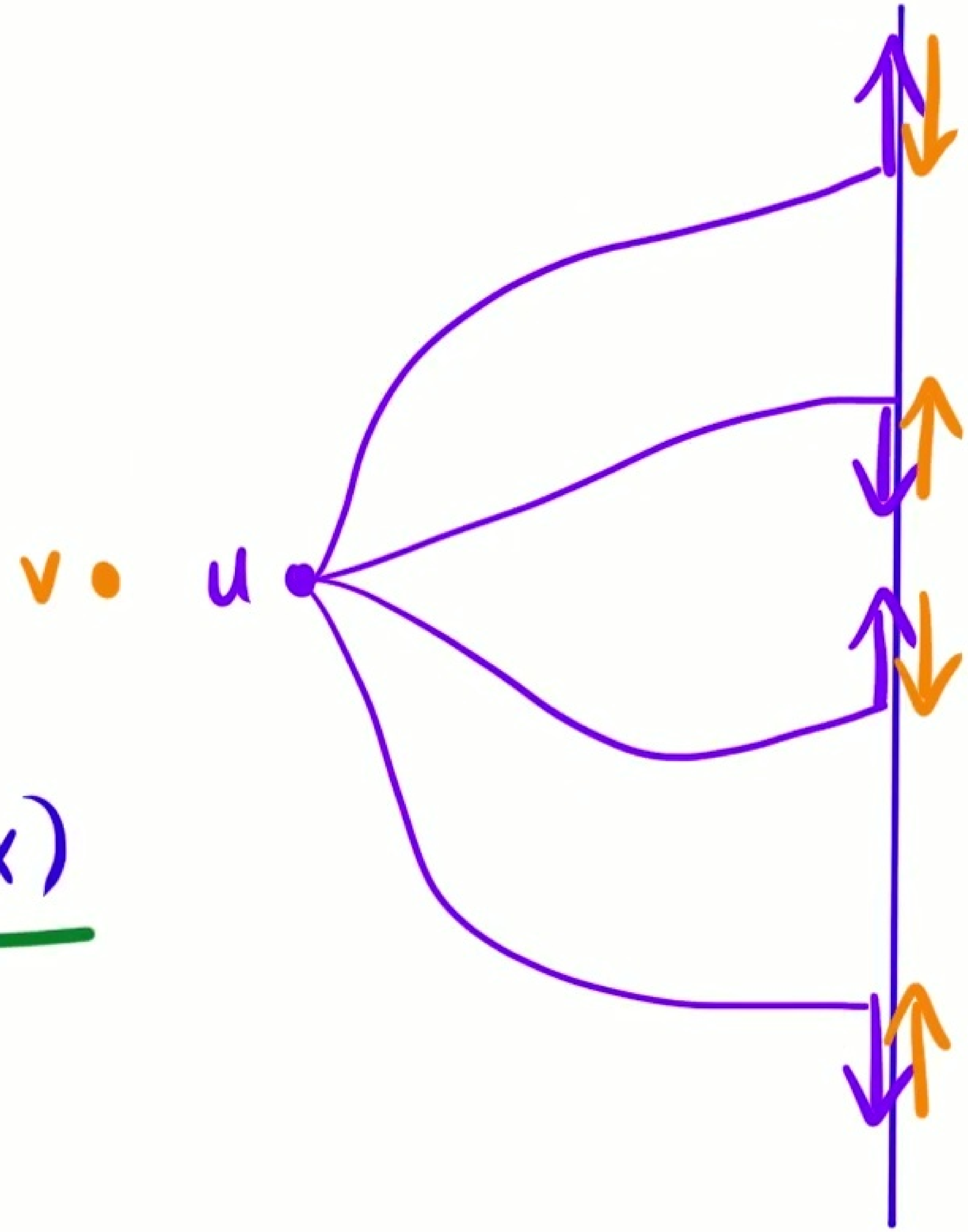

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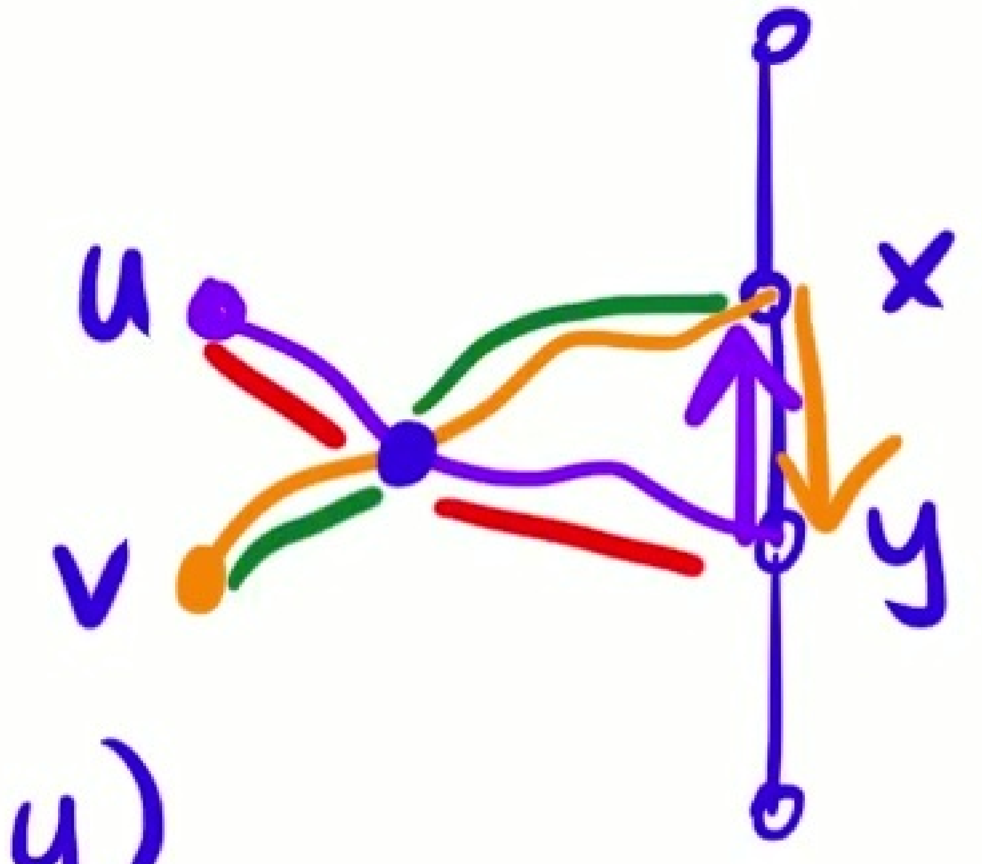

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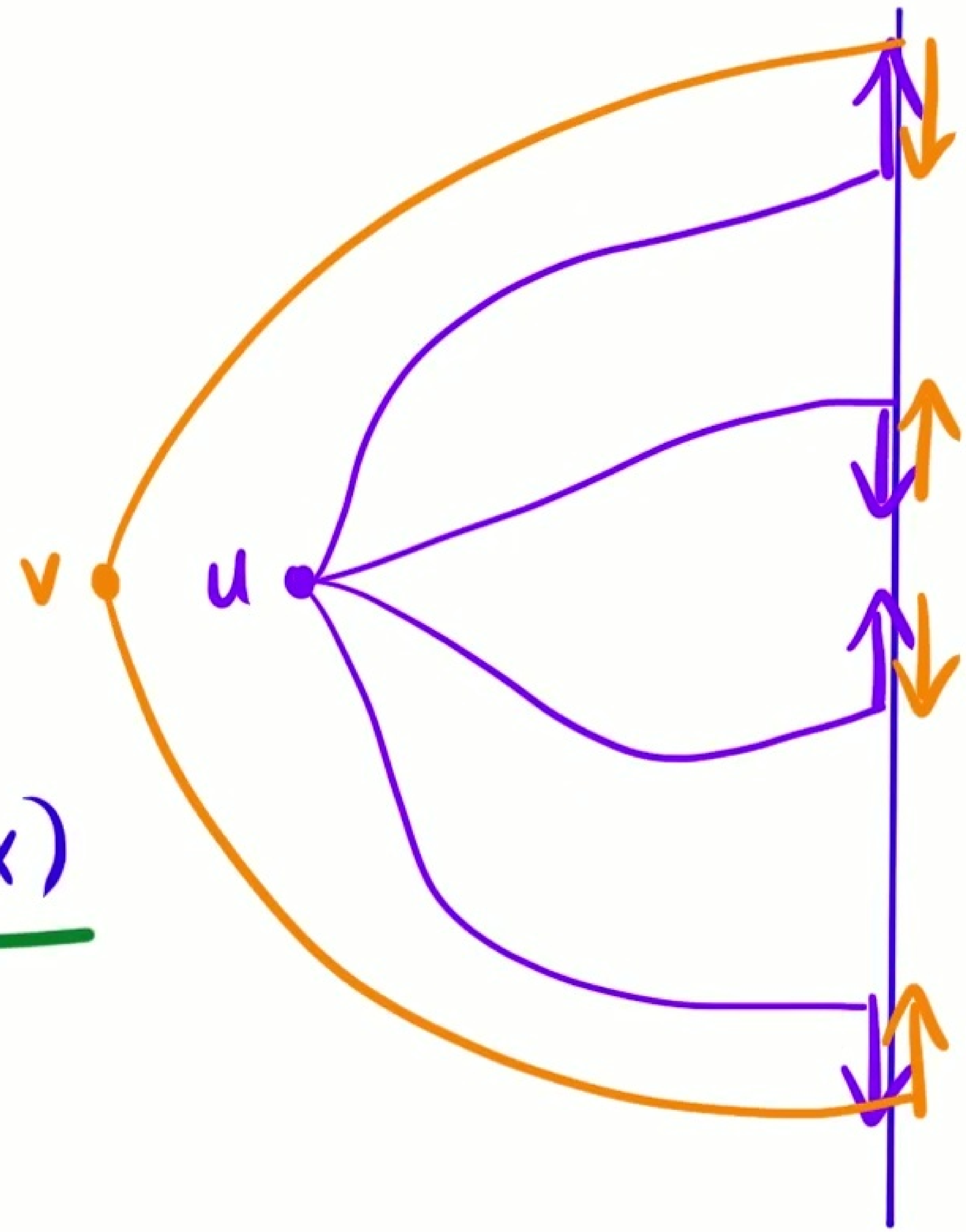

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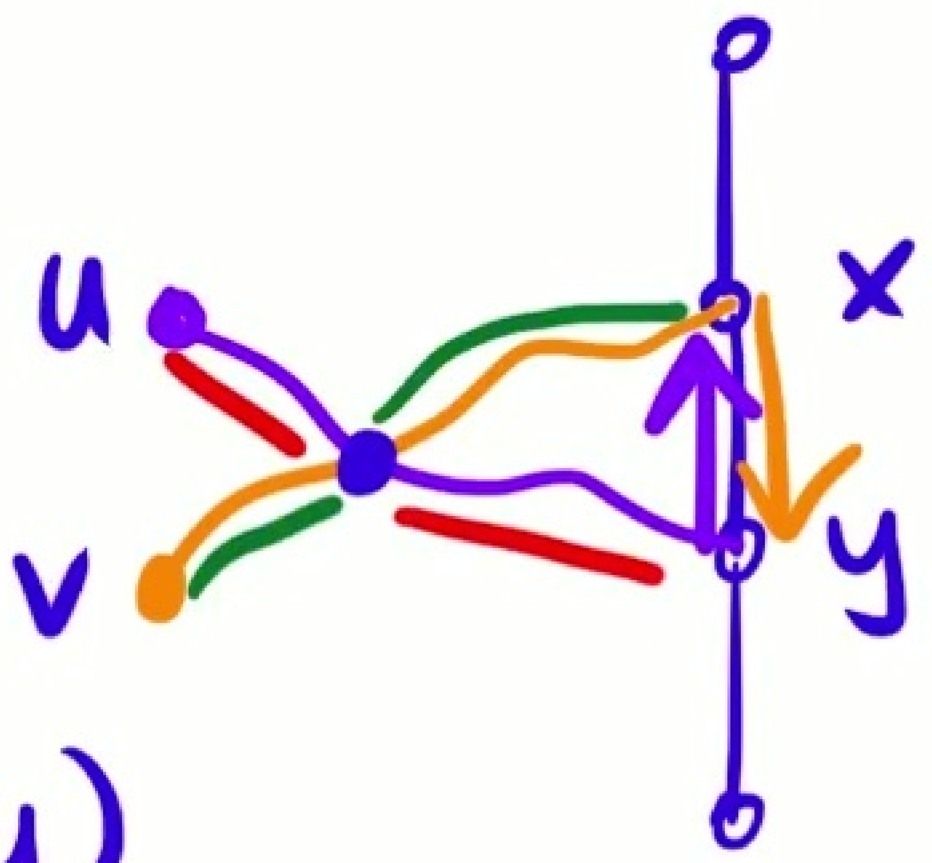

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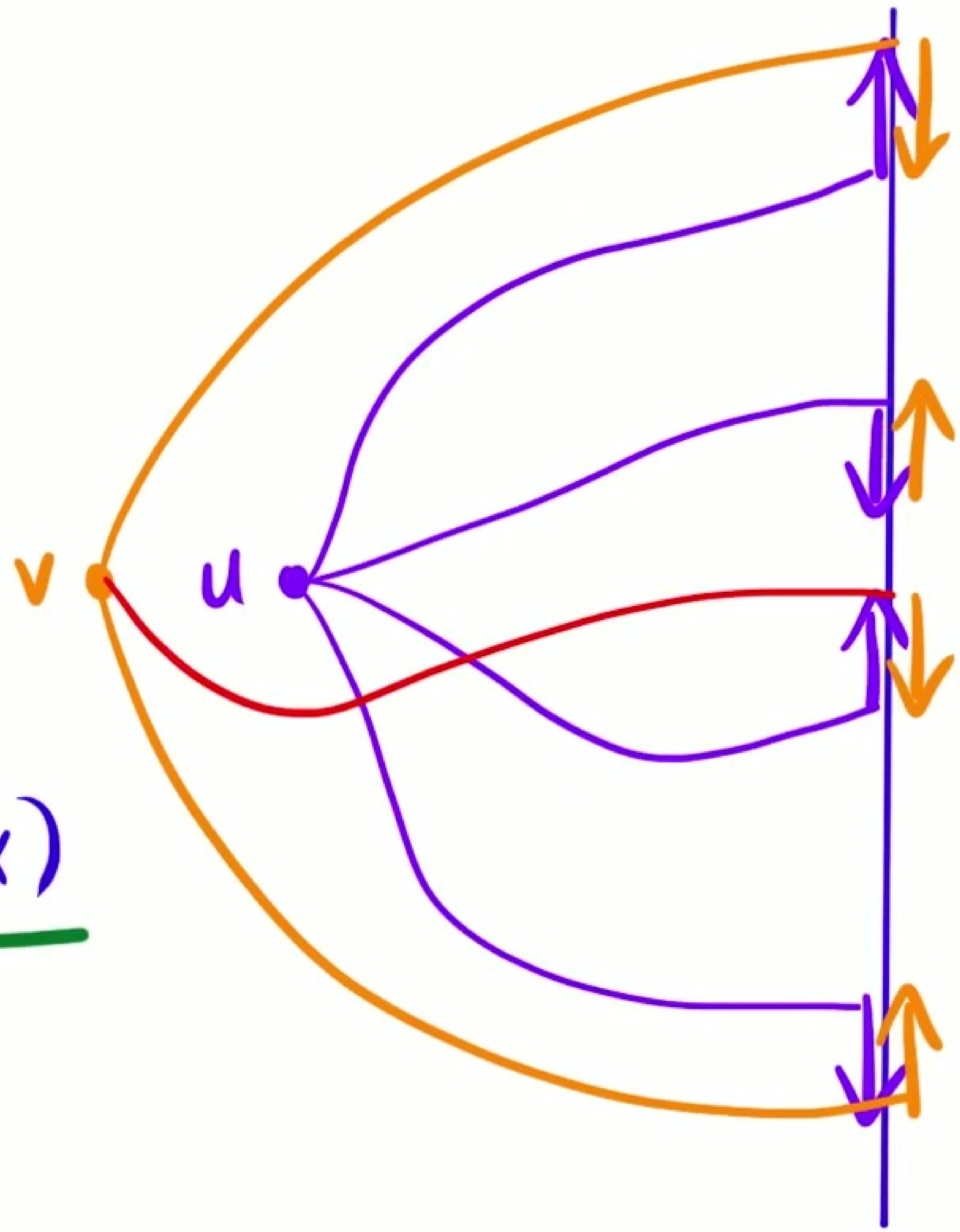

