The Number of Minimum k-cuts: Beating the Karger-Stein Bound

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Joint work with Anupam Gupta (CMU), Euiwoong Lee (NYU)

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Introduction

minimum k-cut: delete min weight edges to cut graph into $\geq k$ connect components

Setting: exact algorithm, k constant

Q: How fast to compute min k-cut?

Q: How <u>many</u> min k-cuts are there?

Prior Work

Goldschmidt–Hochbaum 1994: $O(n^{(1/2-o(1))k^2})$ time deterministic

Karger–Stein 1994: $\tilde{O}(n^{2(k-1)})$ time randomized

Thorup 2008: $\tilde{O}(mn^{2k-2})$ time deterministic

Chekuri et al. 2018: $\tilde{O}(mn^{2k-3})$ time deterministic

This work: $O_k(n^{(1.981+o(1))k})$ time randomized

All of these algorithms can <u>enumerate</u> all min k-cuts

$\Longrightarrow$ get corresponding extremal bound

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Lower bound: as hard as min weight k-clique: $\Omega(n^{(1-o(1))k})$

Our Approach

- Combines Karger-Stein's randomized contraction with Thorup's tree packing algorithm
- Makes new connection to extremal set theory in context of graph cut algorithms

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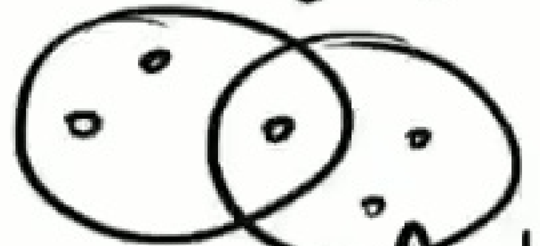

"If $|\mathcal{A}| \geq 2n$, then $\mathcal{A}$ has dual VC dimension ≥ 2"

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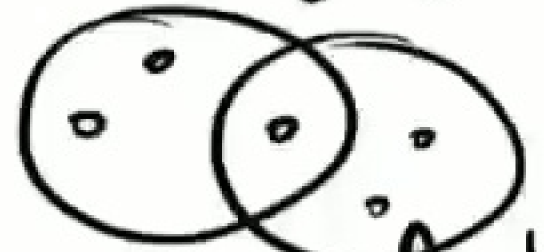
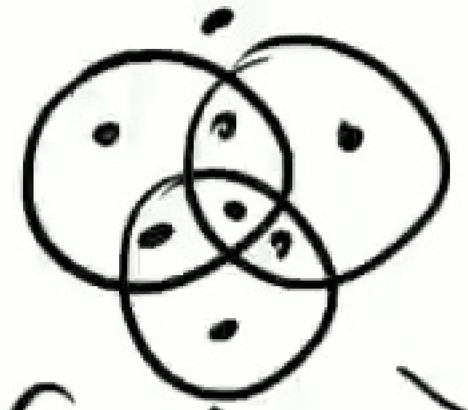
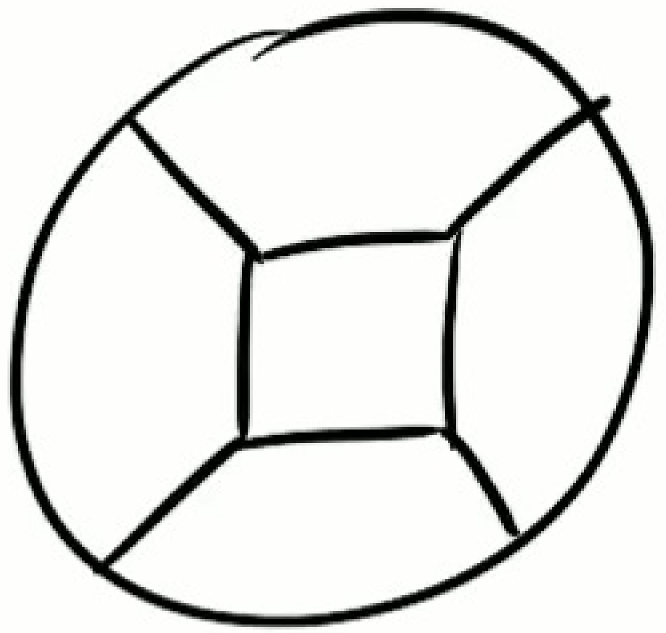
[Folklore] If $\mathcal{A}$ is a collection of distinct sets on $[n]$ (a set system), and $|\mathcal{A}| \geq 2n$, then $\exists A, B \in \mathcal{A}$ that cross: 

"If $|\mathcal{A}| \geq 2n$, then $\mathcal{A}$ has dual VC dimension ≥ 2"

[This work] If $|\mathcal{A}| \geq \Omega(n^{3.75})$, then $\exists A, B, C \in \mathcal{A}$: 

"If $|\mathcal{A}| \geq \Omega(n^{3.75})$, then $\mathcal{A}$ has dual VC dim ≥ 3"

Branch and Bound



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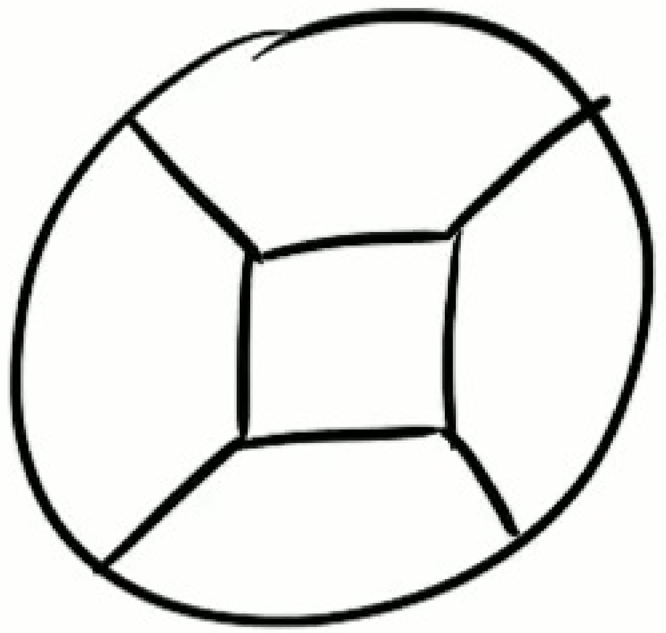

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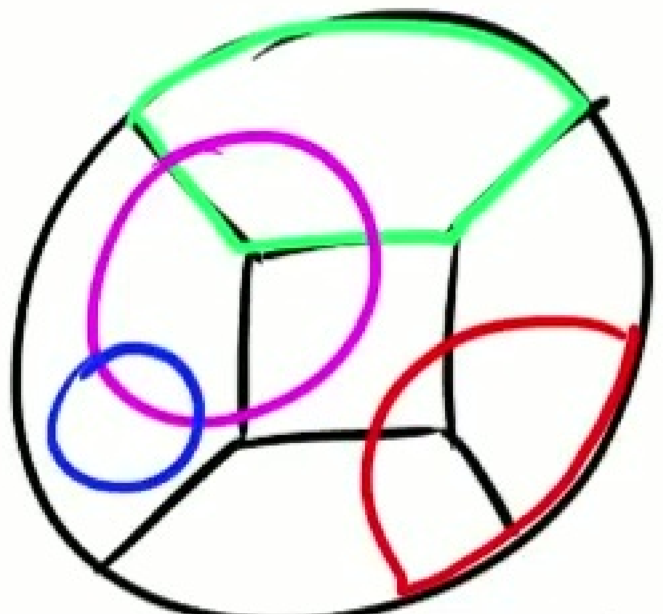
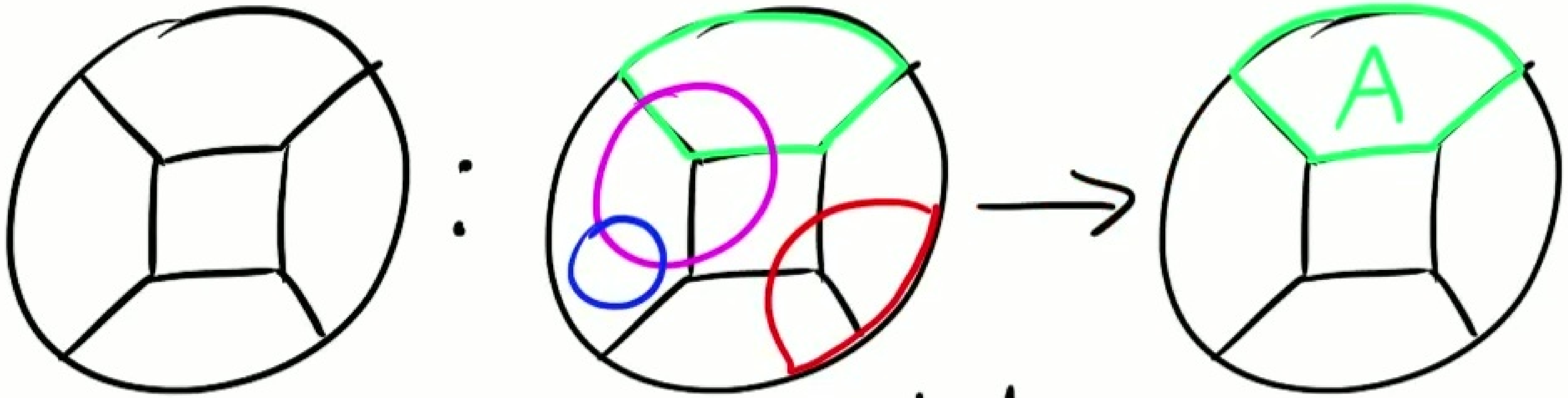


