The Connectivity Threshold for Dense Graphs

Jason Li (CMU)

Joint work with Anupam Gupta (CMU),
Euiwoong Lee (UMichigan)

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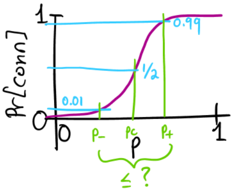

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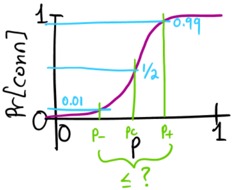

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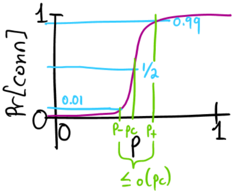

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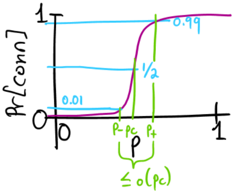

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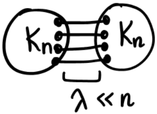
- Threshold width of $O(\frac{\ln \lambda}{\lambda})$ for general graphs
  - $p_c, p_+, p_- = \frac{\ln \beta}{\lambda} \pm O(\frac{\ln \lambda}{\lambda})$ for a parameter β based on the <u>number of approximate mincuts of the graph</u>

Example graphs

CT and Approx. Mincuts

Example graphs

- Barbell graph



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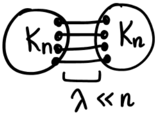

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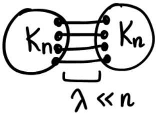

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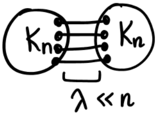

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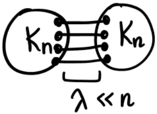

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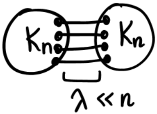

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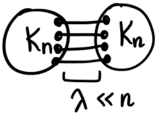

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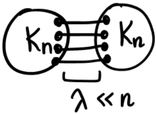

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CT and Approx. Mincuts

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- Proof: standard union bound over all α-mincuts and all α

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- Proof inspired by Karger-Stein's randomized contraction algorithm

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- So $p_+, p_- = 1 - \exp\left(\frac{\beta \pm O(\ln\lceil\beta\rceil)}{\lambda}\right)$. Some more work to get additive $\pm O\left((\ln\lambda)/\lambda\right)$

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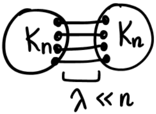

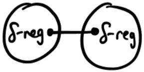

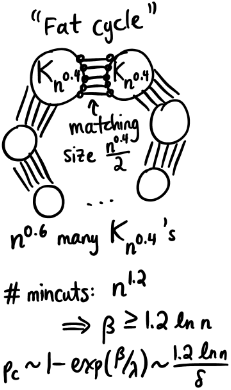

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  - Structural representation of all α-mincuts based on decomp.

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- Tight connectivity threshold for all δ-regular, δ-connected simple graphs for $\delta \gtrsim \sqrt{n}$: $p_c, p_+, p_- = \frac{\ln n}{\delta} \pm O(\frac{1}{\delta})$
- All three conditions (δ-regular, δ-connected, $\delta \geq \sqrt{n}$) are necessary!
- Main tool: Kawarabayashi-Thorup graph decomposition [KT'18]
  - Structural representation of all α-mincuts based on decomp.
- Thm: for any simple graph with minimum degree $\geq \sqrt{n}$, the number of α-mincuts is at most $\tilde{O}(n)^{\alpha}$ for all $\alpha \geq 1$

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- For $p = \frac{\ln n + c}{\delta}$, $\Pr[\text{conn}] = e^{-e^{-c}} \pm o(1)$, generalizing Erdos-Renyi for the complete graph

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  - Related to the network (un)reliability problem: given p, compute the approximate probability of connectivity