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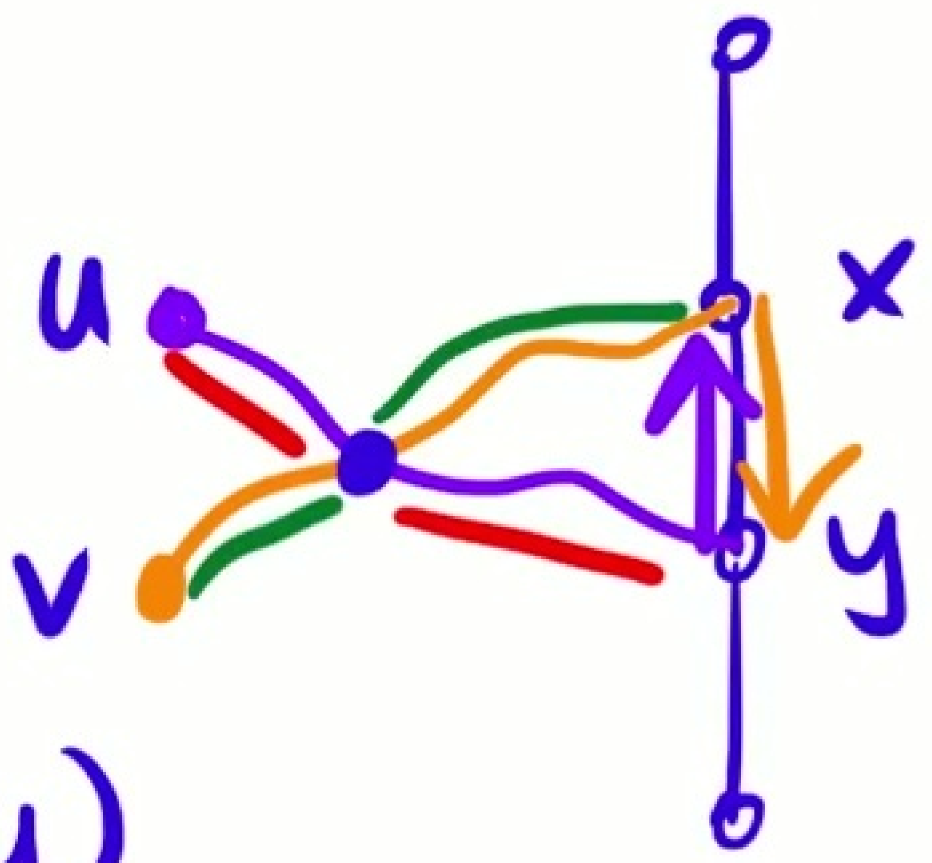

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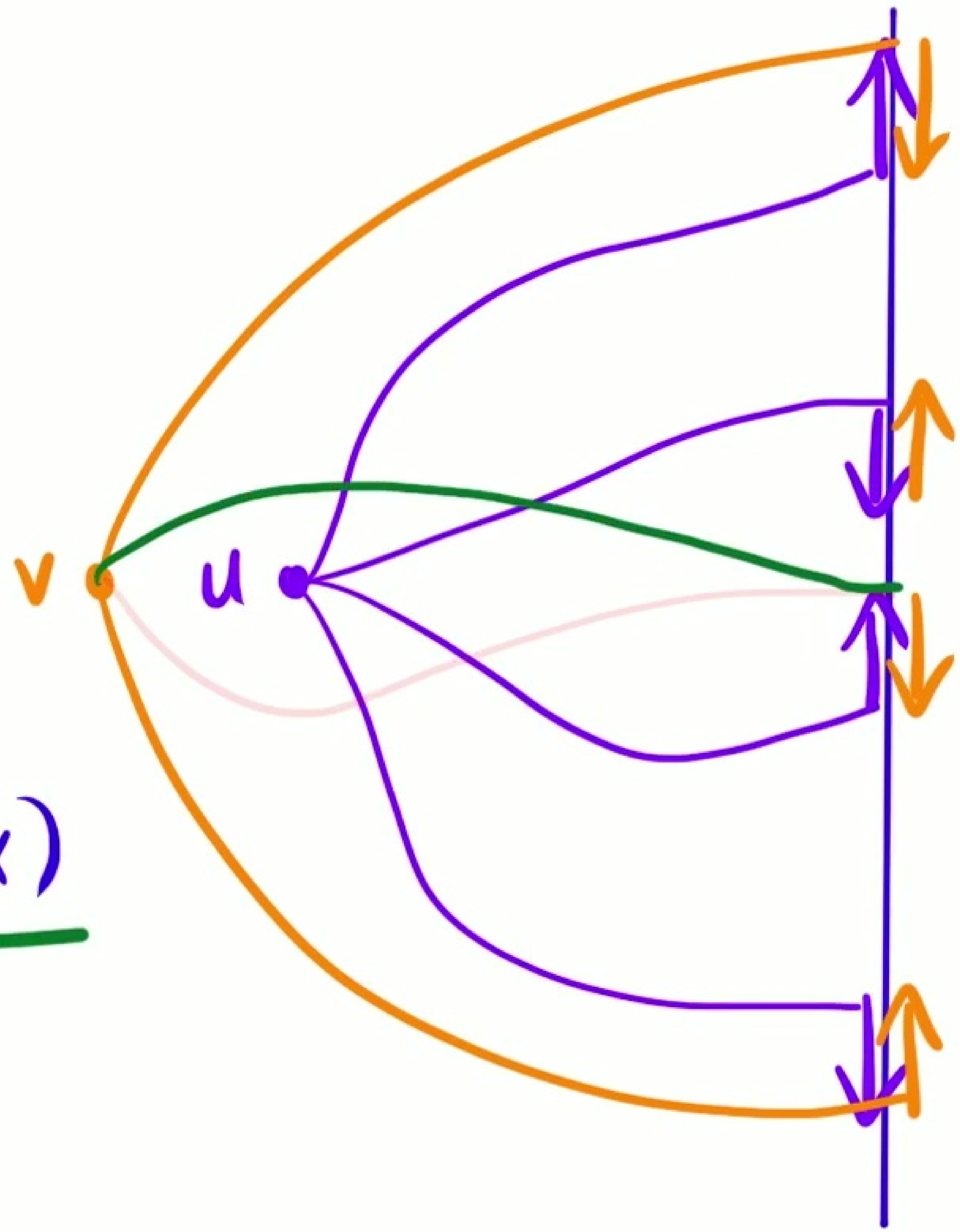

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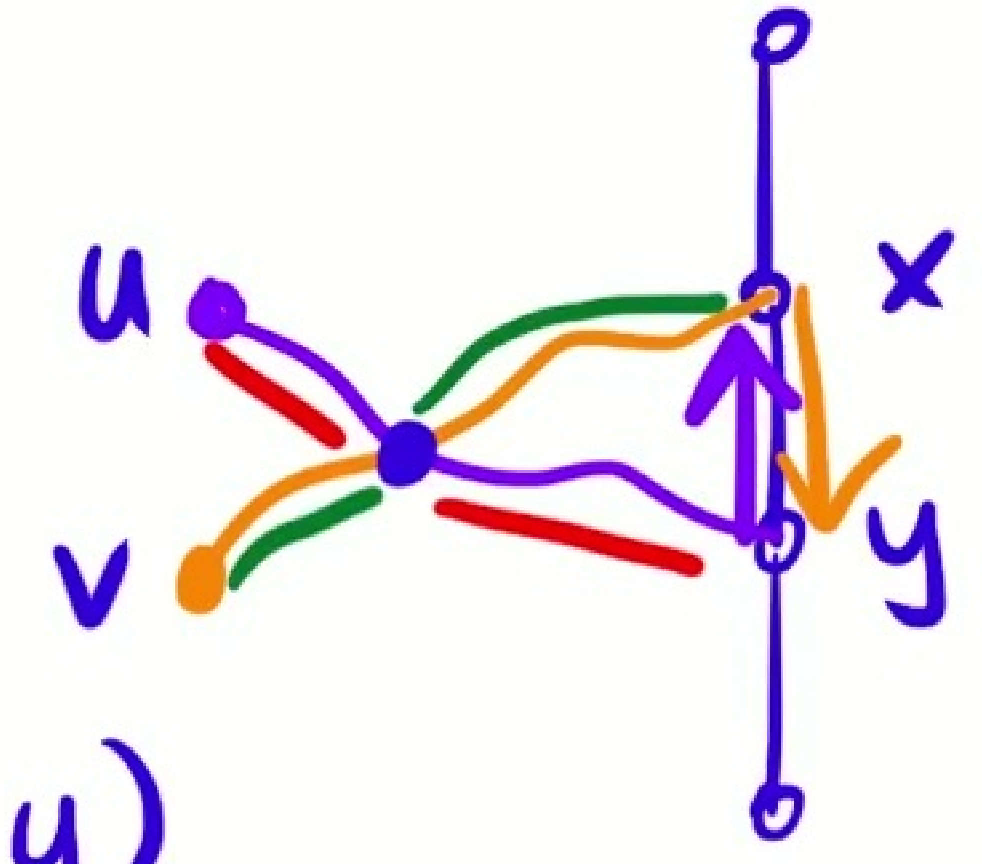

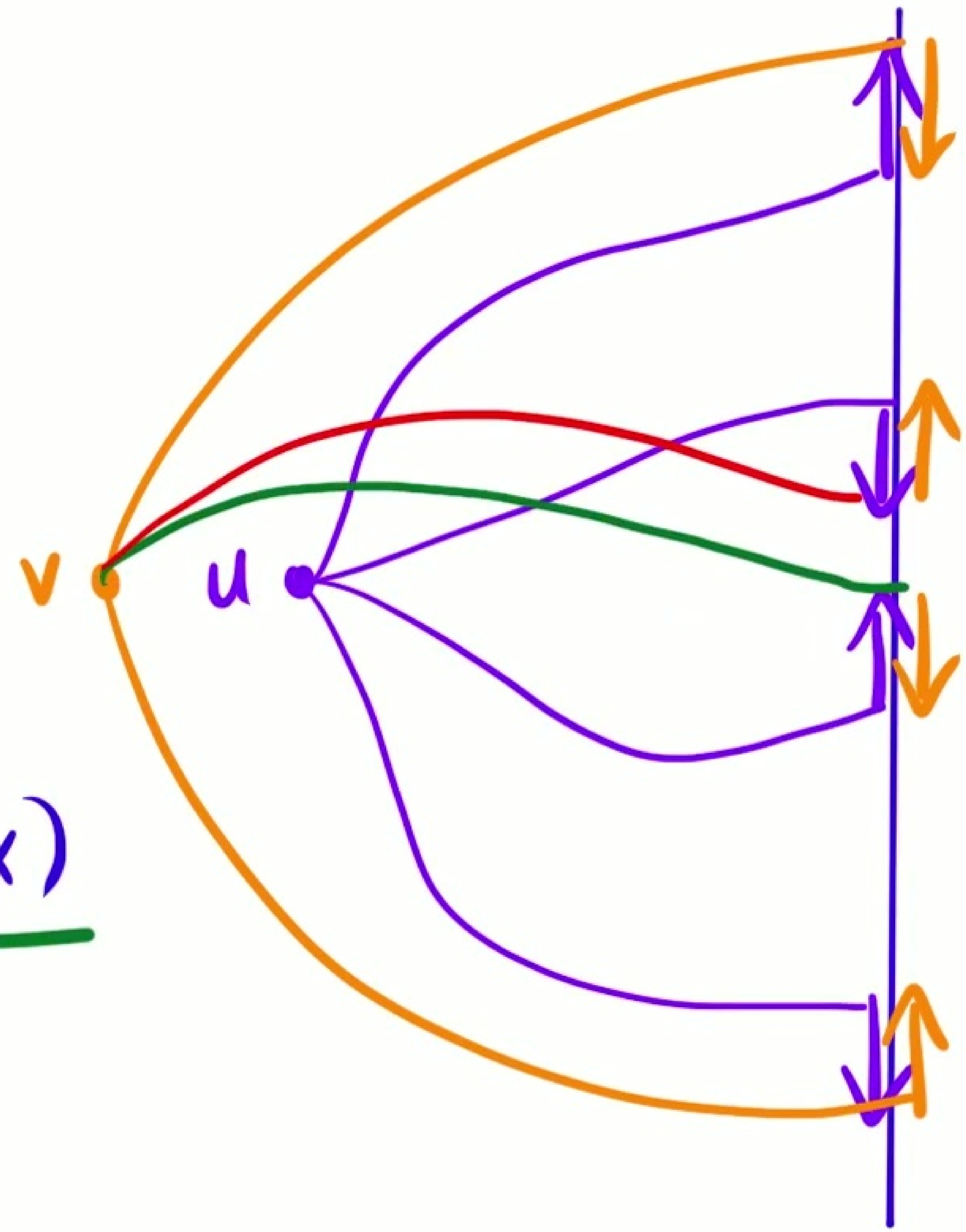

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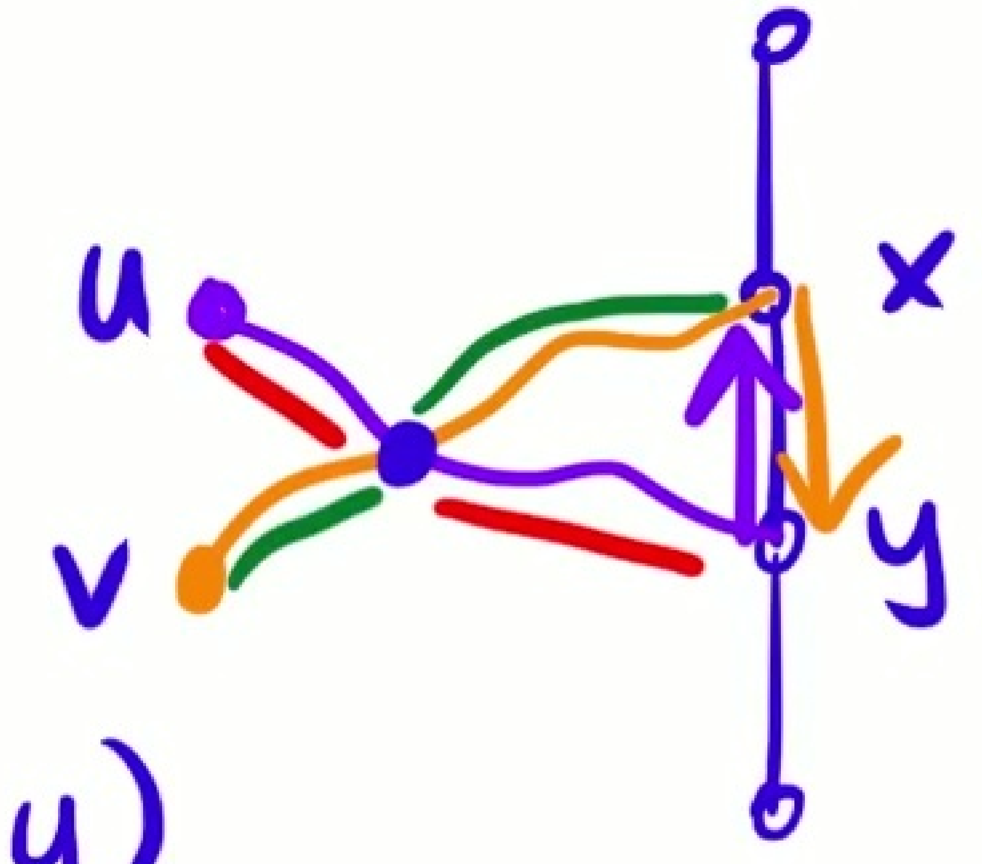