Guess a set $\mathcal{A}$ of subsets of V

Branch and Bound



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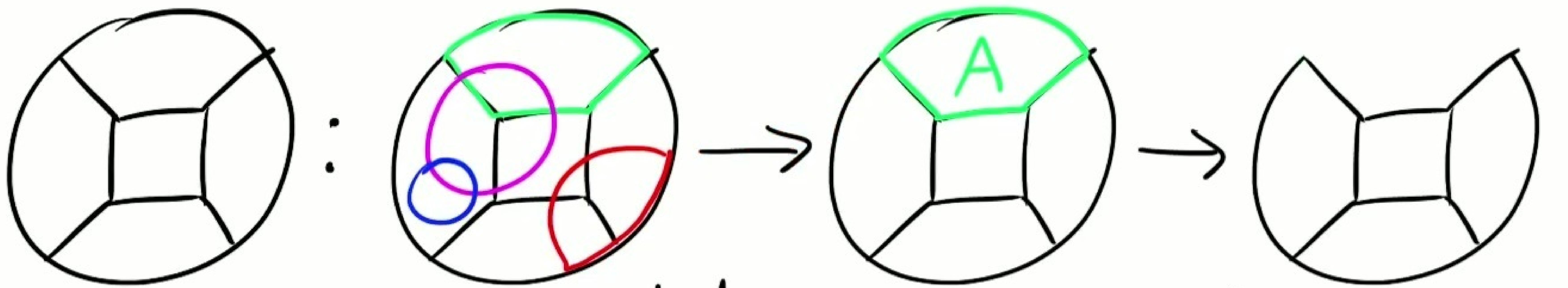

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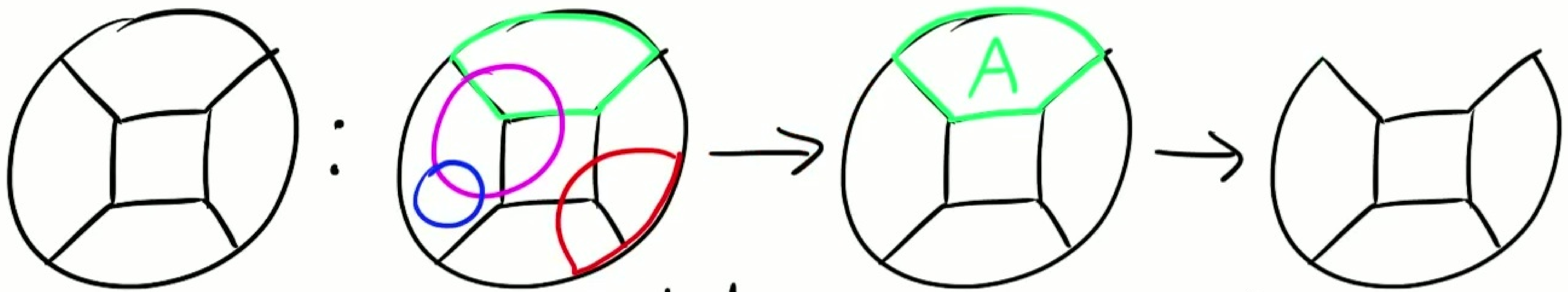

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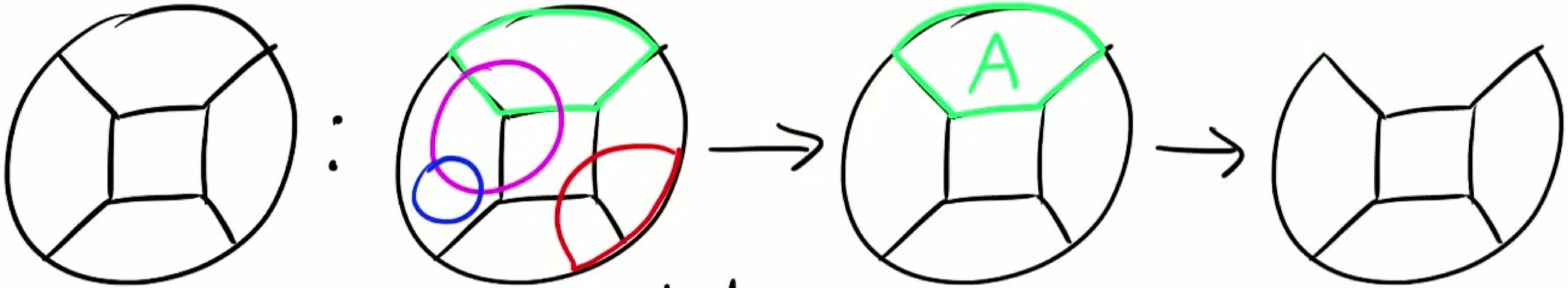

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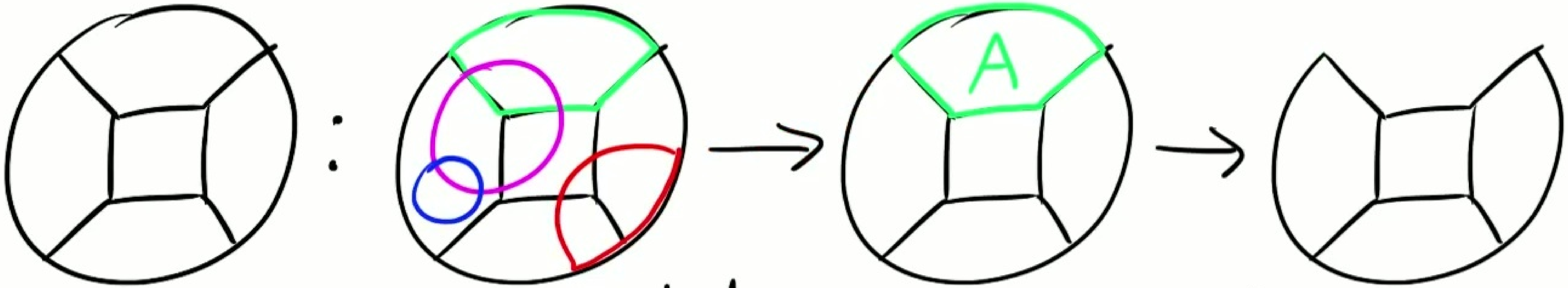

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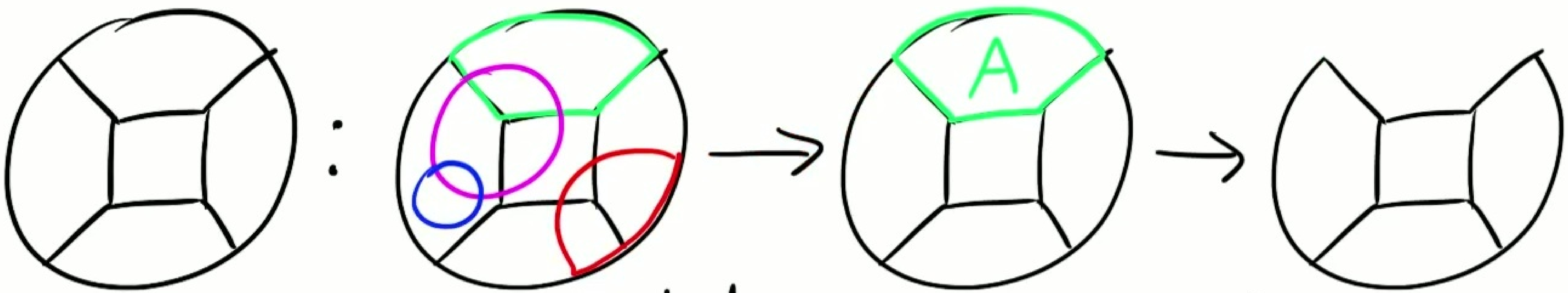

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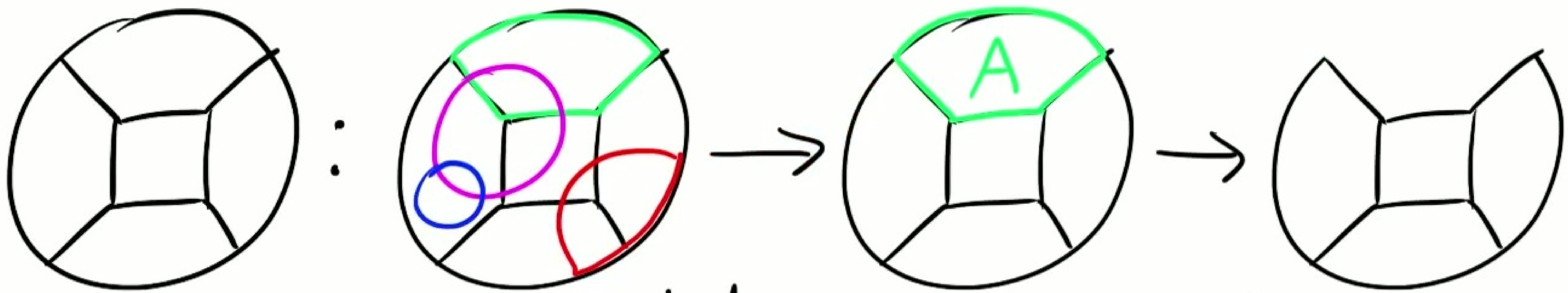

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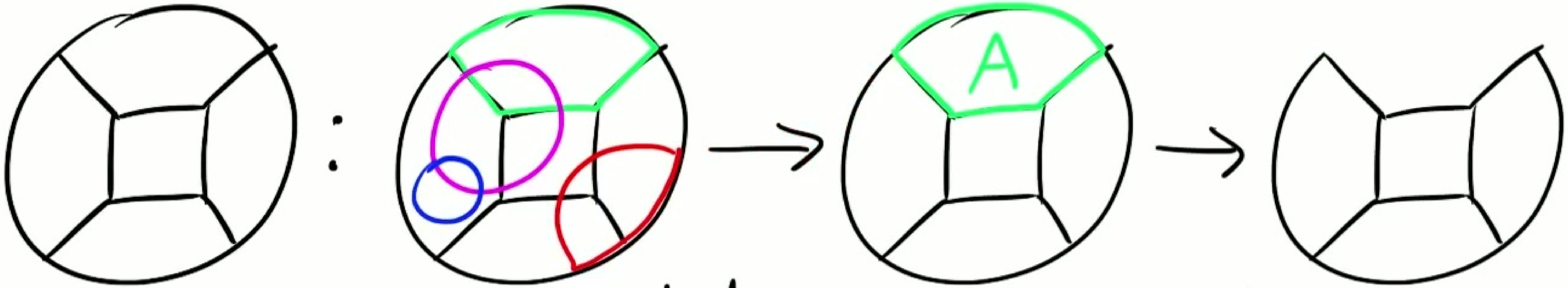

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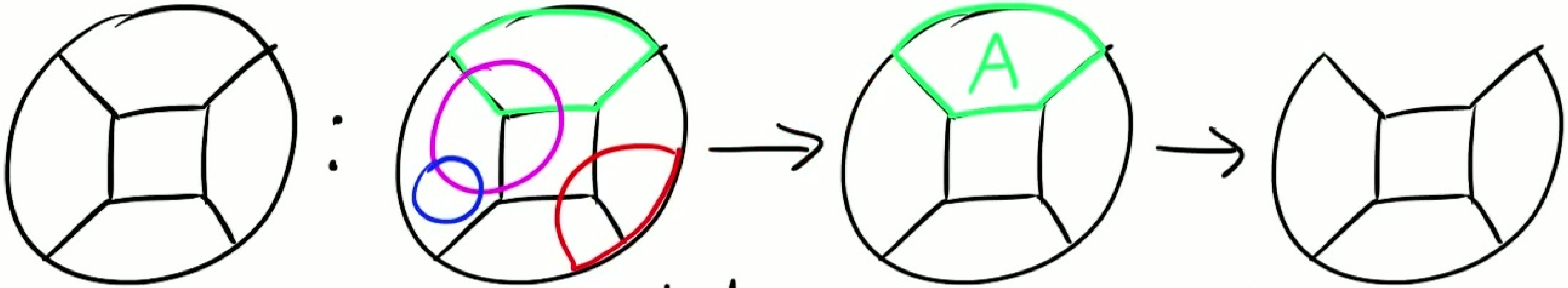

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"much smaller than average"

Let $\mathcal{A}' := \left\{A \subseteq V : w(\partial_G A) \le \frac{1.49}{k} OPT\right\}$

Note: average $S \in \mathcal{S}$ is $\frac{2}{k} OPT$, since $\sum_{S \in \mathcal{S}} w(\partial_G S) = 2OPT$.

