Tight FPT Approximations for k-Median and k-Means

Jason Li

Joint work with
Vincent Cohen-Addad, Anupam Gupta, Amit Kumar, Euiwoong Lee

ICALP 2019

k-median problem

- Metric space (V, d)
- Clients $C \subseteq V$, Facilities $\mathbb{F} \subseteq V$
- Find set F of k facilities minimizing

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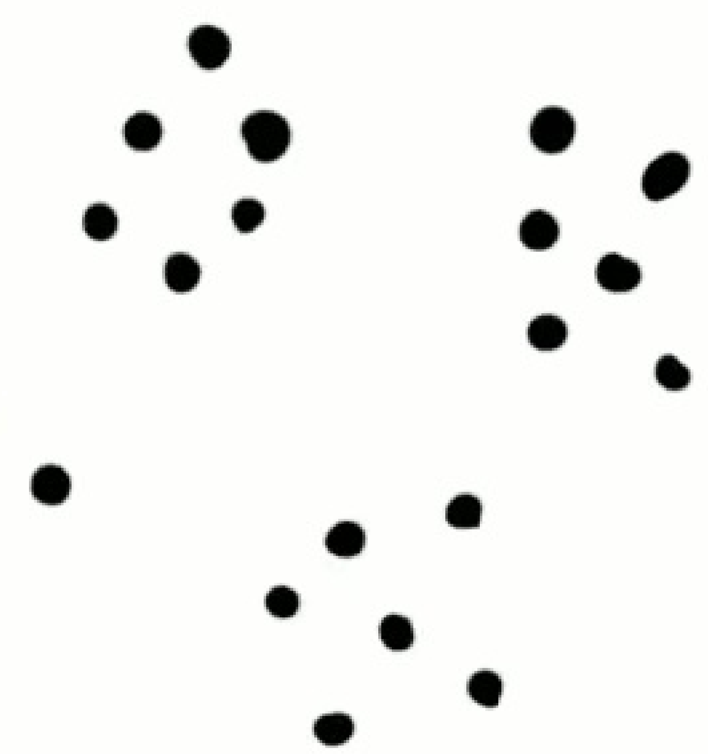

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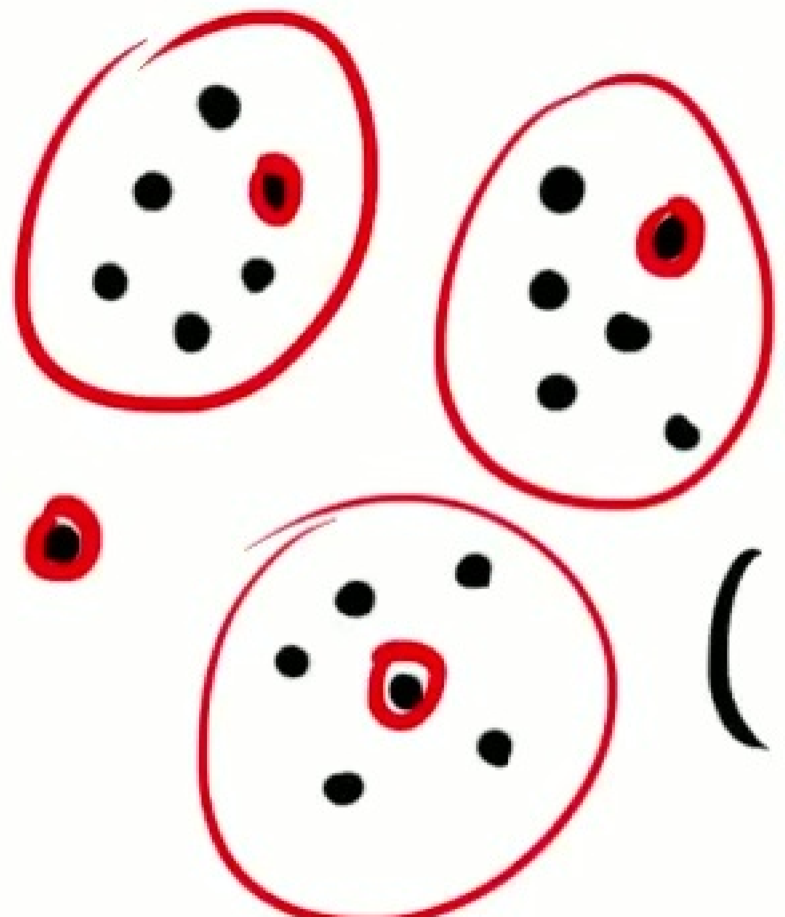

$k=4$

$(C = \mathbb{F})$

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Core-sets

Def[Coreset]: a subset $S \subseteq C$ with weights $w(v): v \in S$

s.t.