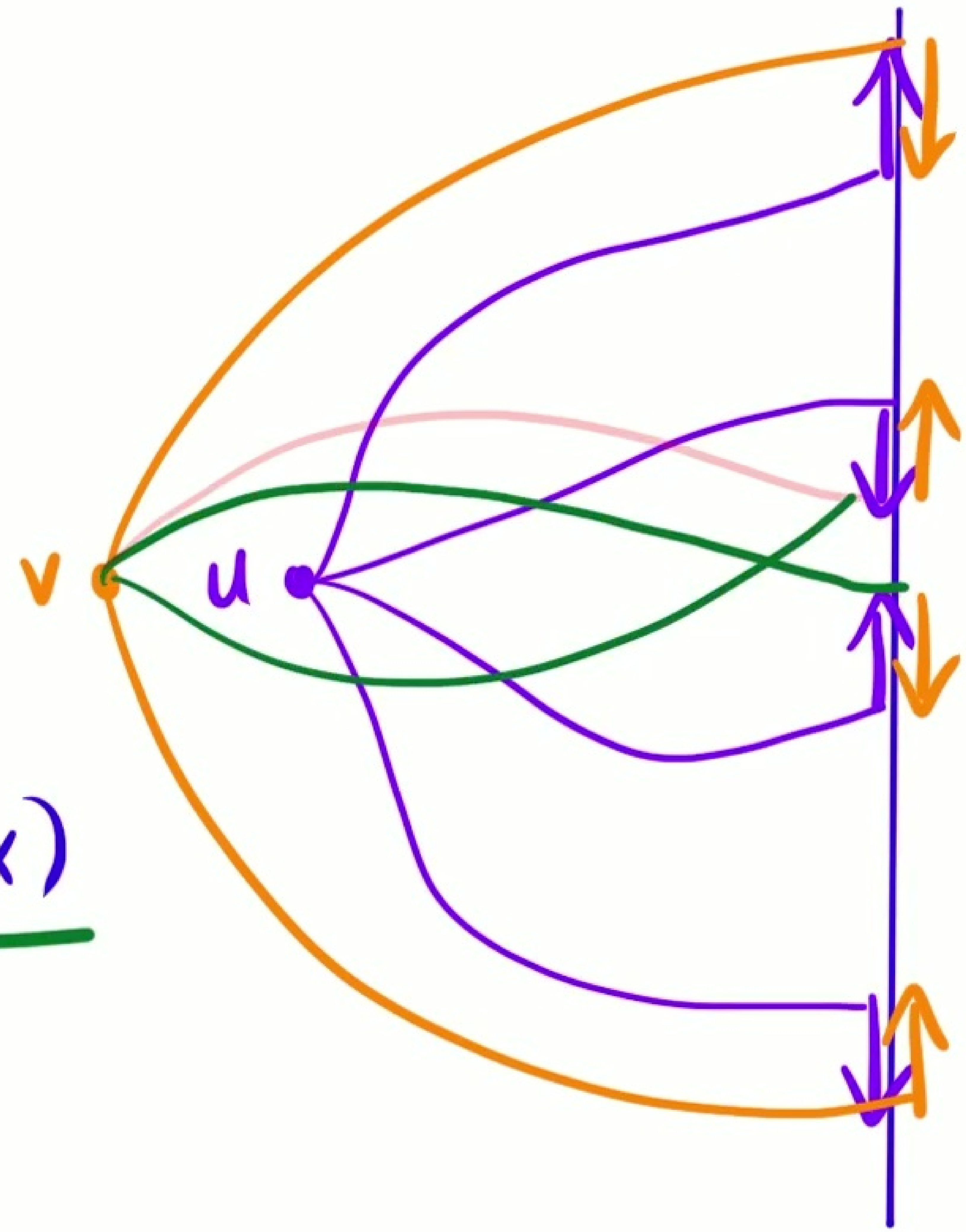

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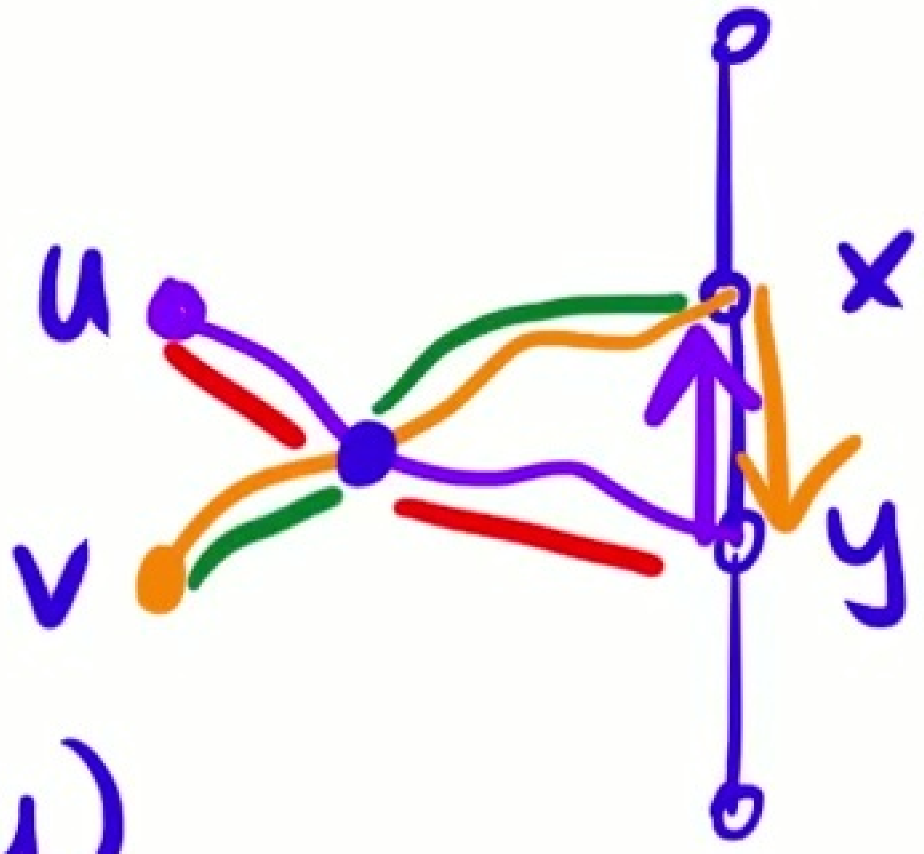

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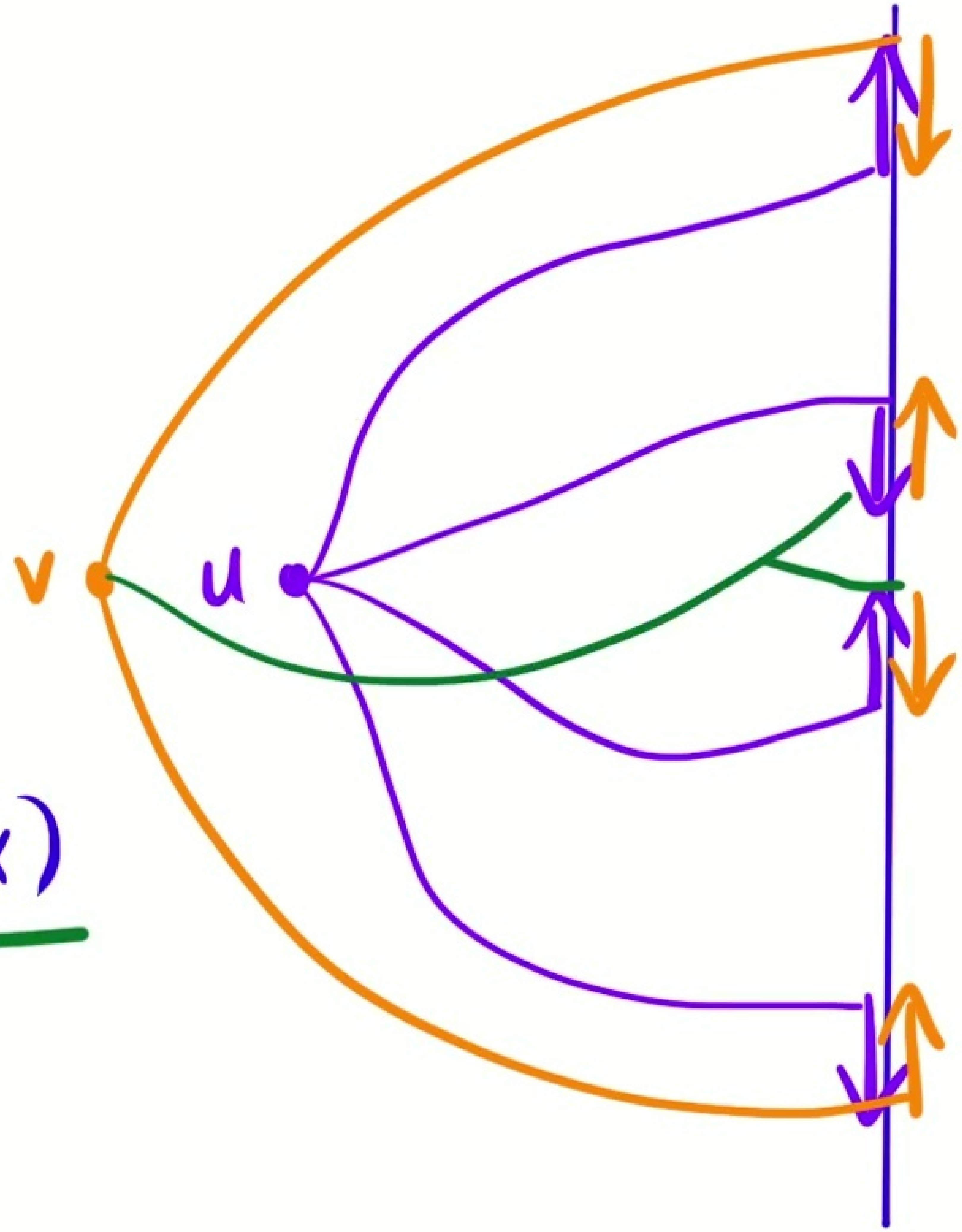
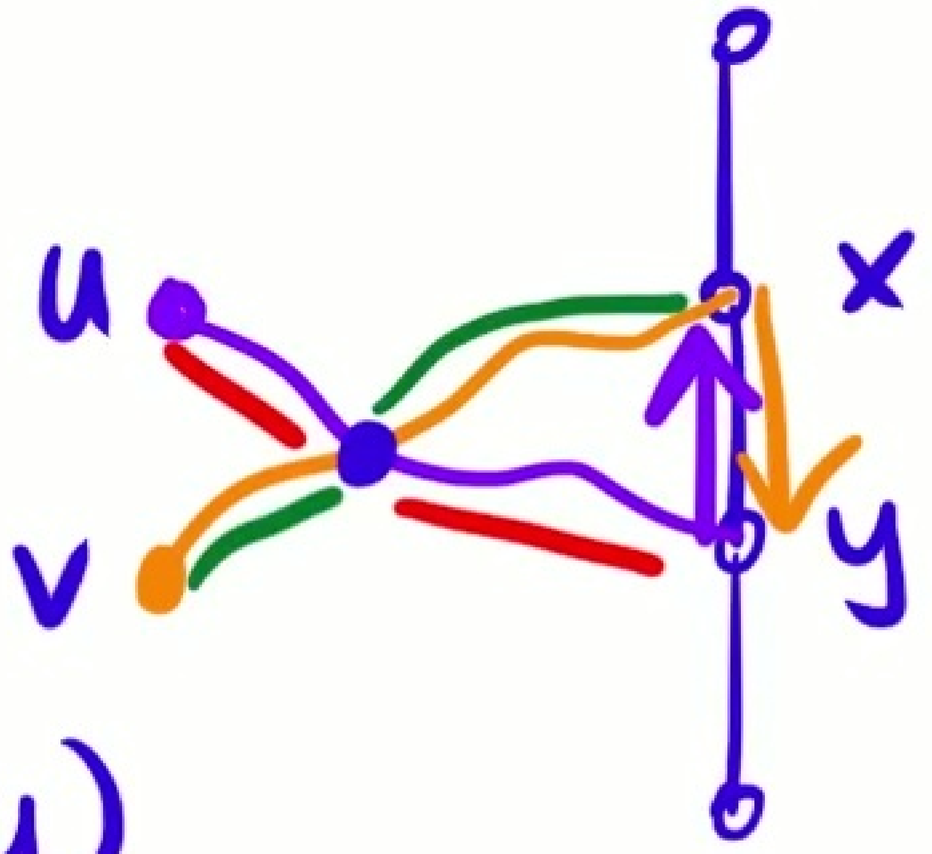


$|\mathcal{L}| = O(Dk^3)$

Monge property:

Cannot have:



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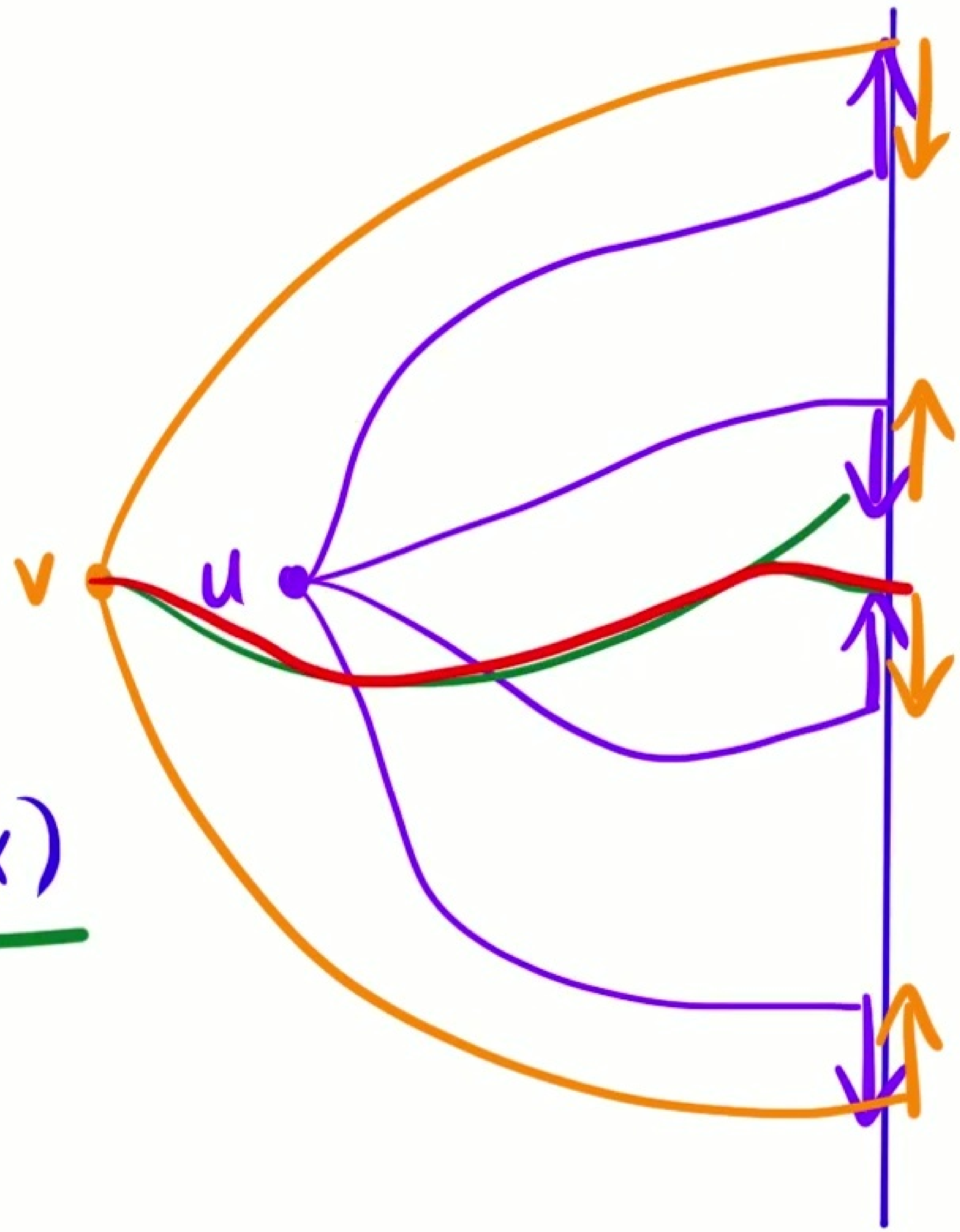