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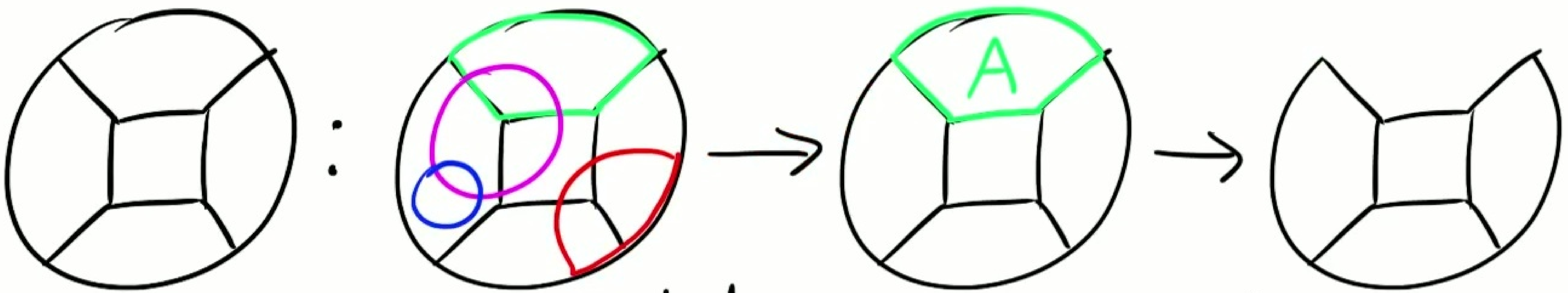

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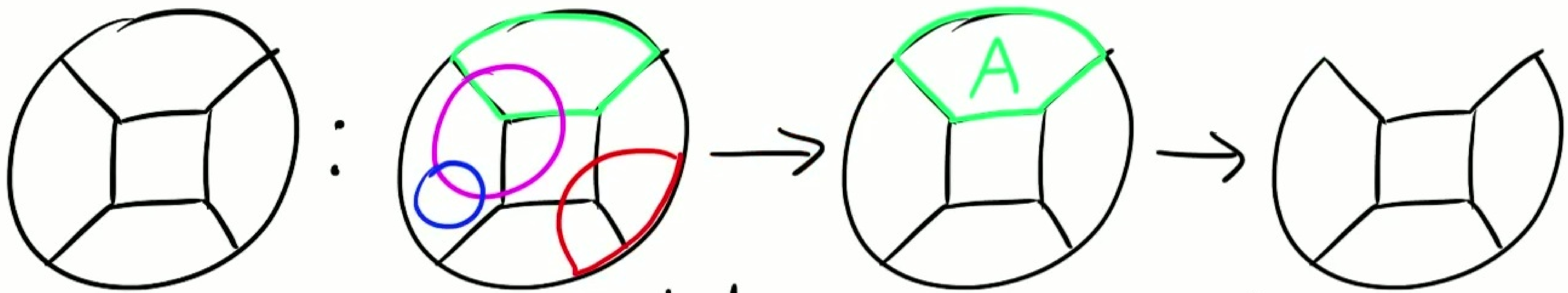

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Claim (bound):

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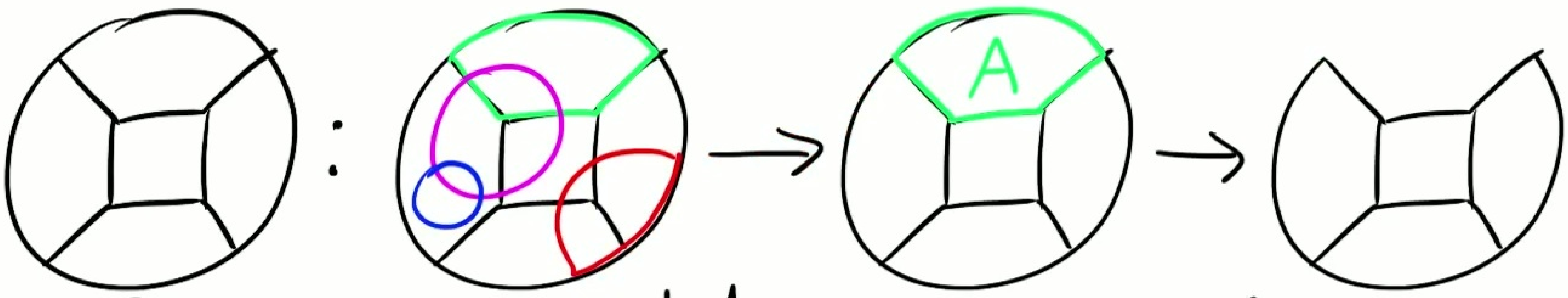

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Claim (bound): $|\mathcal{A}'| = O_k(n)$.

Claim (time): $\mathcal{A}'$ can be computed in poly(n).

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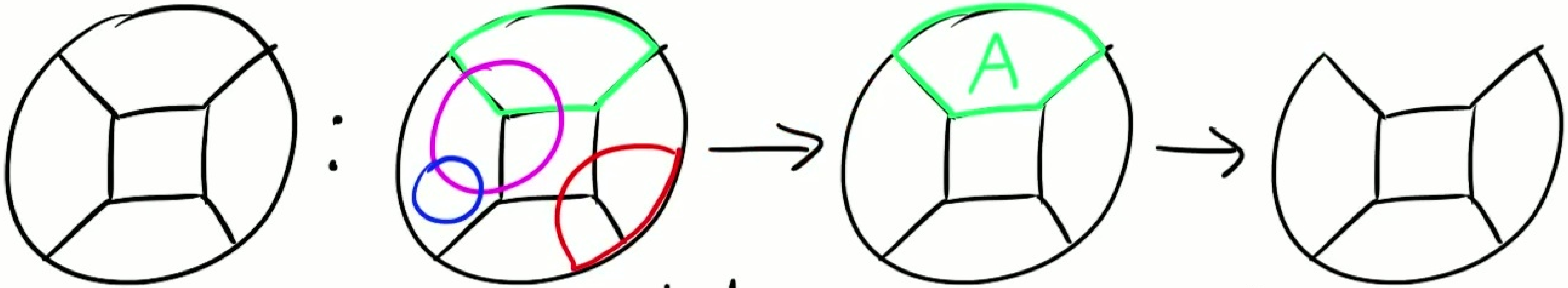

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  $+ |\mathcal{A}| \cdot T(n, k-1)$ n

  $\Rightarrow O_k(n^{k+O(1)})$!

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Extremal Bound

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$\to |\mathcal{A}'| = O_k(n)$

$\mathcal{A}'$ computed in polytime

- Strategy: if $|\mathcal{A}'| \gg n$, i.e. too many cheap cuts, then can construct a k-cut solution $<$ OPT, contradiction.

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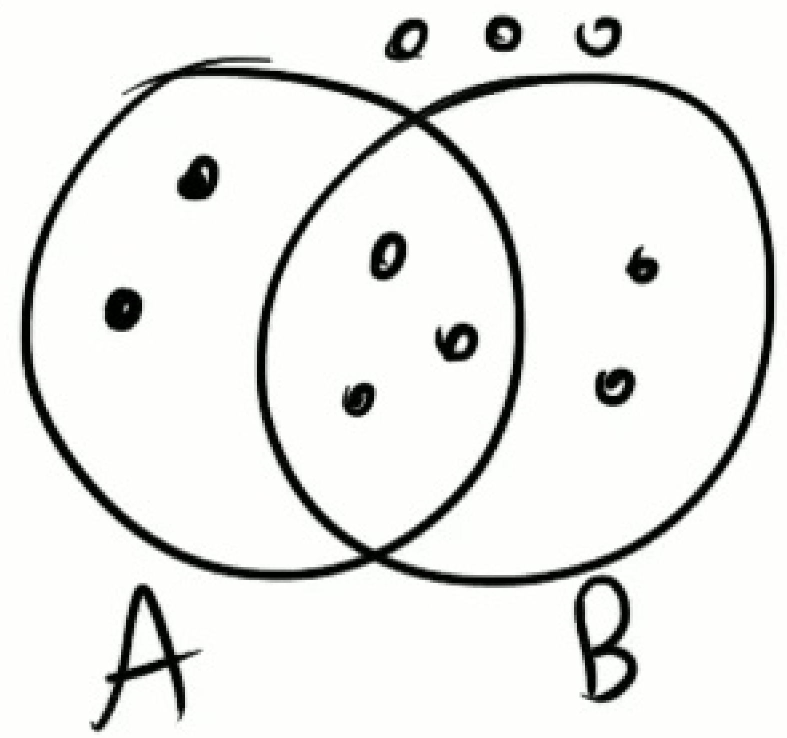
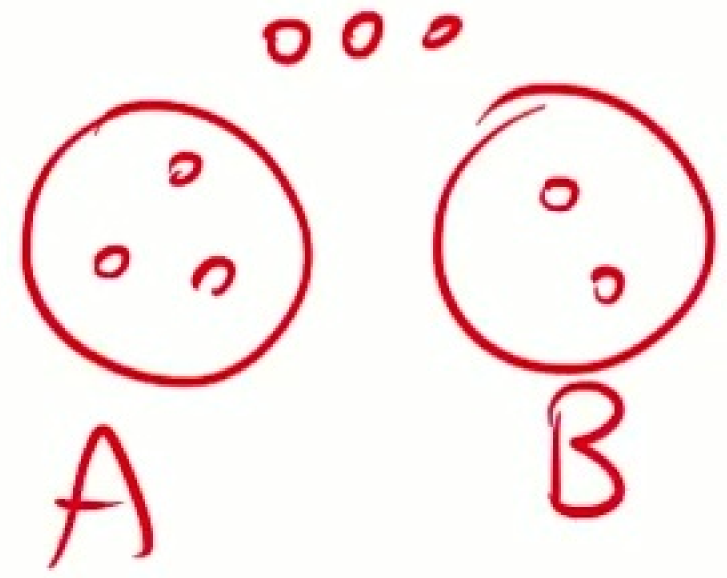


don't cross:



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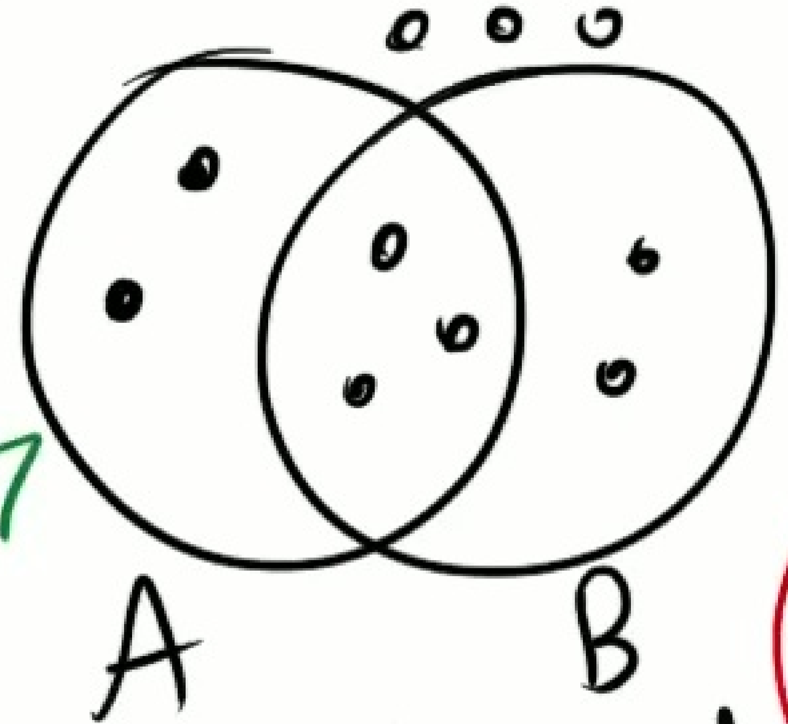

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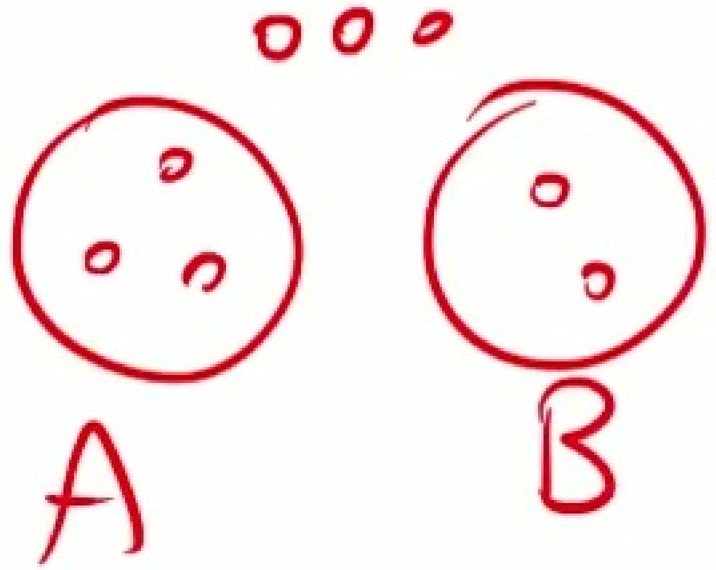

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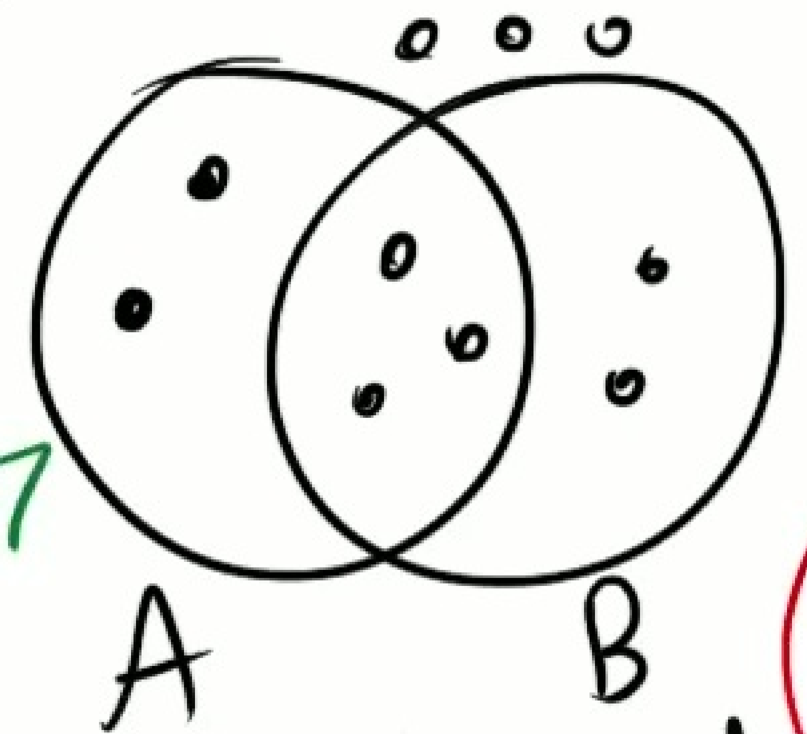

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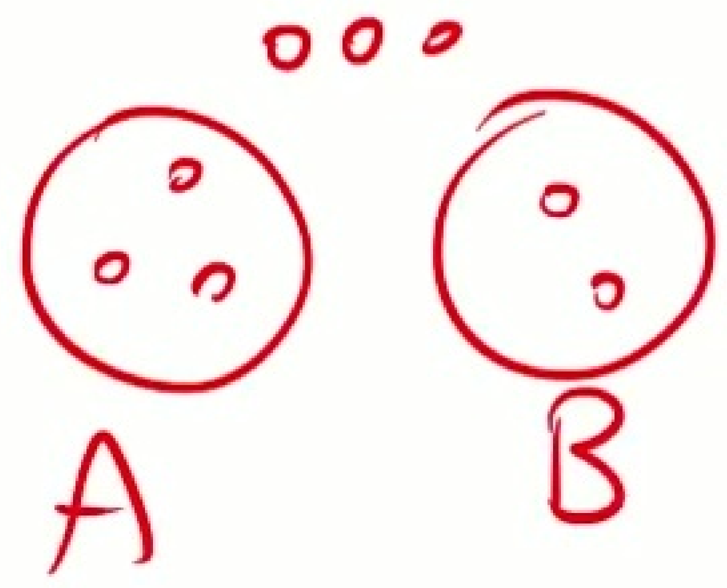

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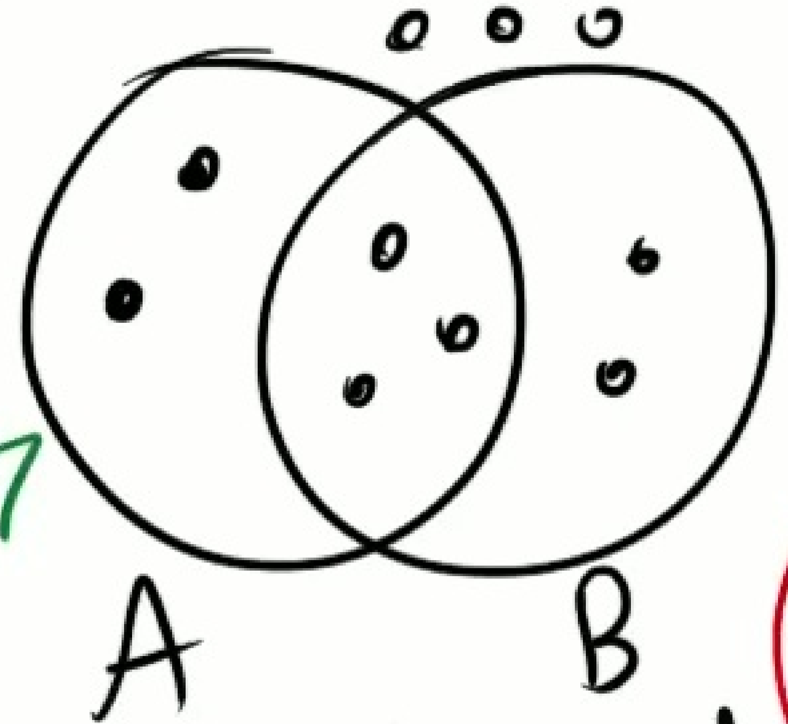

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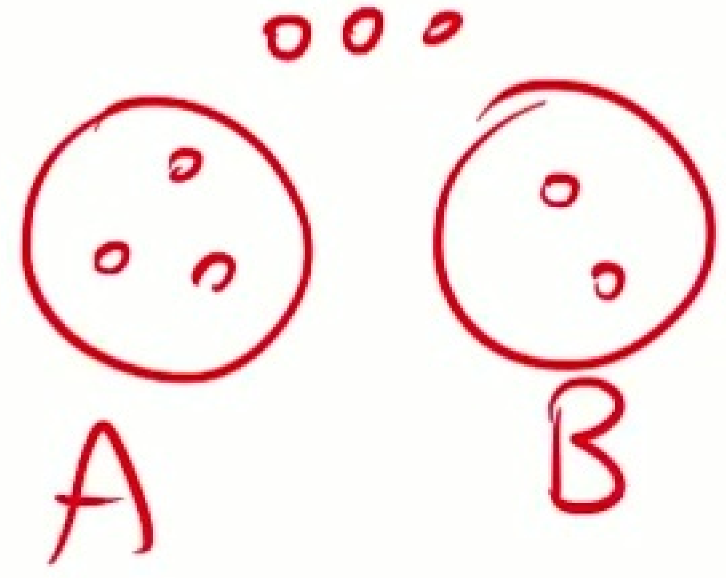