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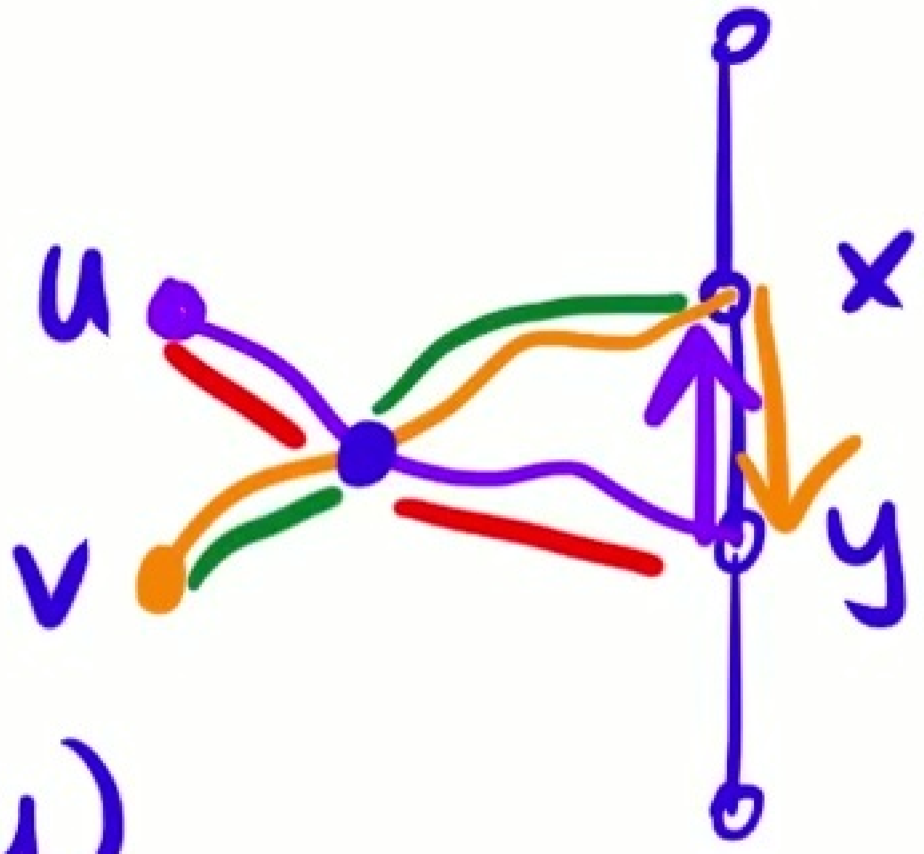

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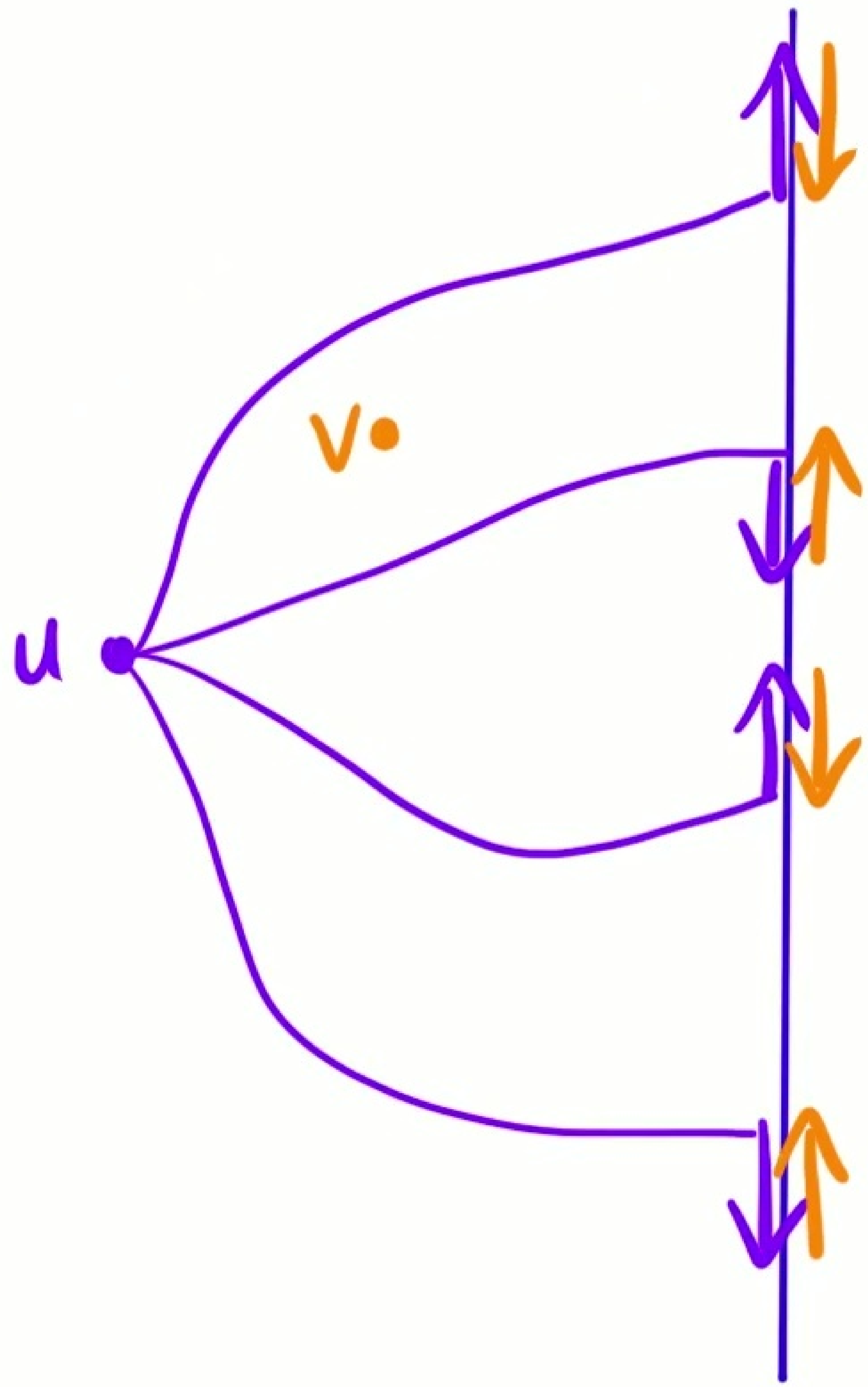

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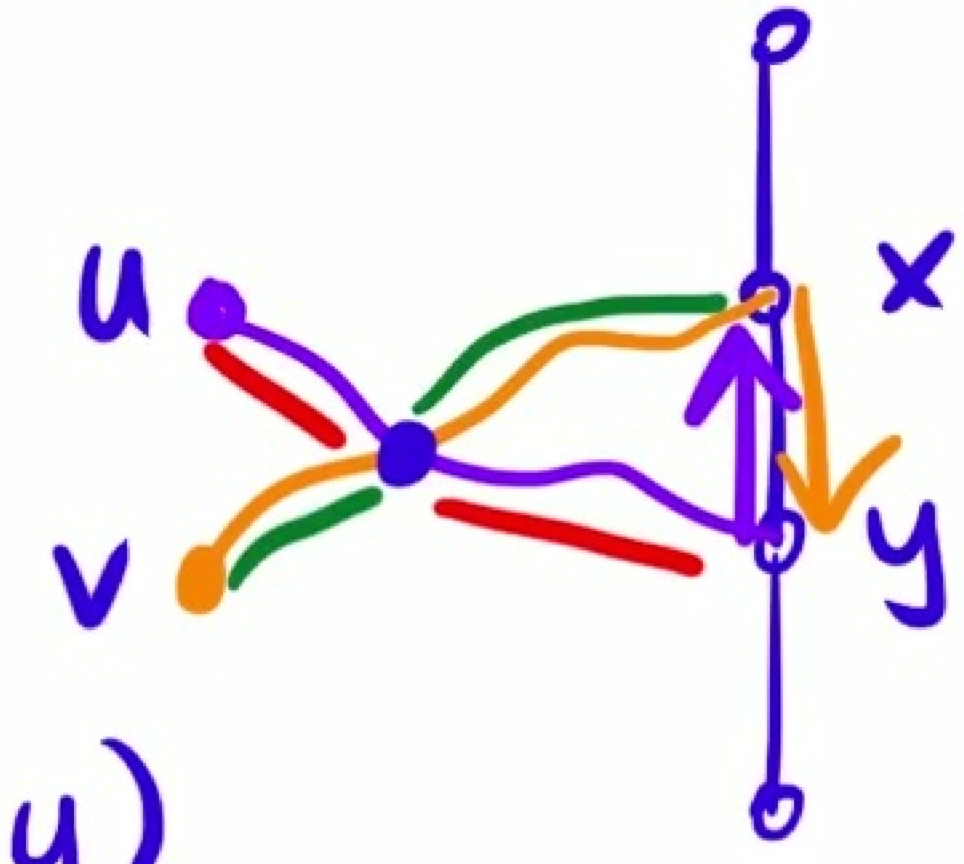

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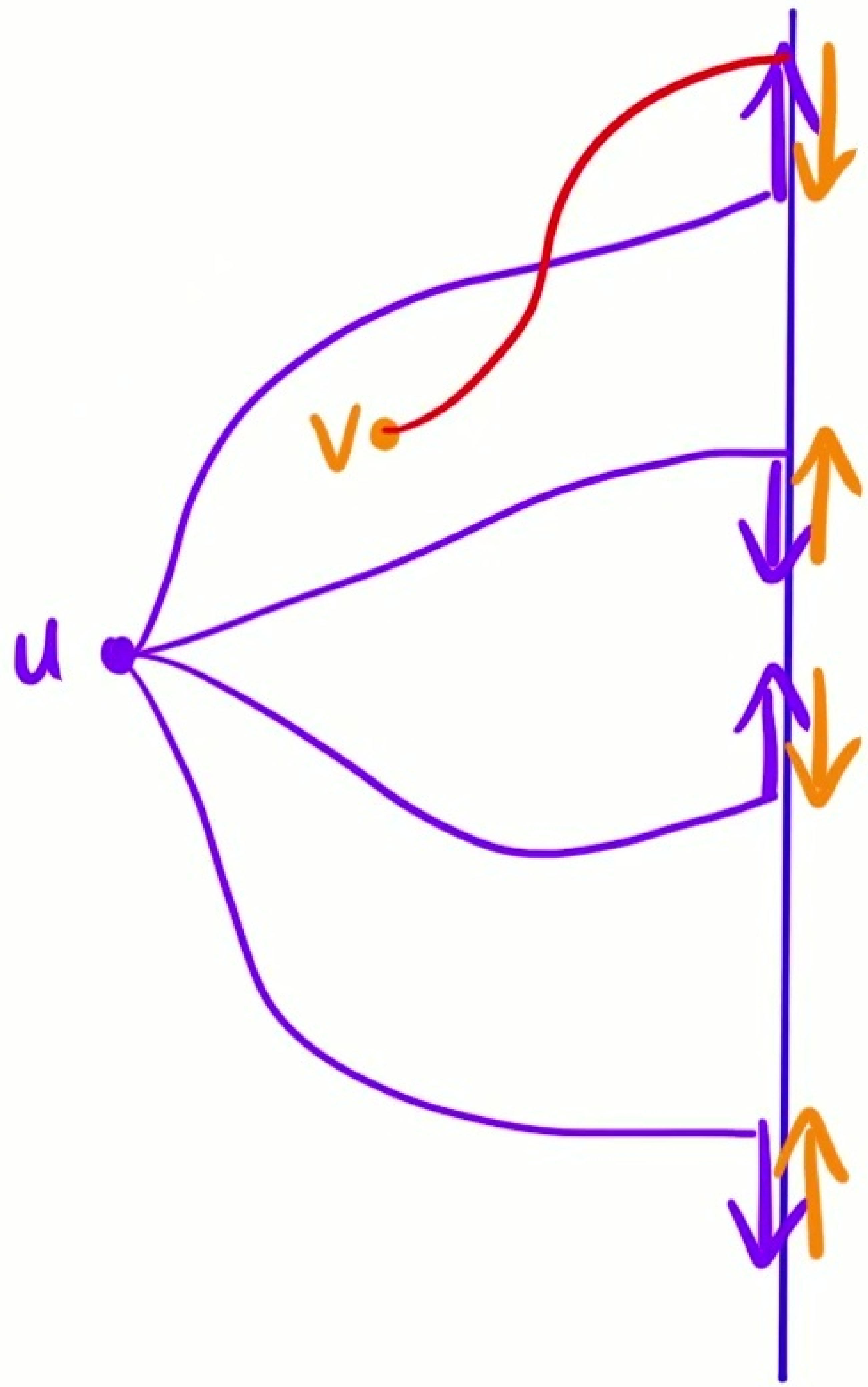

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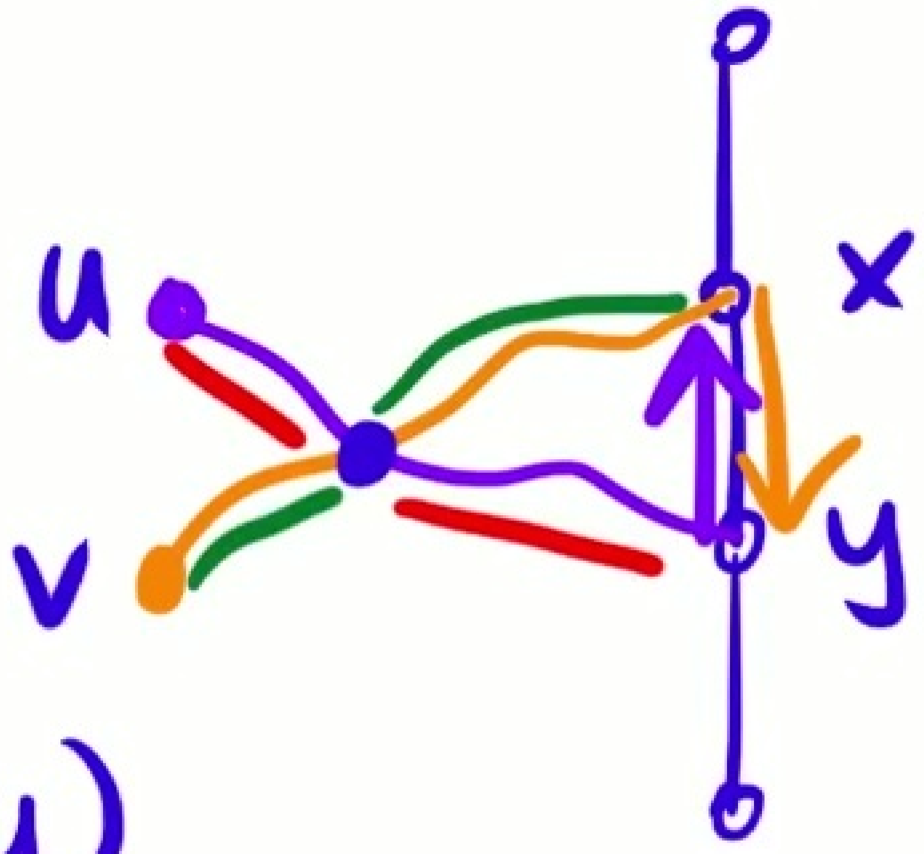

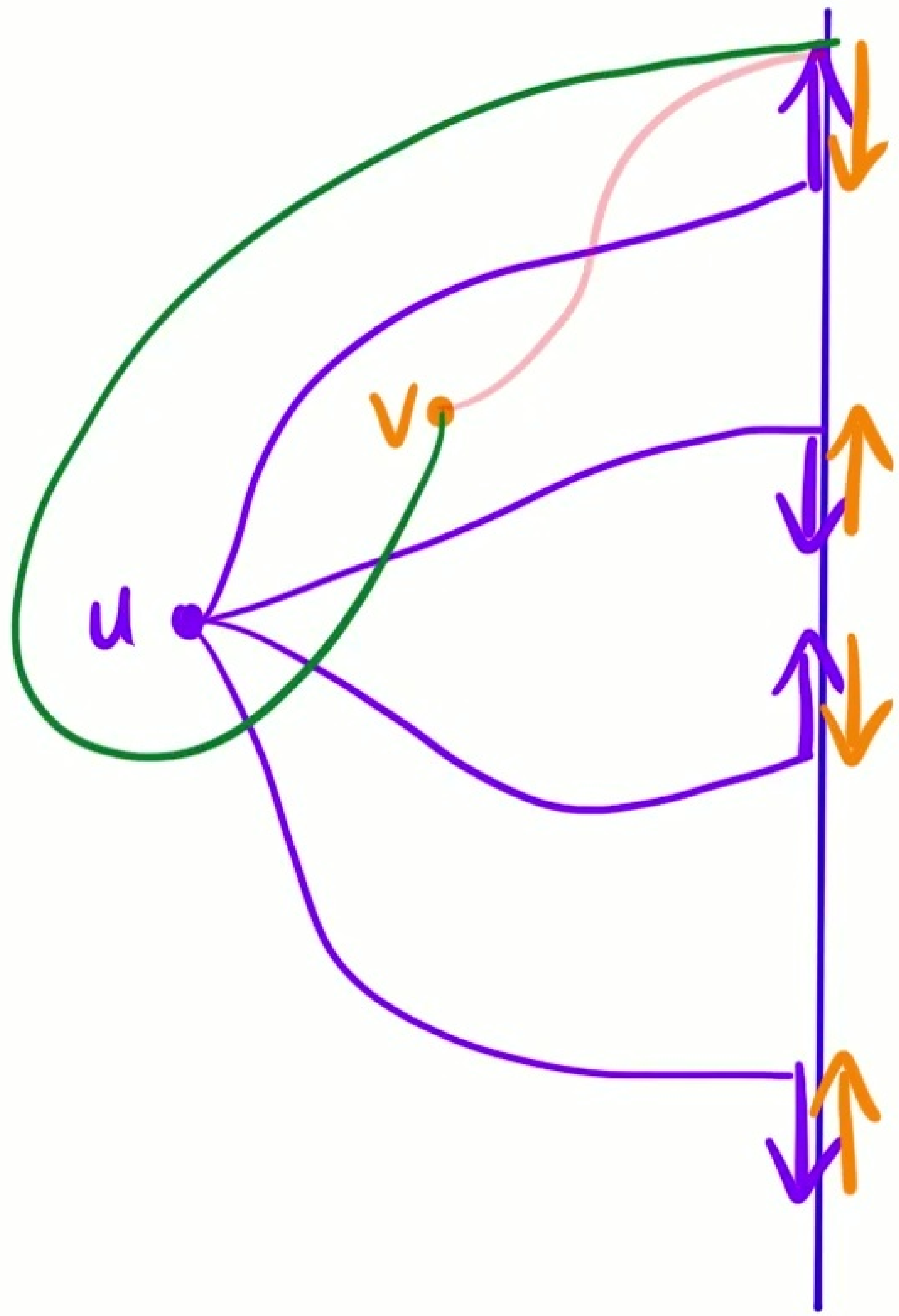

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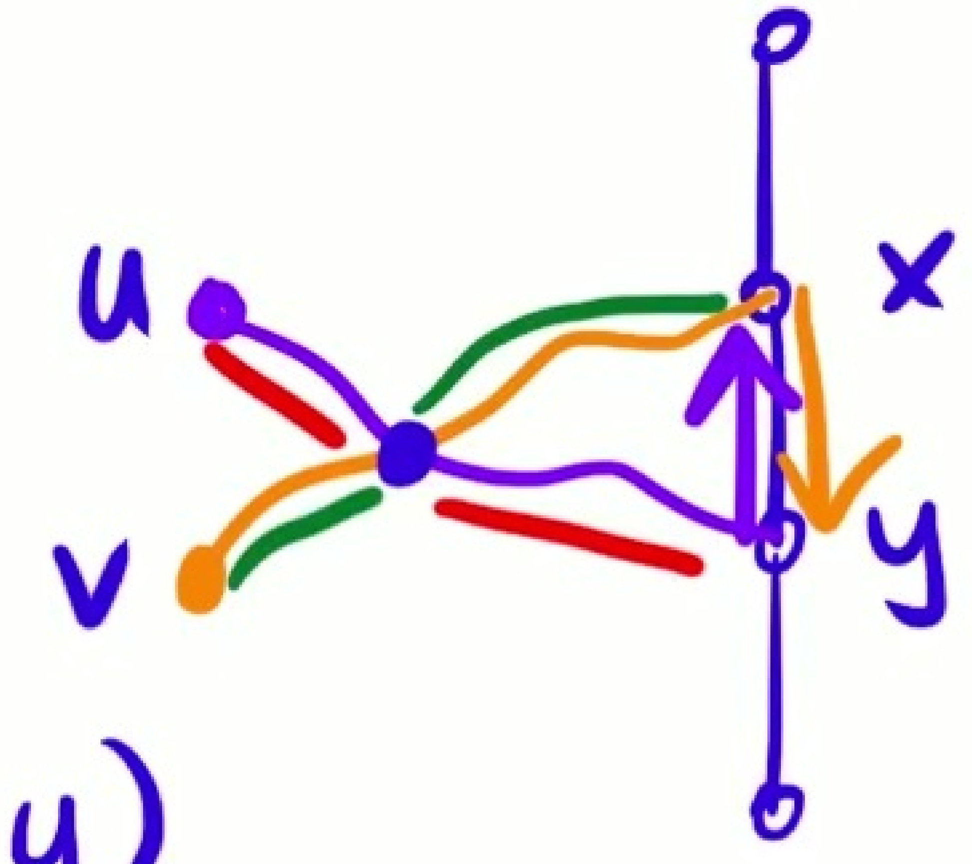

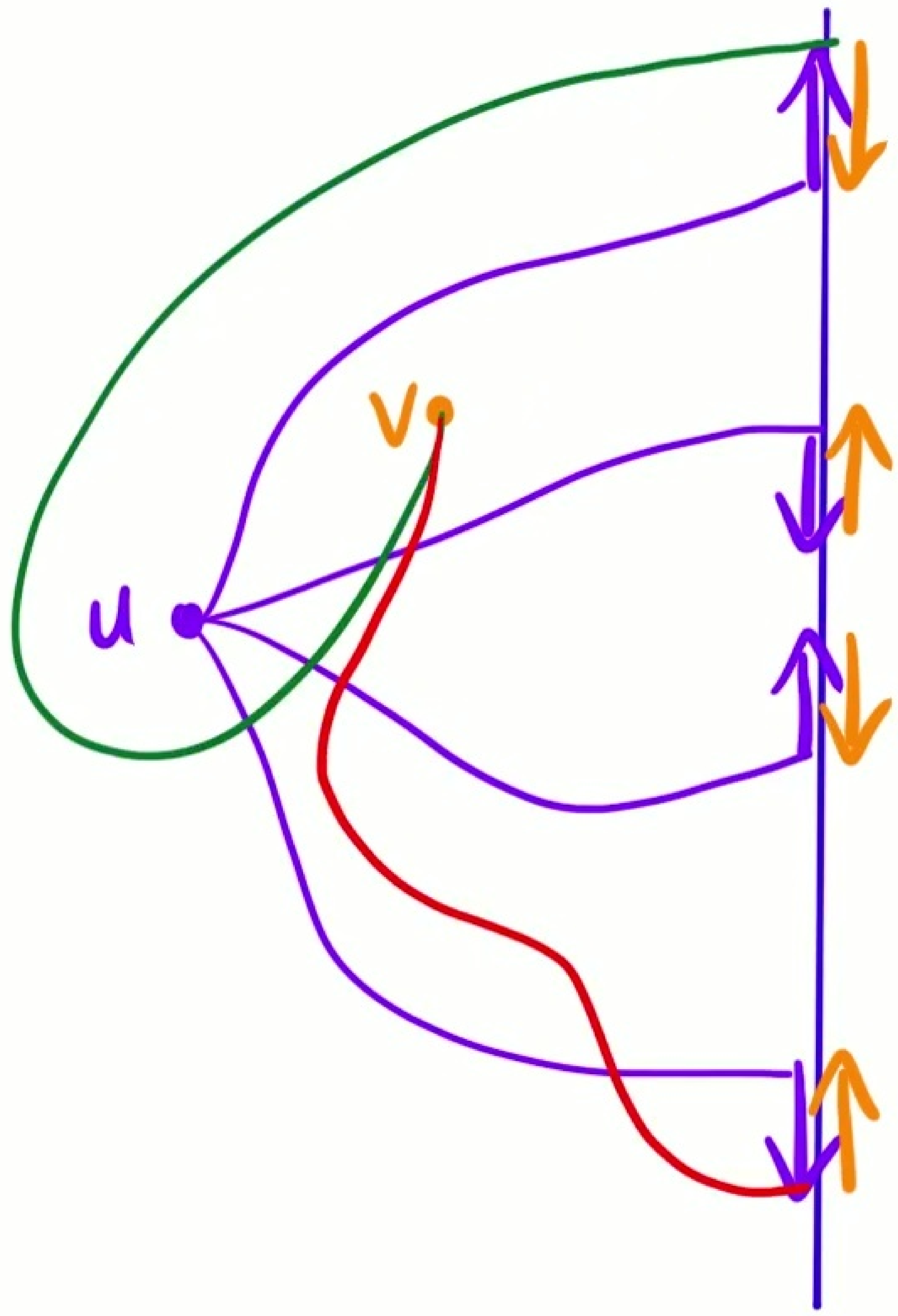

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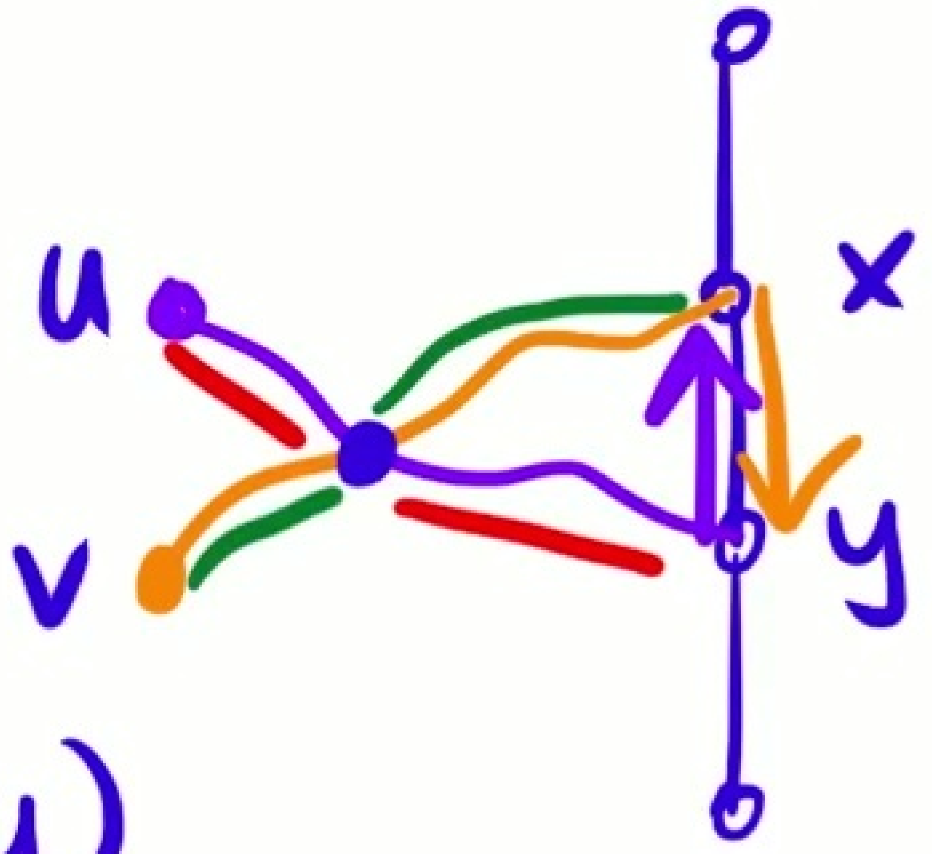

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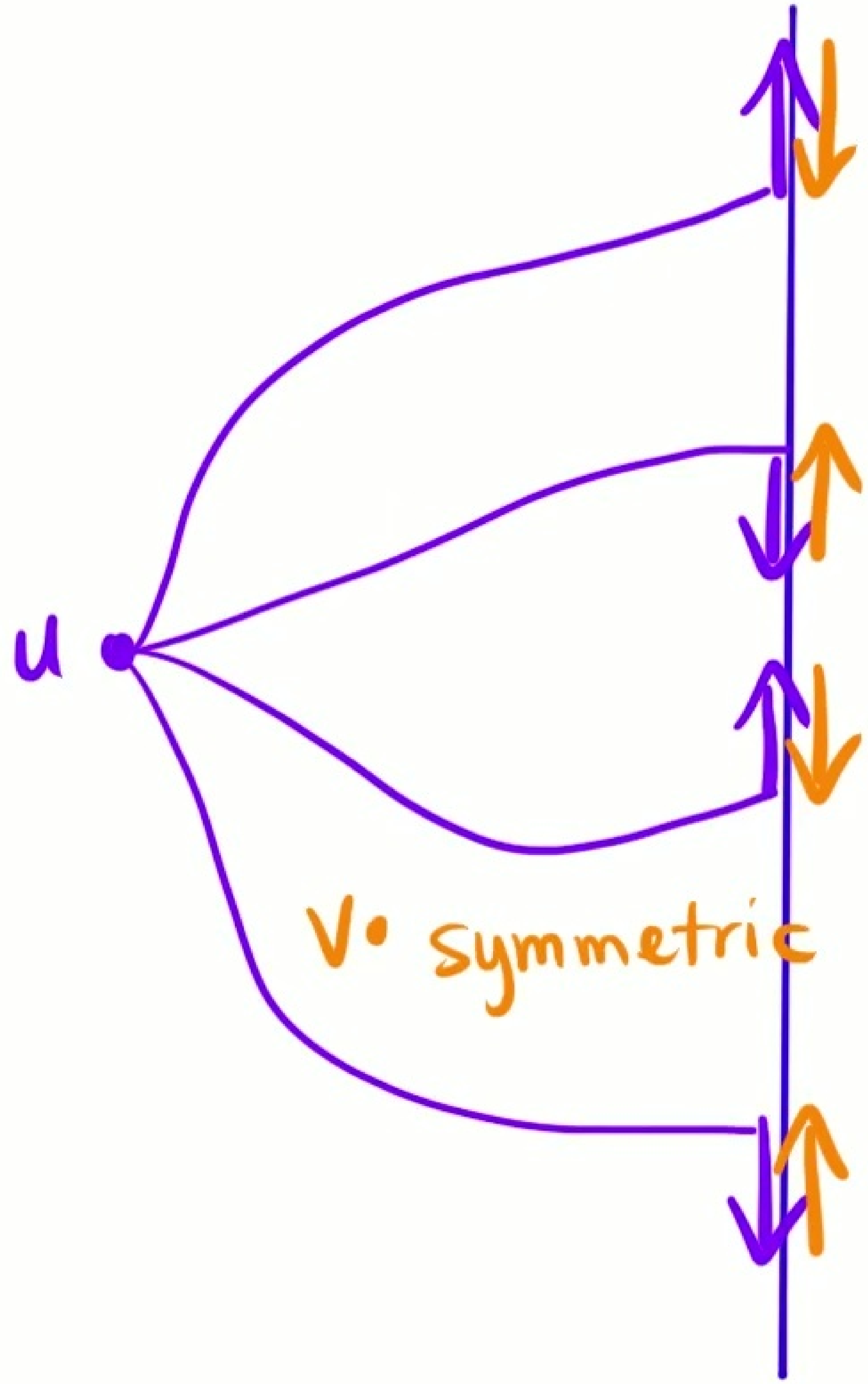