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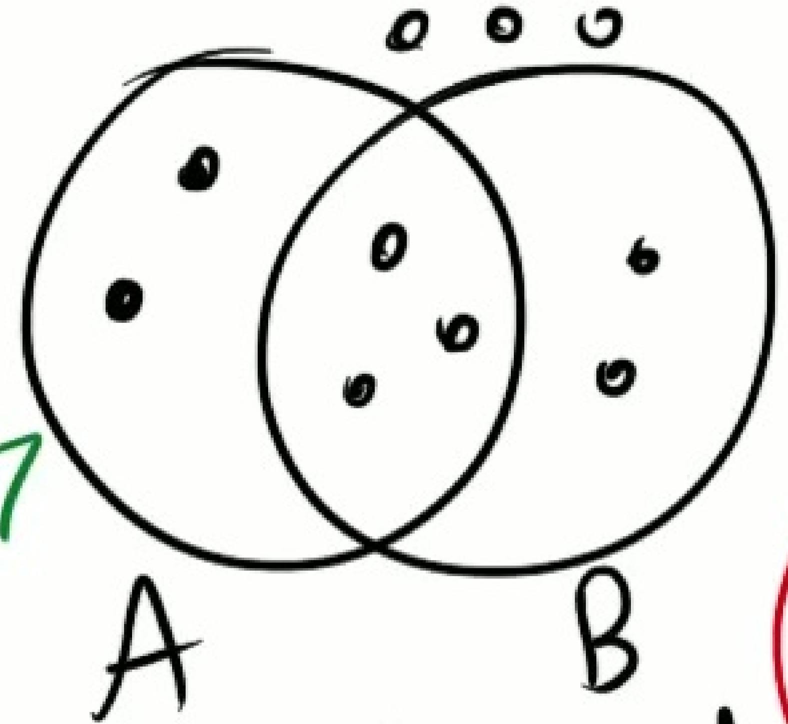

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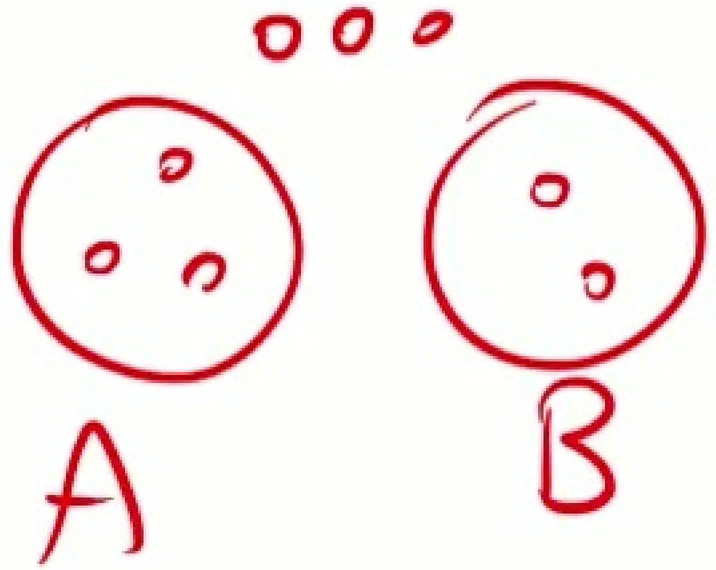

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$$\Longrightarrow \frac{1-\varepsilon}{k} OPT \text{ price per } +1 \text{ component.}$$

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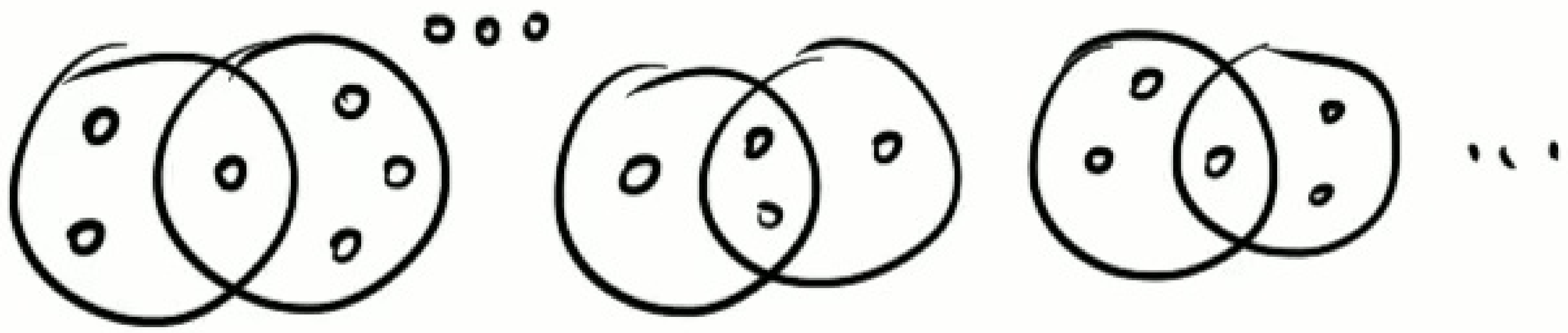

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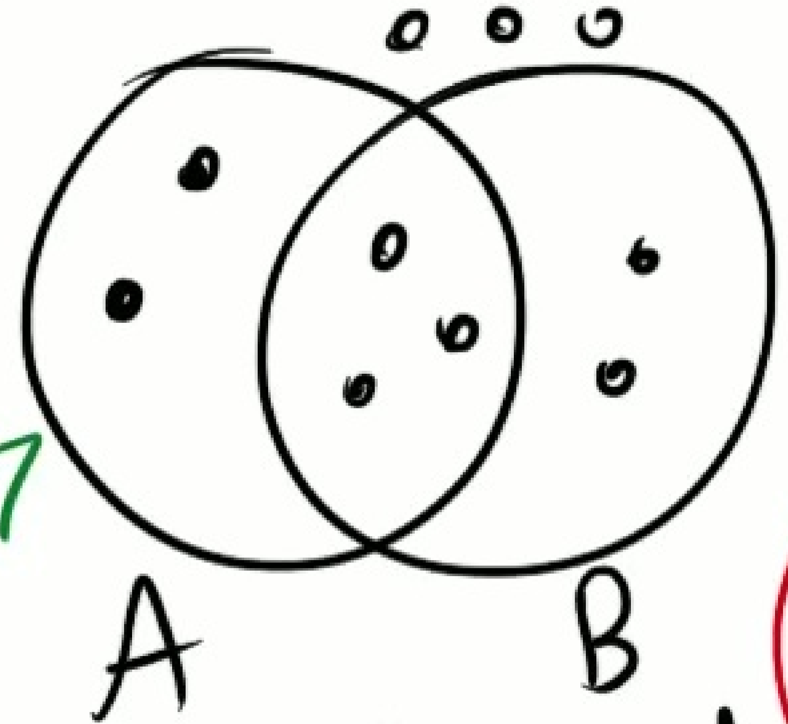

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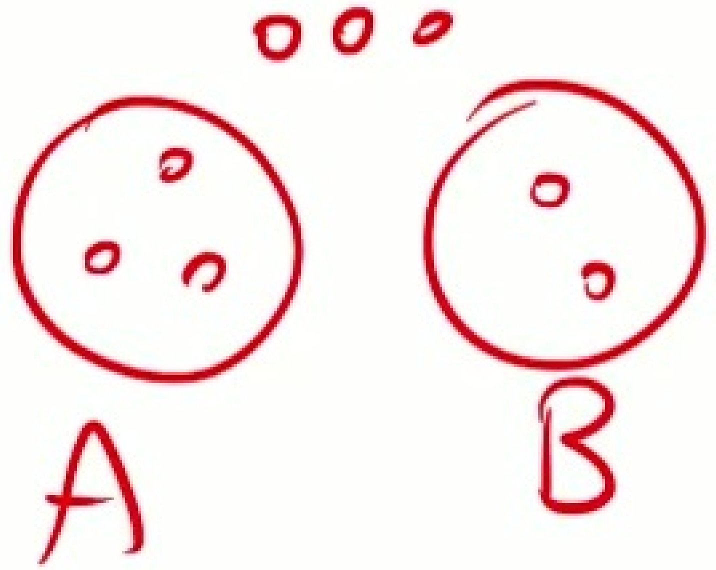

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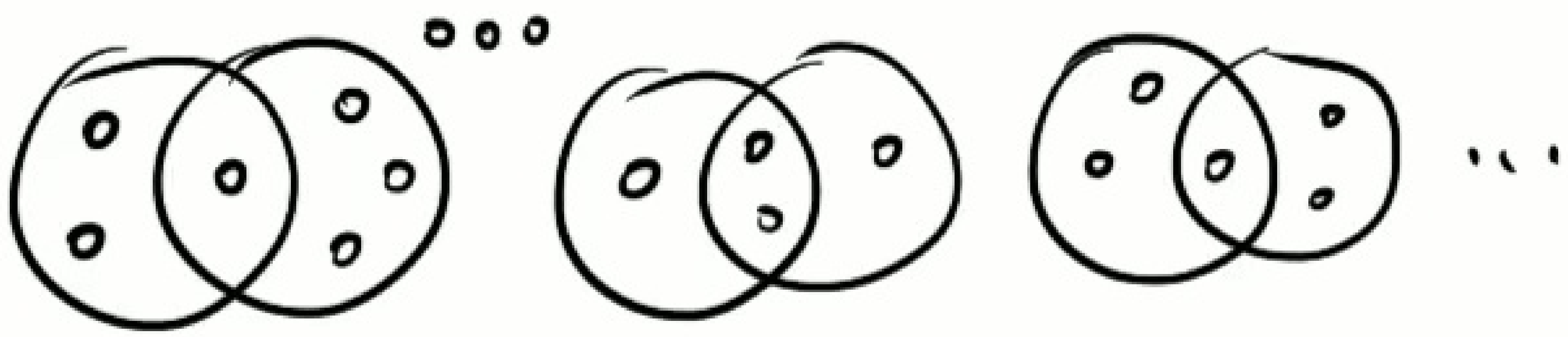
- Think of it as +3 additional components for price of $\le 2 \cdot \frac{1.49}{k} OPT$

  $\Longrightarrow \frac{1-\varepsilon}{k} OPT$ price per +1 component.
- Repeat until k comps?



pay $\lceil \frac{k}{3} \rceil \cdot 2 \cdot \frac{1.49}{k} OPT < OPT$

for k large enough, contradiction.

Computing $\mathcal{A}'$

- Idea: modified Karger-Stein algorithm.

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$$\Longrightarrow \Pr[\text{success}] = \left(1 - \frac{2.98}{n}\right)\left(1 - \frac{2.98}{n-1}\right)\cdots \gtrsim \frac{1}{n^{2.98}}$$

- Repeat $\Theta(n^{2.98} \log n)$ times. Output the $\Theta(2^k n)$ sets with smallest boundaries found.

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- Progress is based on tree packing [Thorup 2008].

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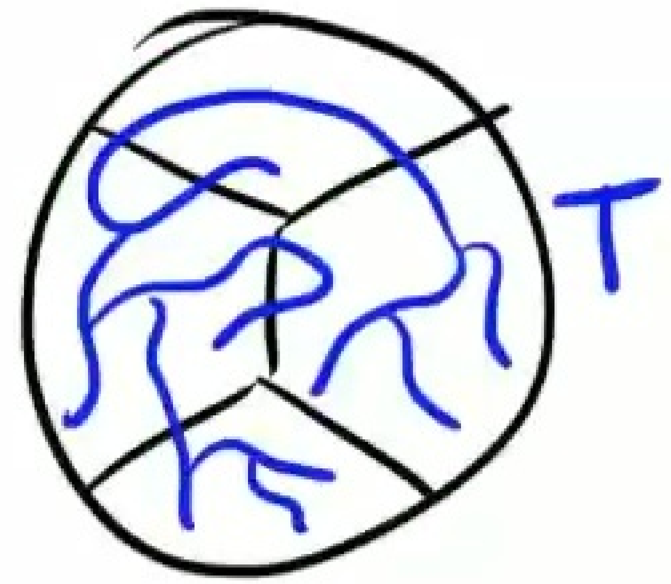
Thm [Thorup '08] Can compute in poly(n) time a set $\mathcal{T}$ of poly(n) spanning trees of G, s.t. for any min k-cut S, there exists $T \in \mathcal{T}$ that crosses S $\le 2k-2$ times.

$k=4$:



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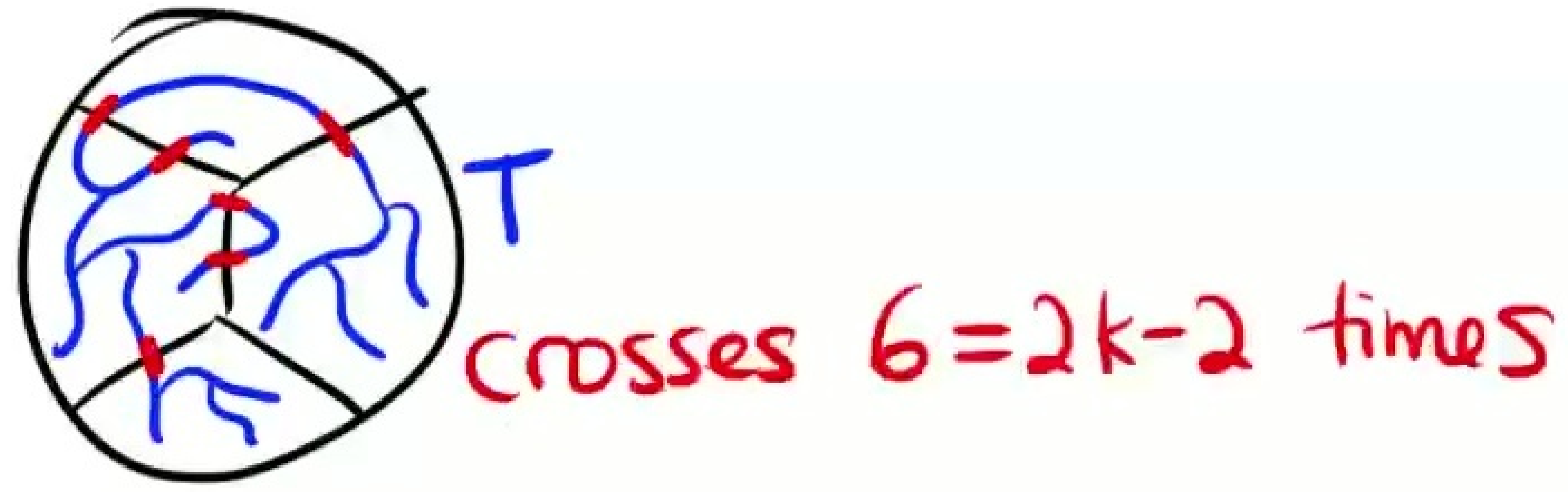
k=4: 

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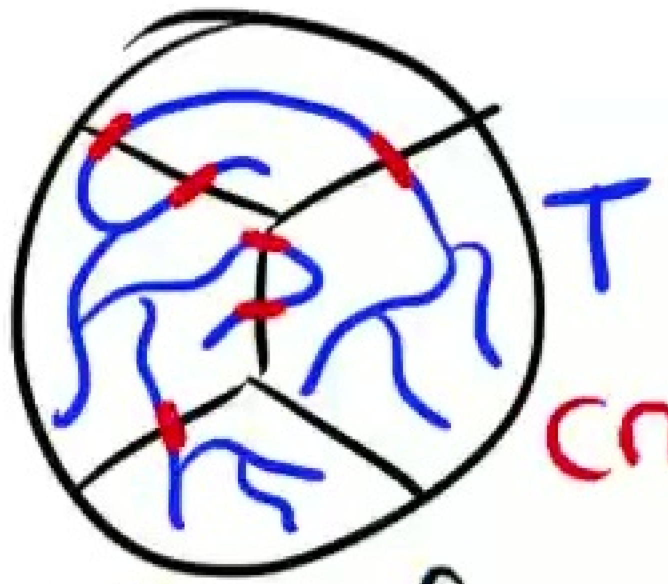
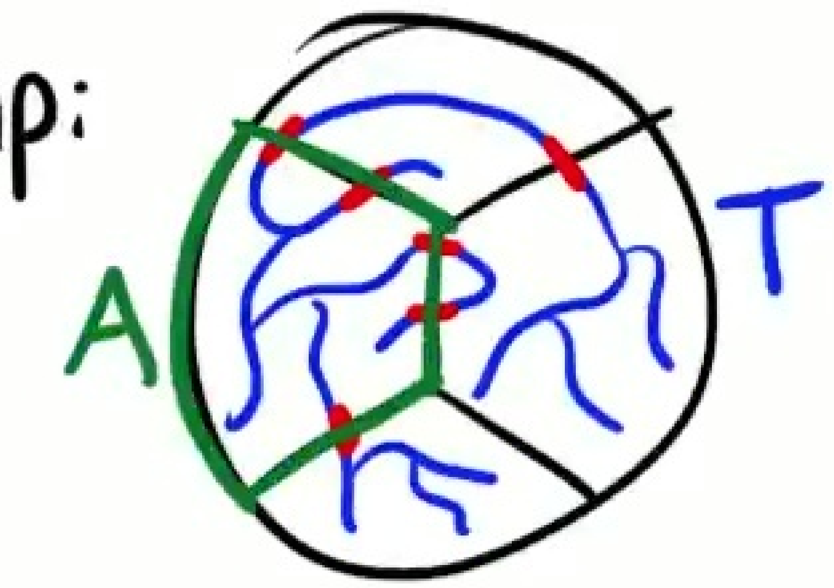


Thorup's min k-cut algo: for each $T \in \mathcal{T}$, for every way to delete $\leq 2k-2$ edges of T and merge the components into k components, compute the min k-cut value.

Tree as a Measure of Progress

- Suppose $\mathcal{A}$ contains this comp:



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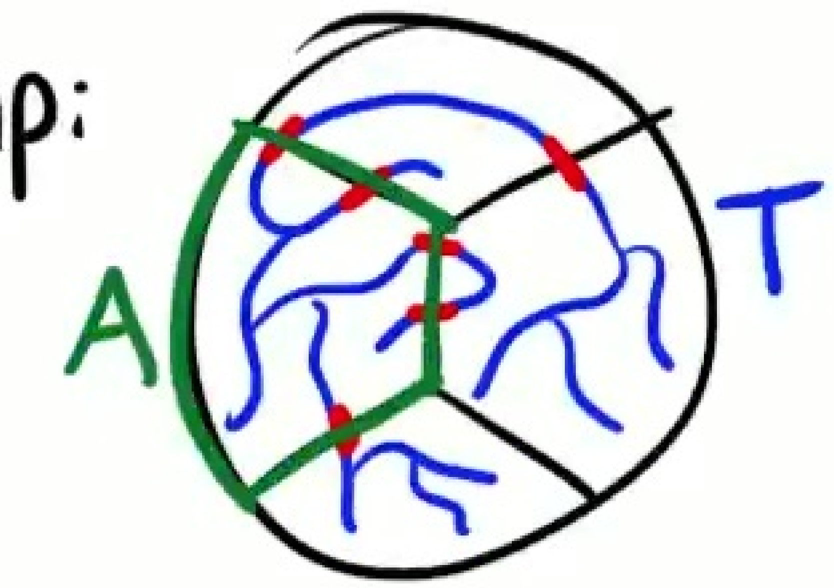