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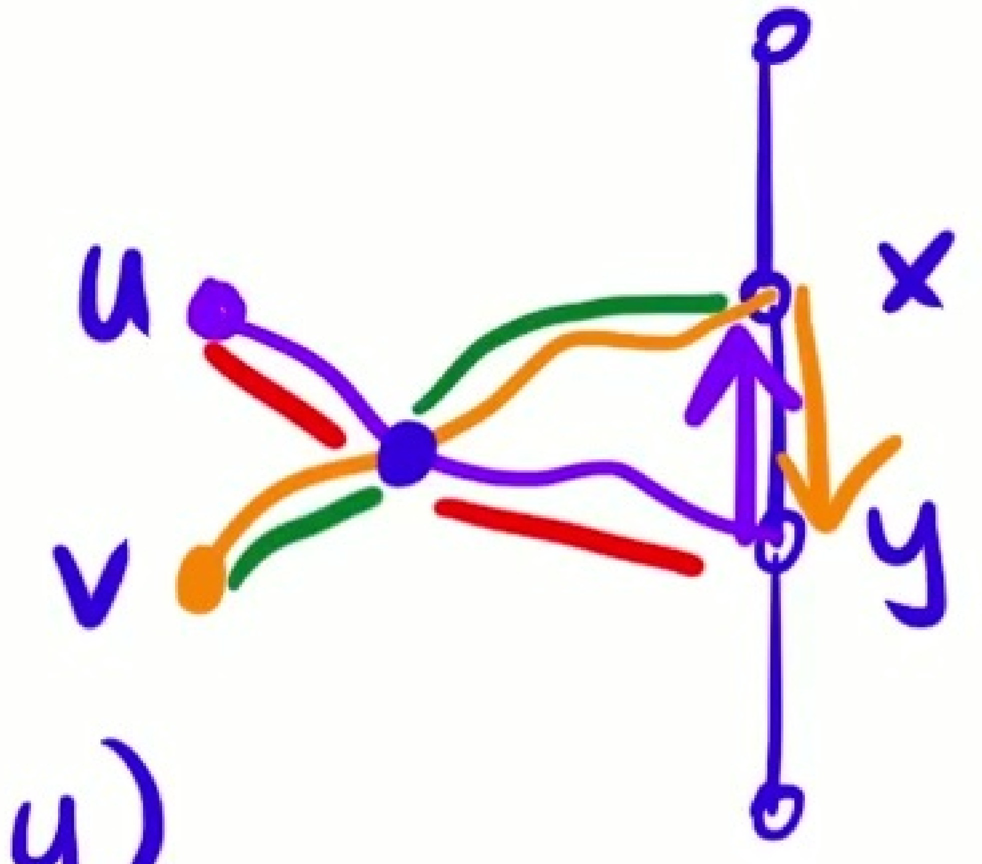

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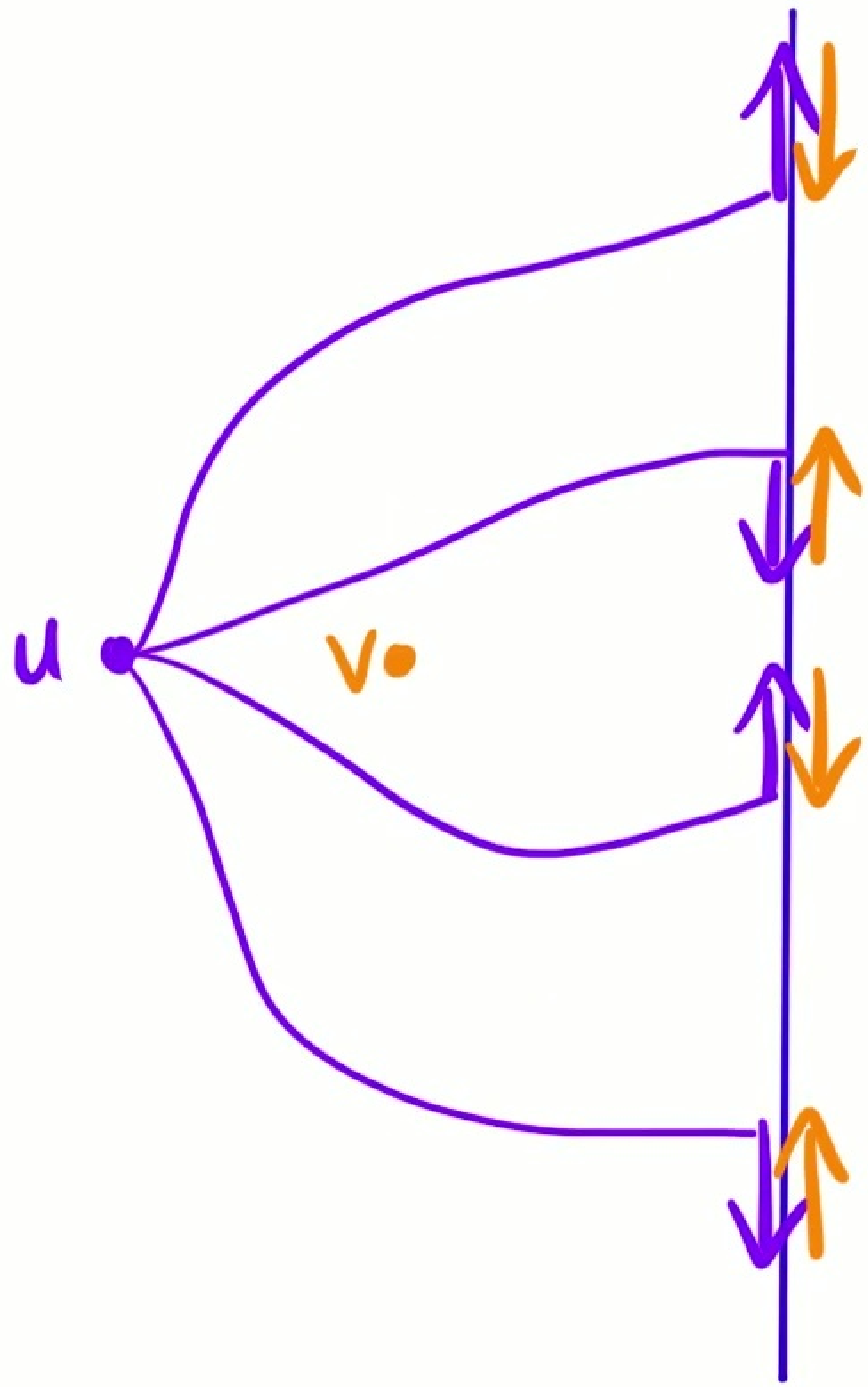

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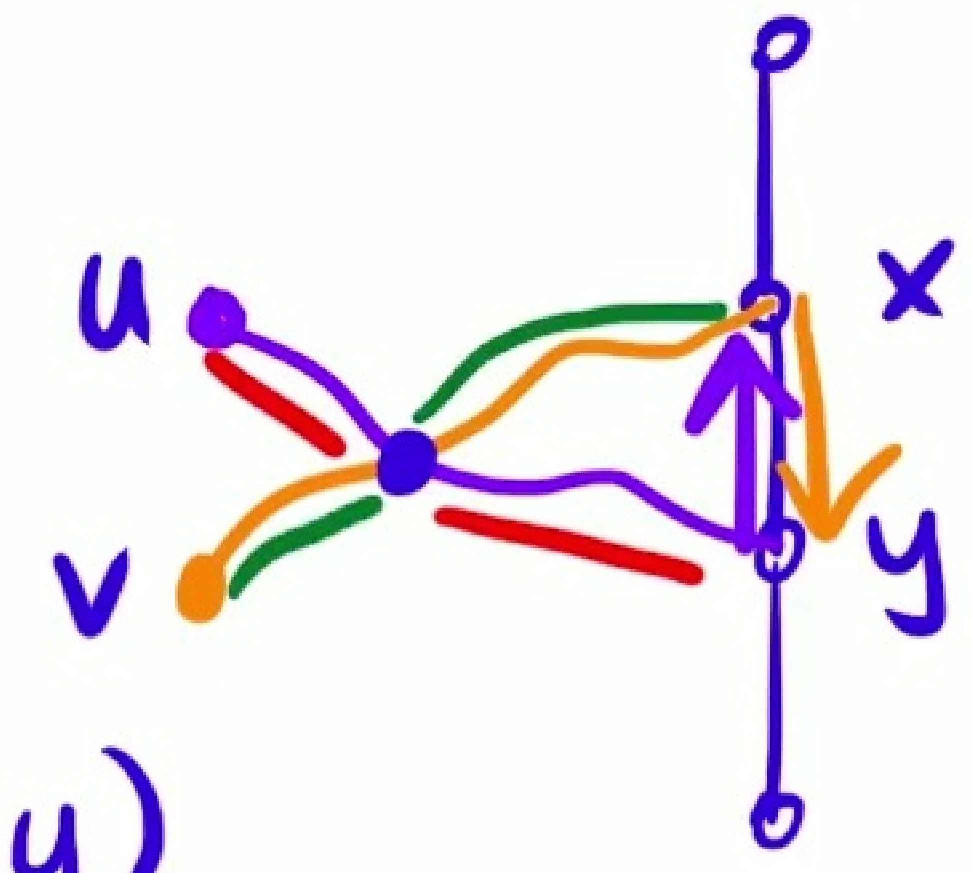

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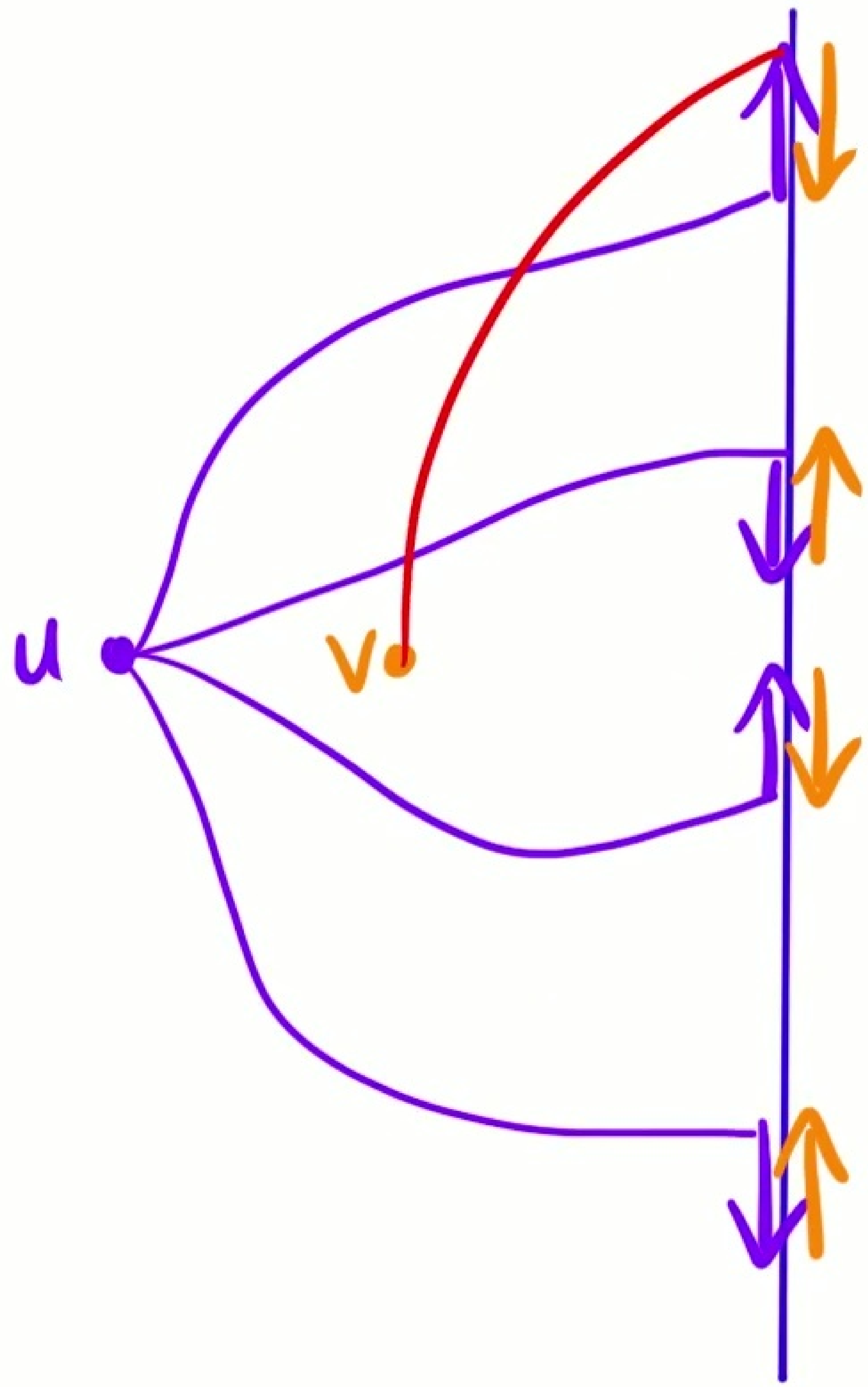

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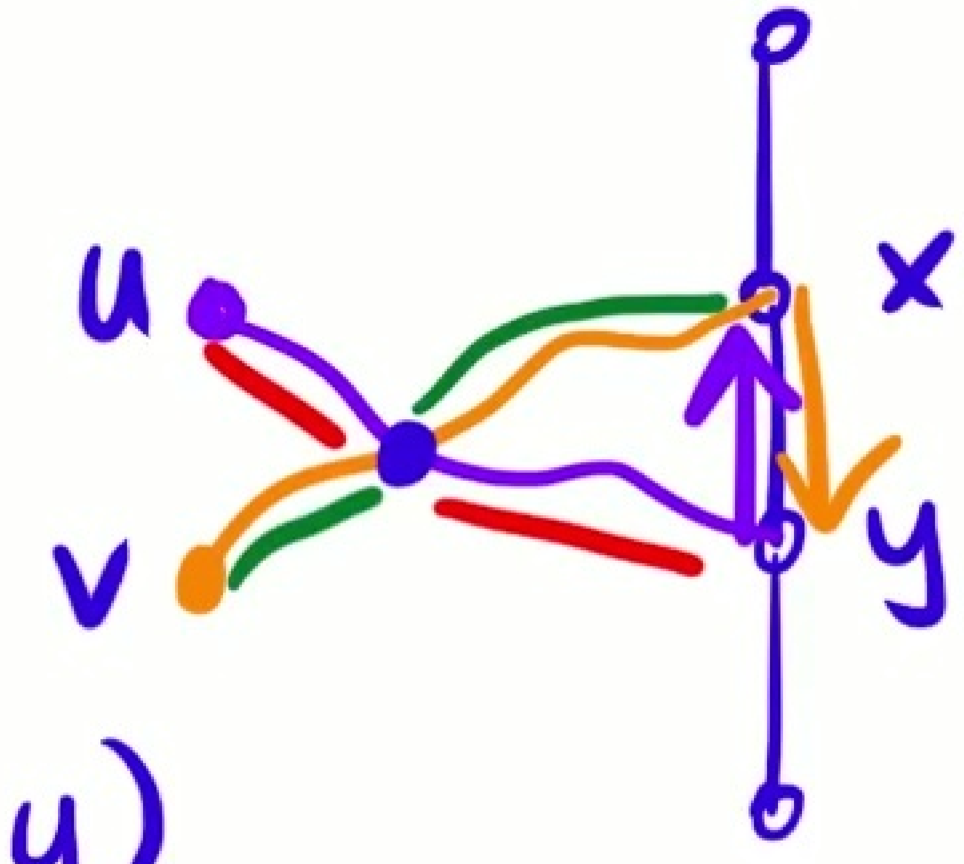