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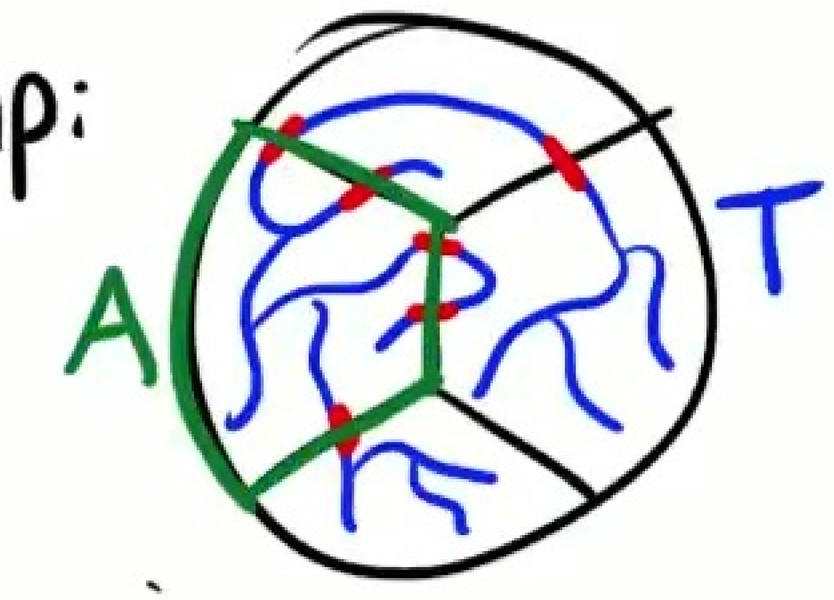

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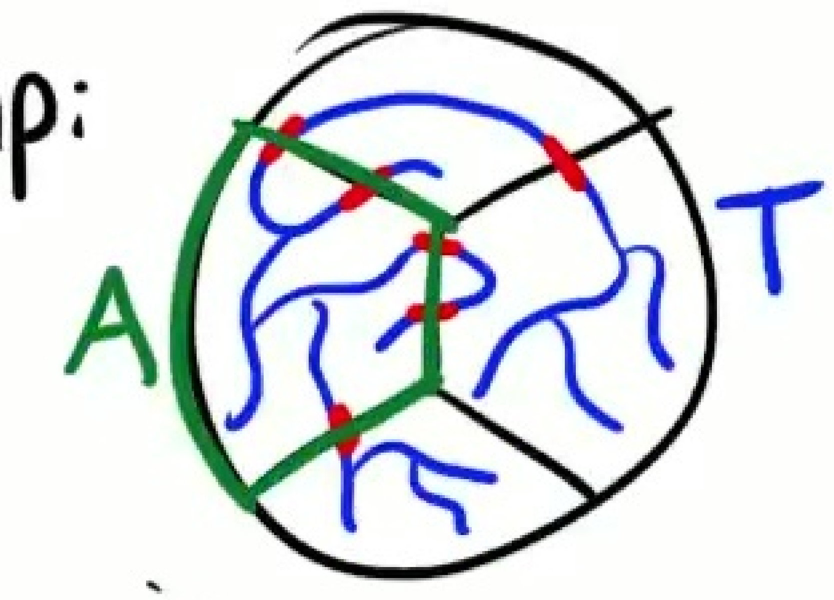

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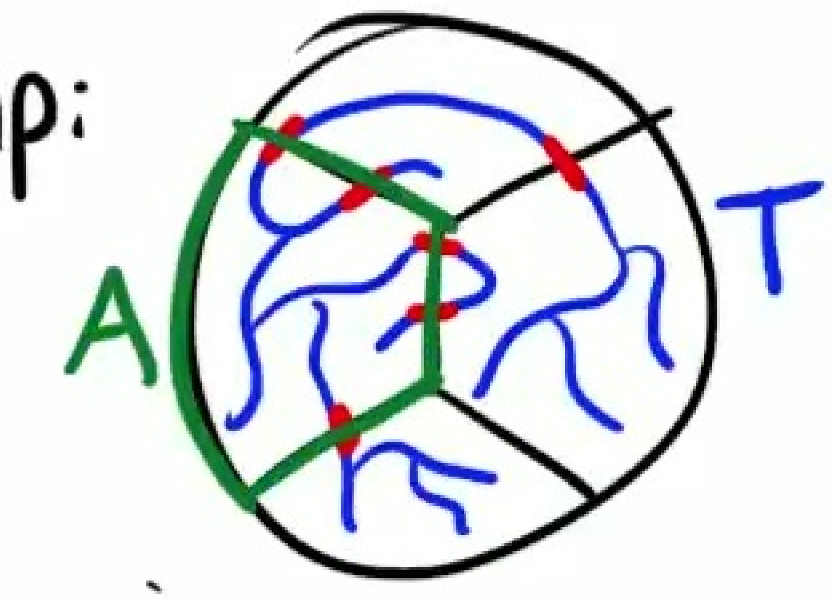

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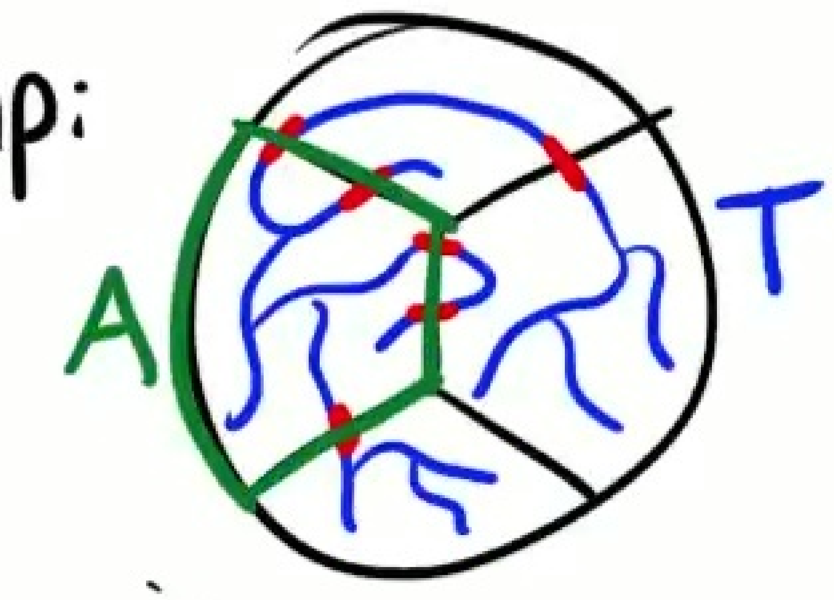

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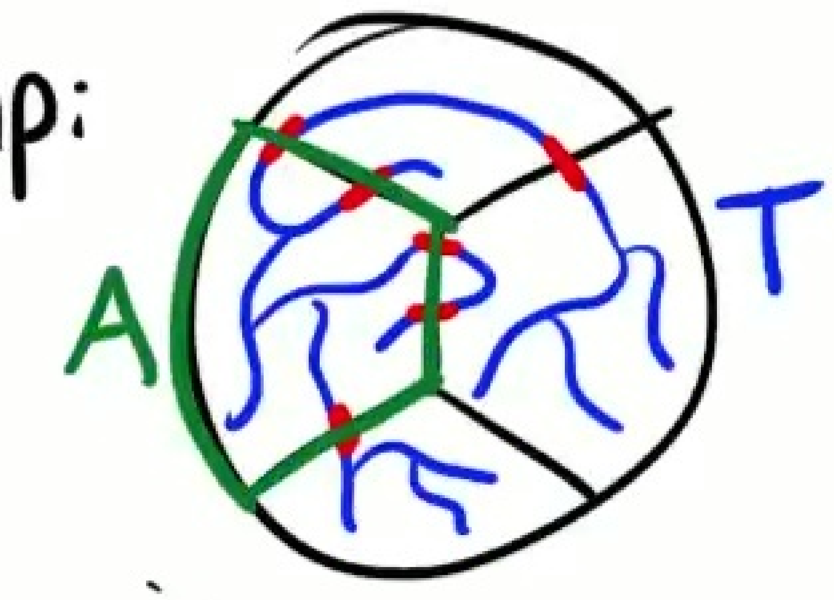

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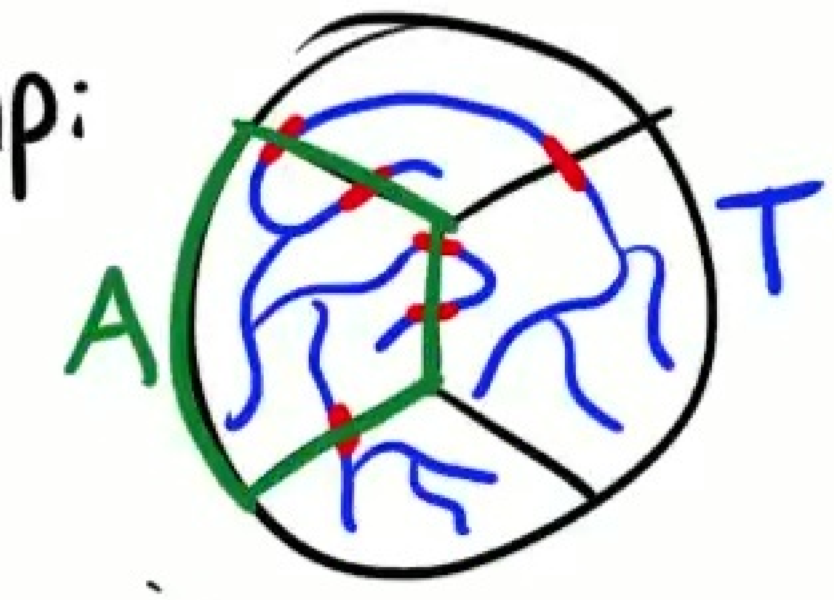

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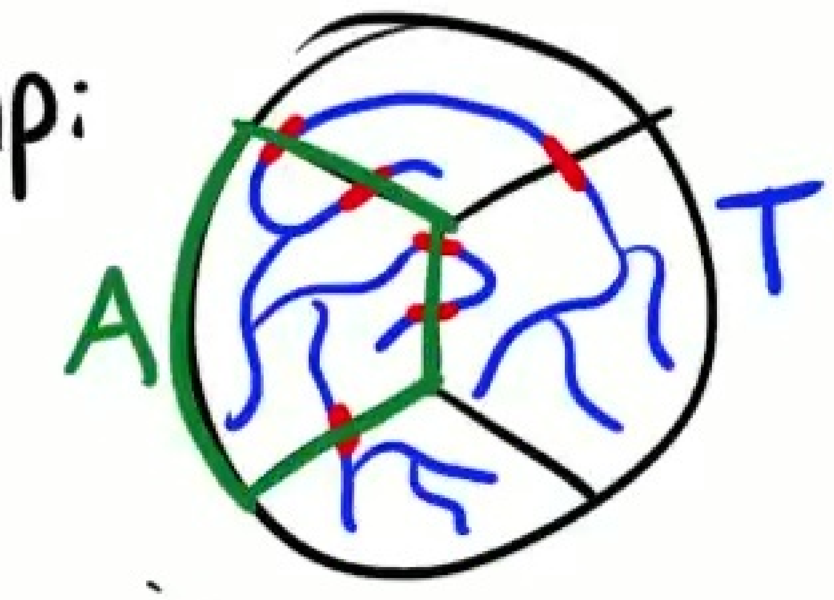

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E.g. $|\mathcal{A}'| = O_k(n)$, so if $\exists S_i \in \mathcal{A}'$ cutting ≥ 2 edges of T, then branch on $\mathcal{A}$.