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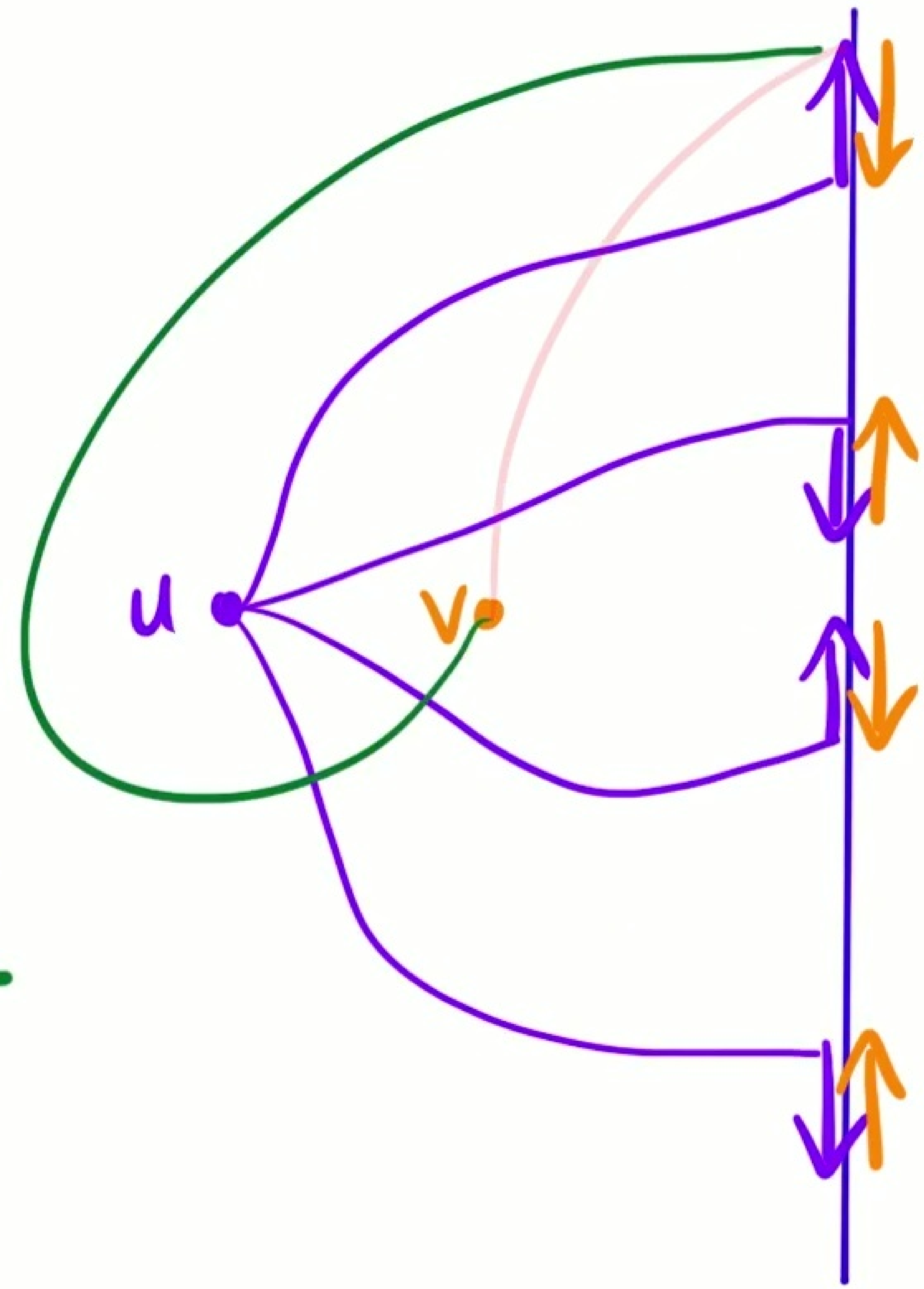

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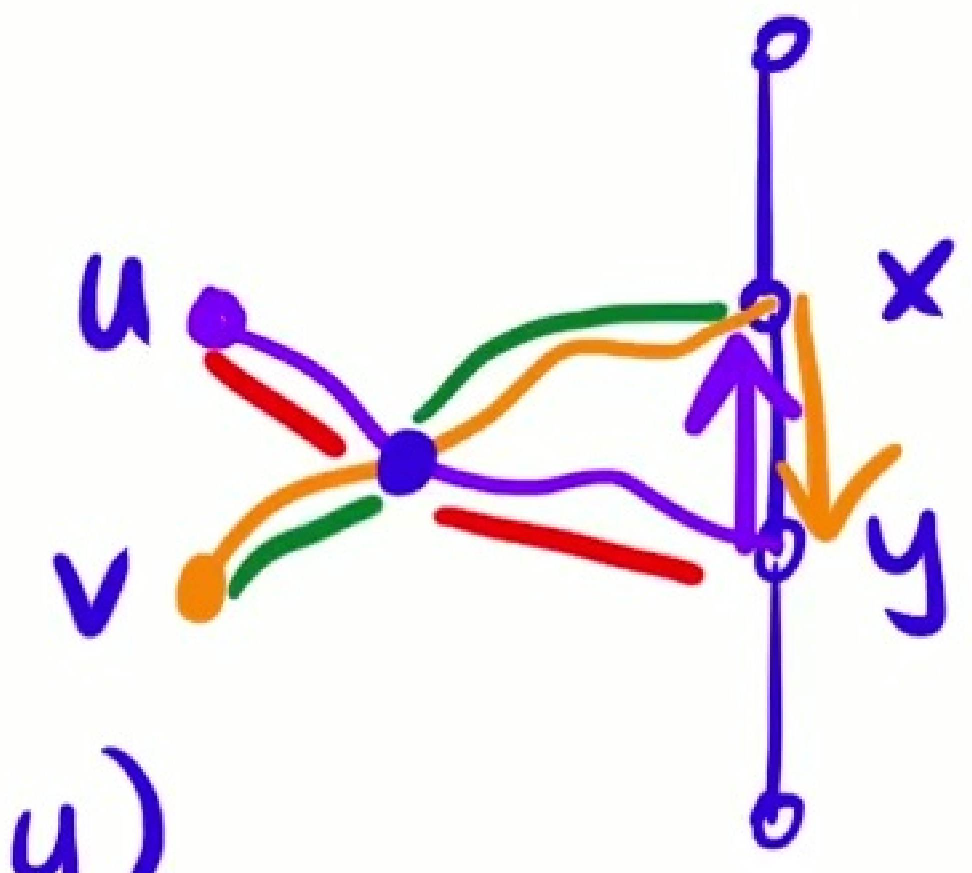

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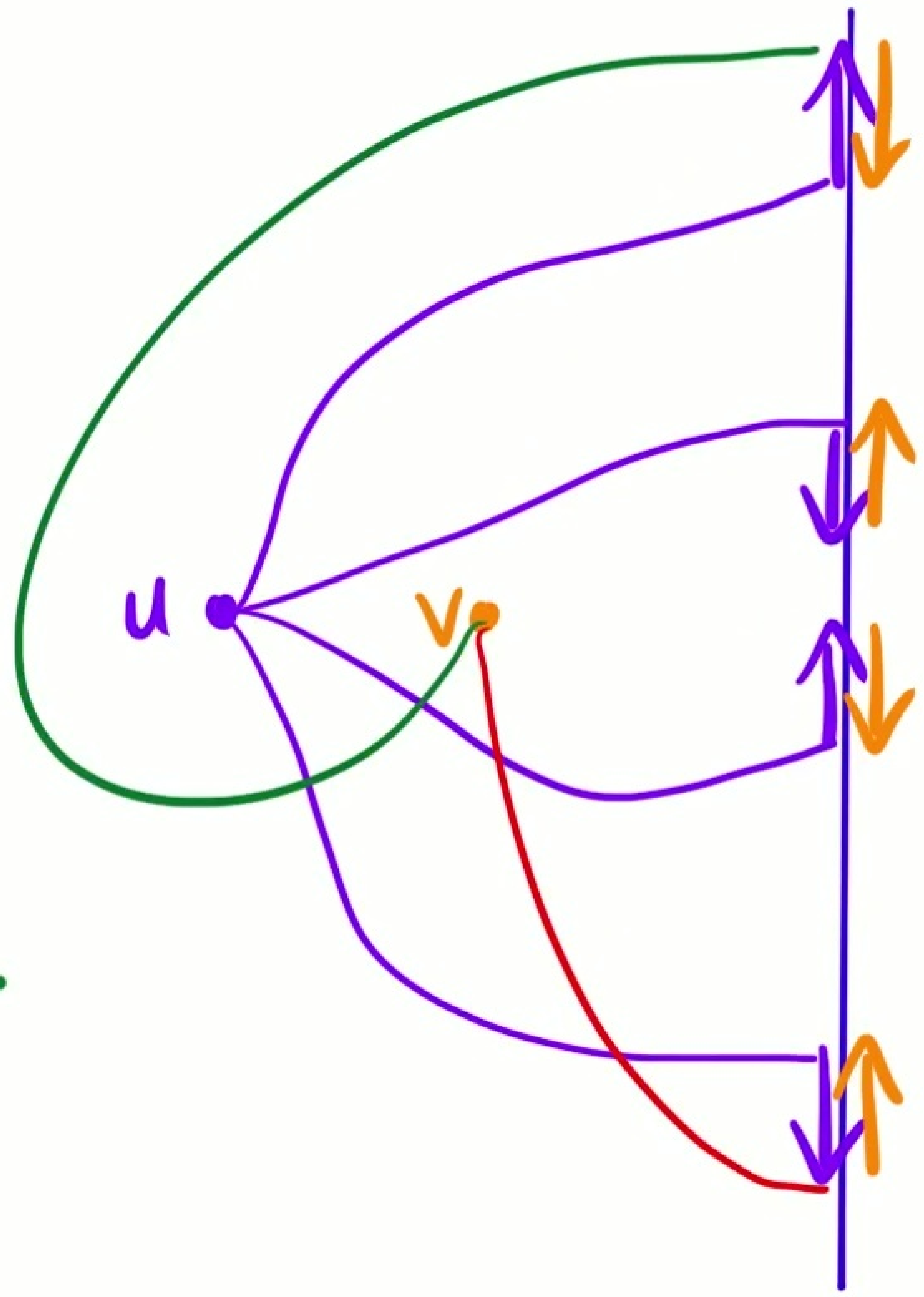

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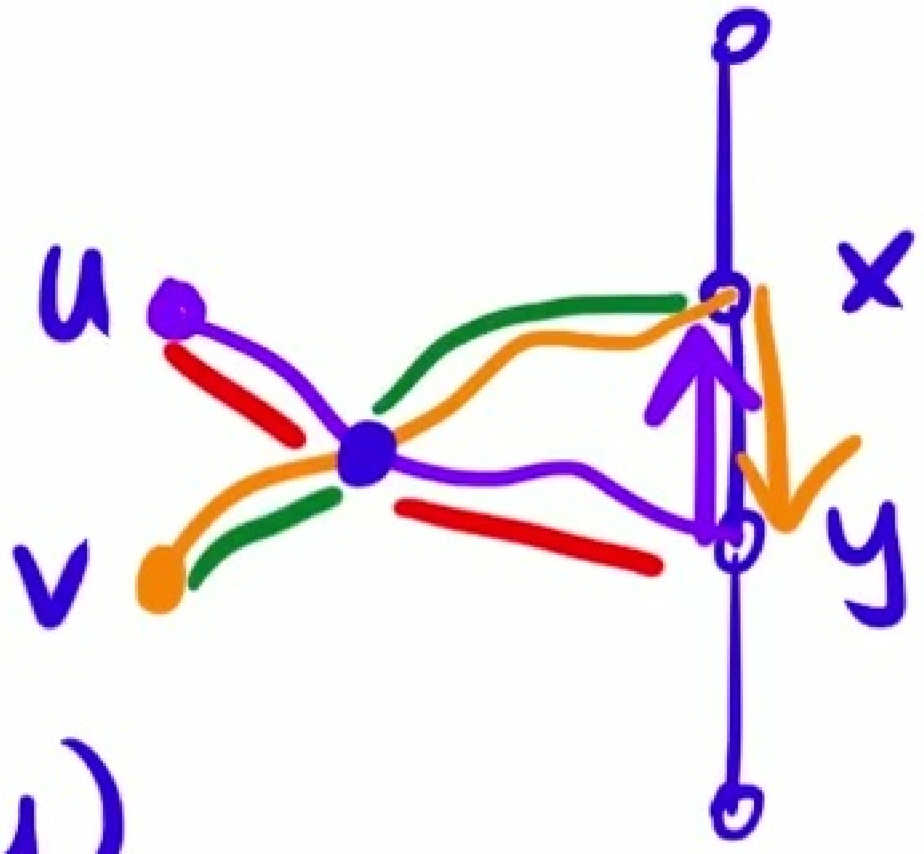

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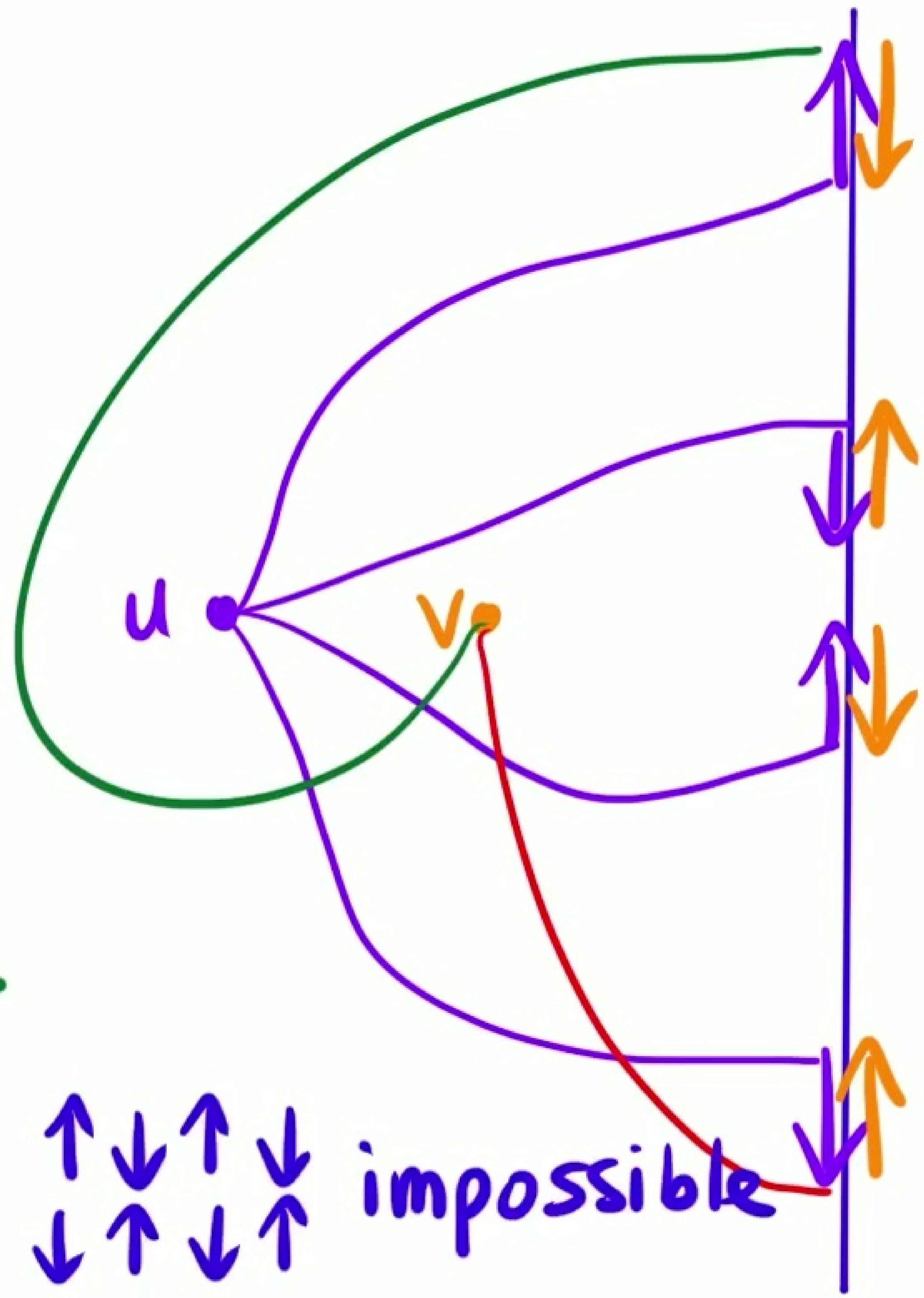