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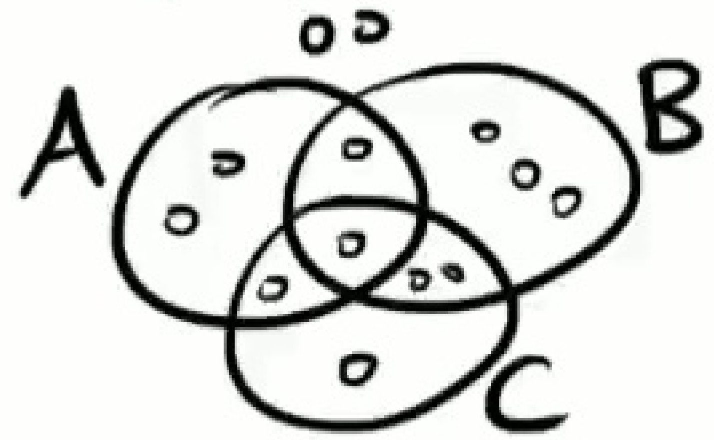

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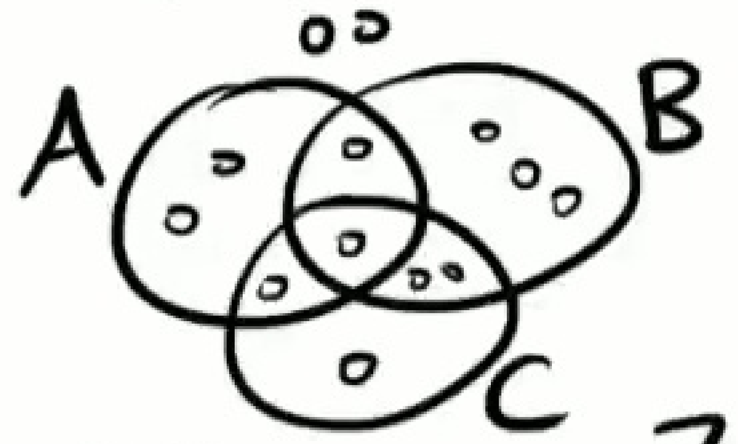

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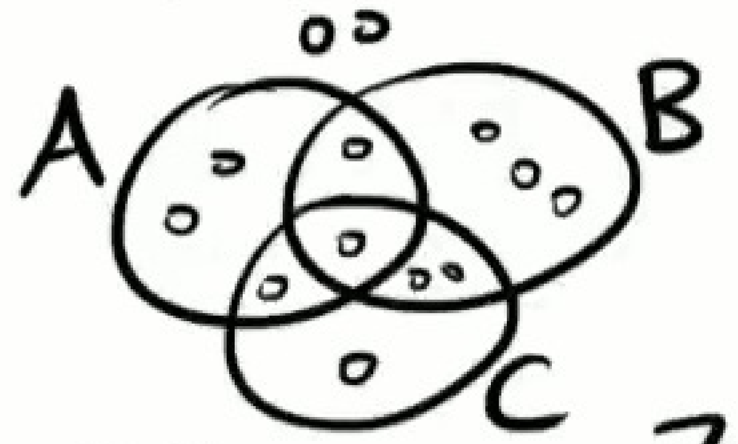

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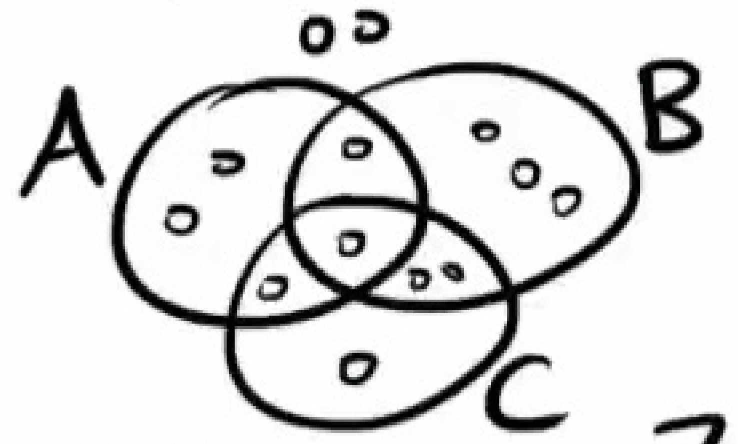
Construct cut iteratively, cost $< \mathrm{OPT}$, contradiction.

Balanced Example

- Suppose G is "balanced": $w(\partial_G S_i) = \frac{2}{k}\mathrm{OPT}$ for all S_i.
- To make progress, find $\mathcal{A}$, $|\mathcal{A}| = O_k(n^{4-\varepsilon})$, s.t. $\exists\, S_i \in \mathcal{A}$ cutting ≥ 4 edges of T: $|\partial_T S_i| \geq 4$
- Such S_i exists because $\sum |\partial_T S_i| = 2(2k-2) \Rightarrow$ avg $|\partial_T S_i|$ is $\frac{4k-4}{k} \approx 4$

Thm: Let $\mathcal{A} := \{ A \subseteq V : w(\partial_G S_i) \leq \frac{2.3}{k}\mathrm{OPT} \}$. Then, $|\mathcal{A}| \leq n^{3.75}$

Pf: Use extremal result: if $|\mathcal{A}| \geq \Omega(n^{3.75})$, then $\exists$ sets A, B, C:



Cut out A, B, C: pay $3 \cdot \frac{2.3}{k}\mathrm{OPT} = \frac{7-\varepsilon}{k}\mathrm{OPT}$ for $+7$ components.

Construct cut iteratively, cost $<$ OPT, contradiction.

To find $\mathcal{A}$: run modified Karger-Stein again, output smallest $\Theta(n^{3.75})$ cuts.

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Thm: 2.3 3.75 A, B, C:

Pf:

Recursion?

ponents.

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Thm: … 2.3 … 3.75

Pf: … A, B, C:

Recursion?
If can do this $\Omega(k)$ times, then save $n^{\Omega(k)}$ runtime
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… ponents.

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Balanced Example $(k' < k)$

- Suppose G is "balanced": $w(\partial_G S_i) = \frac{2}{k'}\mathrm{OPT}'$ for all S_i.
- To make progress, find $\mathcal{A}$, $|\mathcal{A}| = O_k(n^{4-\varepsilon})$, s.t. $\exists S_i \in \mathcal{A}$ cutting ≥ 4 edges of T: $|\partial_T S_i| \geq \cancel{4}\ >3k$
- Such S_i exists because $\sum |\partial_T S_i| = \cancel{2(2k-2)}\ >3k \Rightarrow$ avg $|\partial_T S_i|$ is $\cancel{\frac{4k-4}{k'}}\ \frac{>3k}{k'} > 3$

Thm: $\ldots\ 2.3\ \ldots\ 3.75$

A, B, C:

Pf:

Recursion?

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…ponents.

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Thm: ... 2.3 ... 3.75

Pf: ...

A, B, C:

...ponents.

> **Recursion?**
>
> If can do this $\Omega(k)$ times, then save $n^{\Omega(k)}$ runtime $\Rightarrow n^{(2-\varepsilon)k}$ time.
>
> Suppose $>3k$ edges left.
> Then, can still pay $n^{3.75}$ to cut 4 edges of T.

To find $\mathcal{A}$: run modified Karger-Stein again, output smallest $\Theta(n^{3.75})$ cuts.

General Case

- Morally, have solved two extreme cases

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  1. Unbalanced: $\exists\, S_i$ with $w(\partial_G S_i) < \frac{1.49}{k} OPT$
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  1. Unbalanced: $\exists\, S_i$ with $w(\partial_G S_i) < \frac{1.49}{k} OPT$
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- "Interpolate" between the two cases
  (technical; is where we get 1.98 factor)

Open Questions

WIP: Beating factor 1.98

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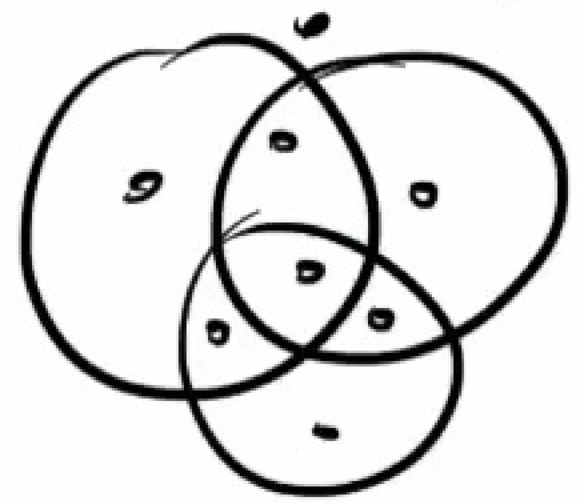
Conjecture: If $|\mathcal{A}| \geq \Theta(n^3)$, then $\exists A, B, C$:



(easy lower bound of $\Omega(n^3)$)

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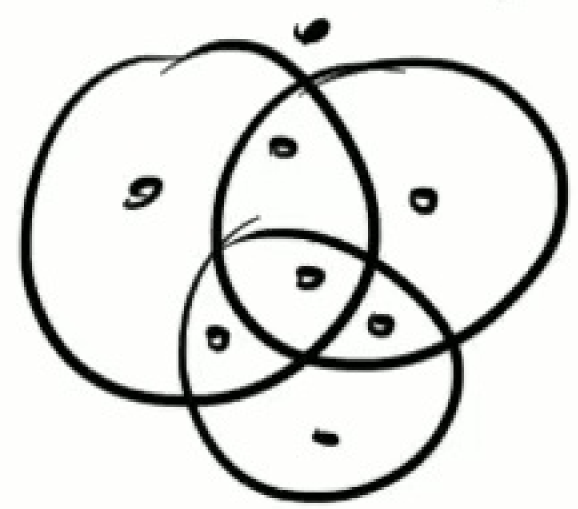

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k-cut for hypergraphs?