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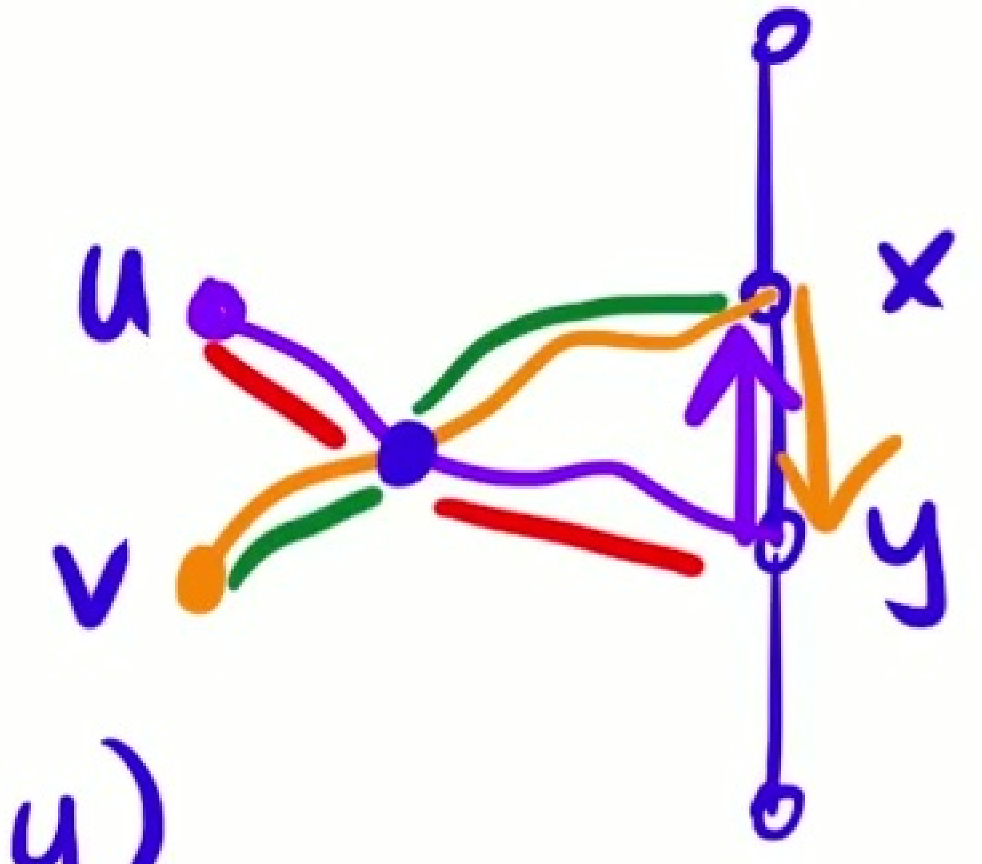

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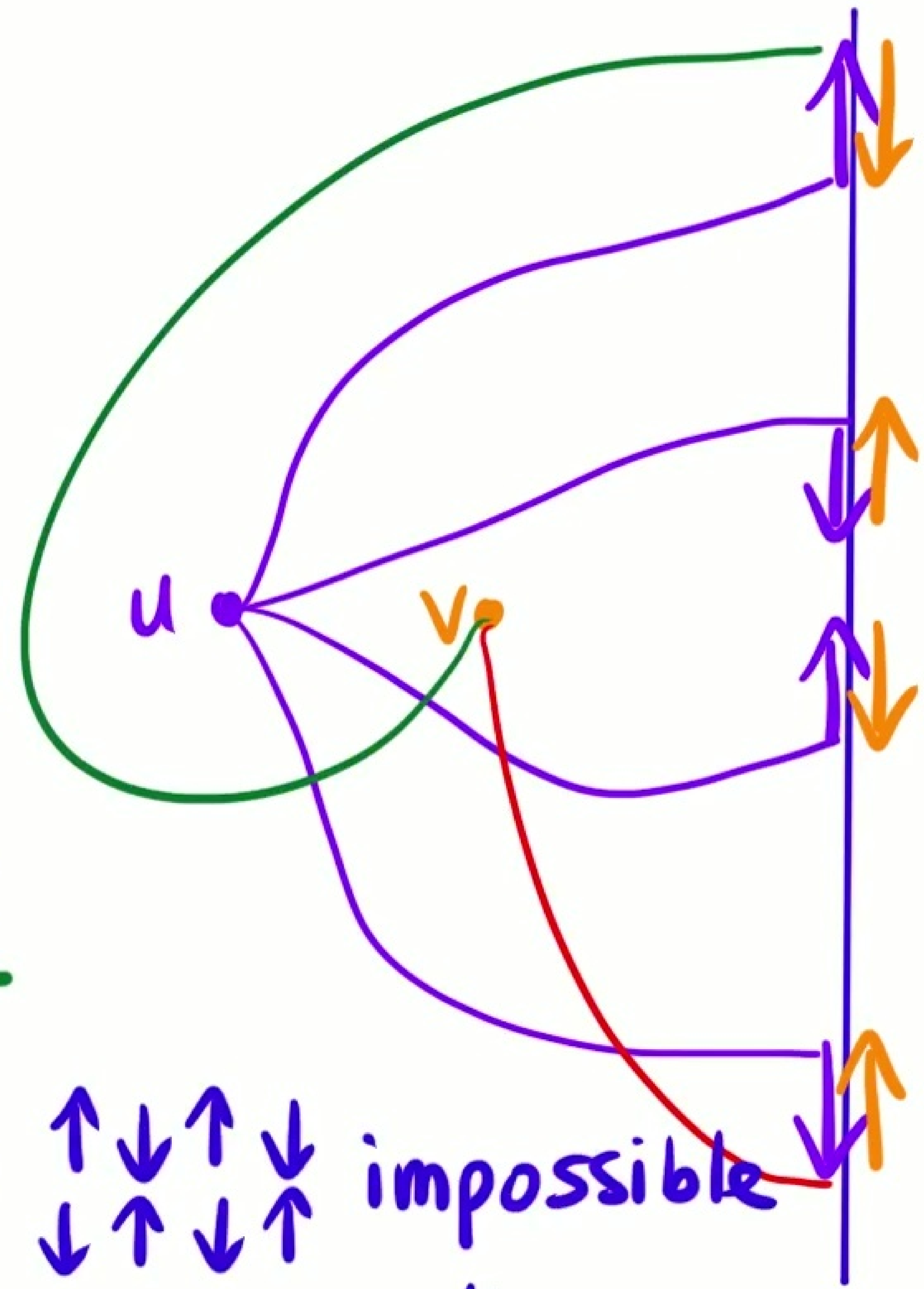

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$\Rightarrow$ VC dim < 4

$\Rightarrow |\mathcal{L}| \le O(Dk^3) = O(D^4)$.

Future directions?

Optimal bound for $|\mathcal{L}|$?

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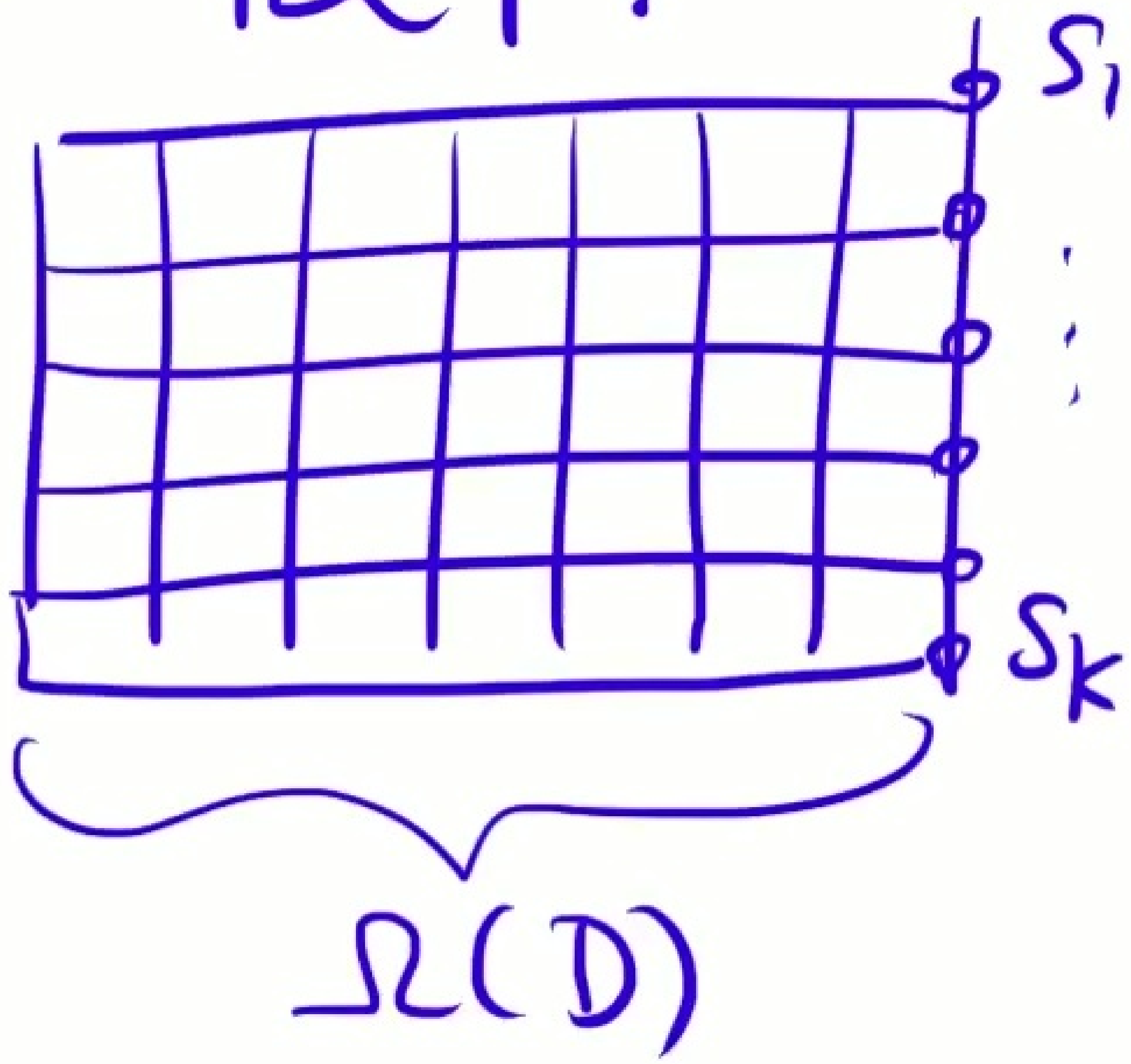
$|\mathcal{L}| = \Omega(kD)$:



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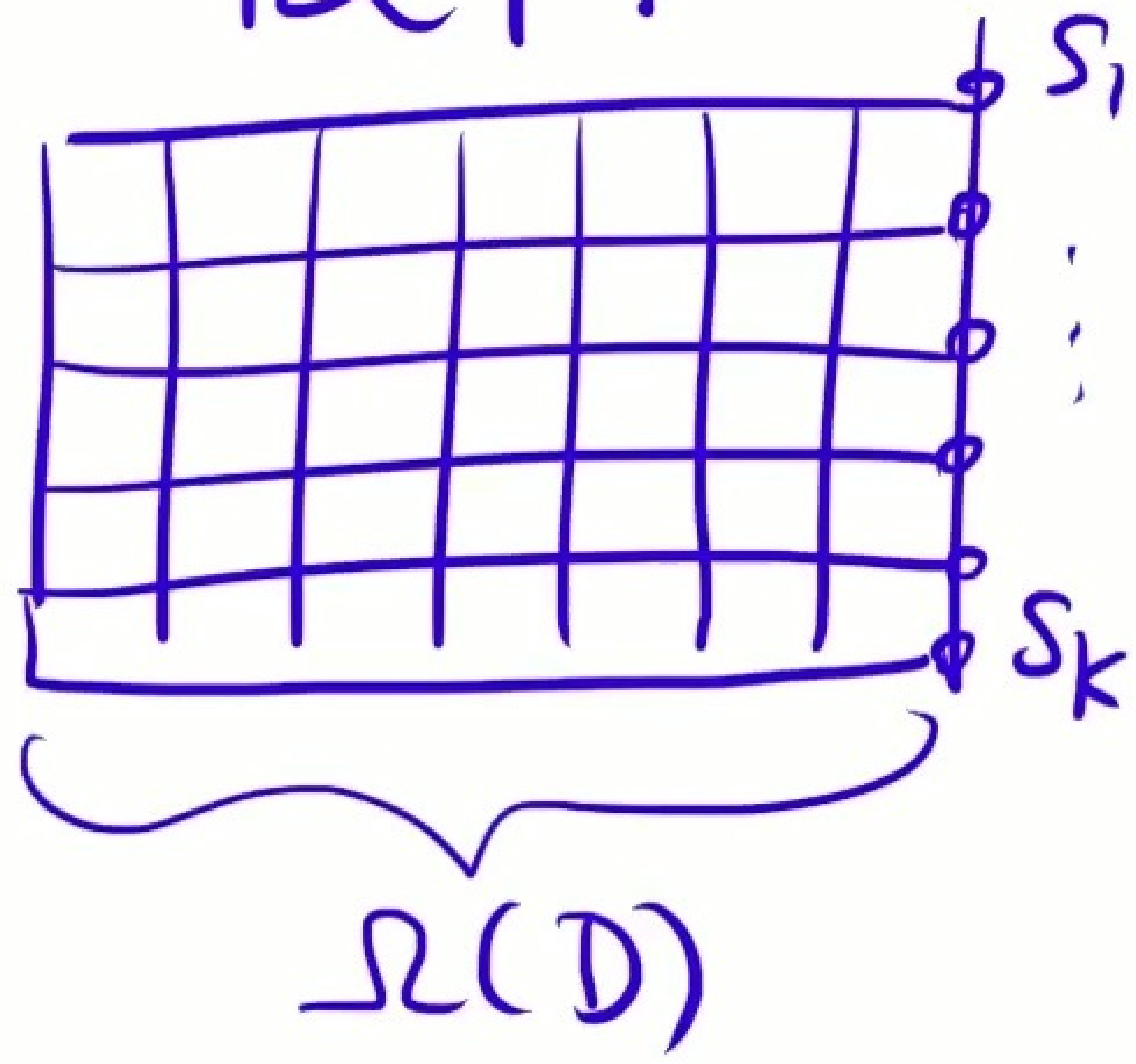

$|\mathcal{L}| = O(k^3 D)$

Bounds for bounded genus? Minor-free?

- $O(k^{O(g)}D)$ seems possible with same techniques