$\forall F$ set of k facilities:

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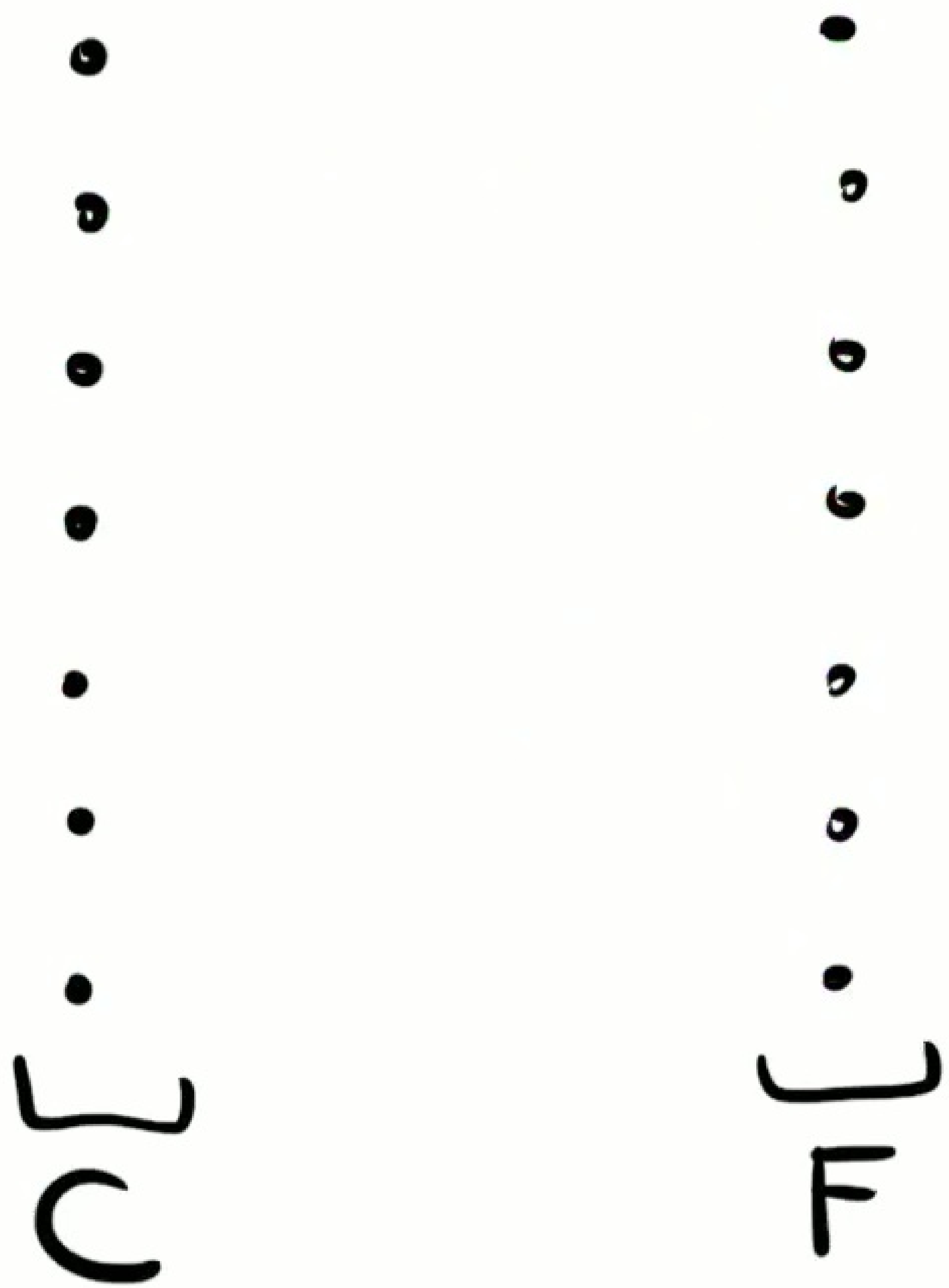
At loss of $(1+\varepsilon)$, can assume $\mathrm{poly}(\varepsilon^{-1} k \log n)$ (weighted) clients

Insight from hardness: why 1+2/e?



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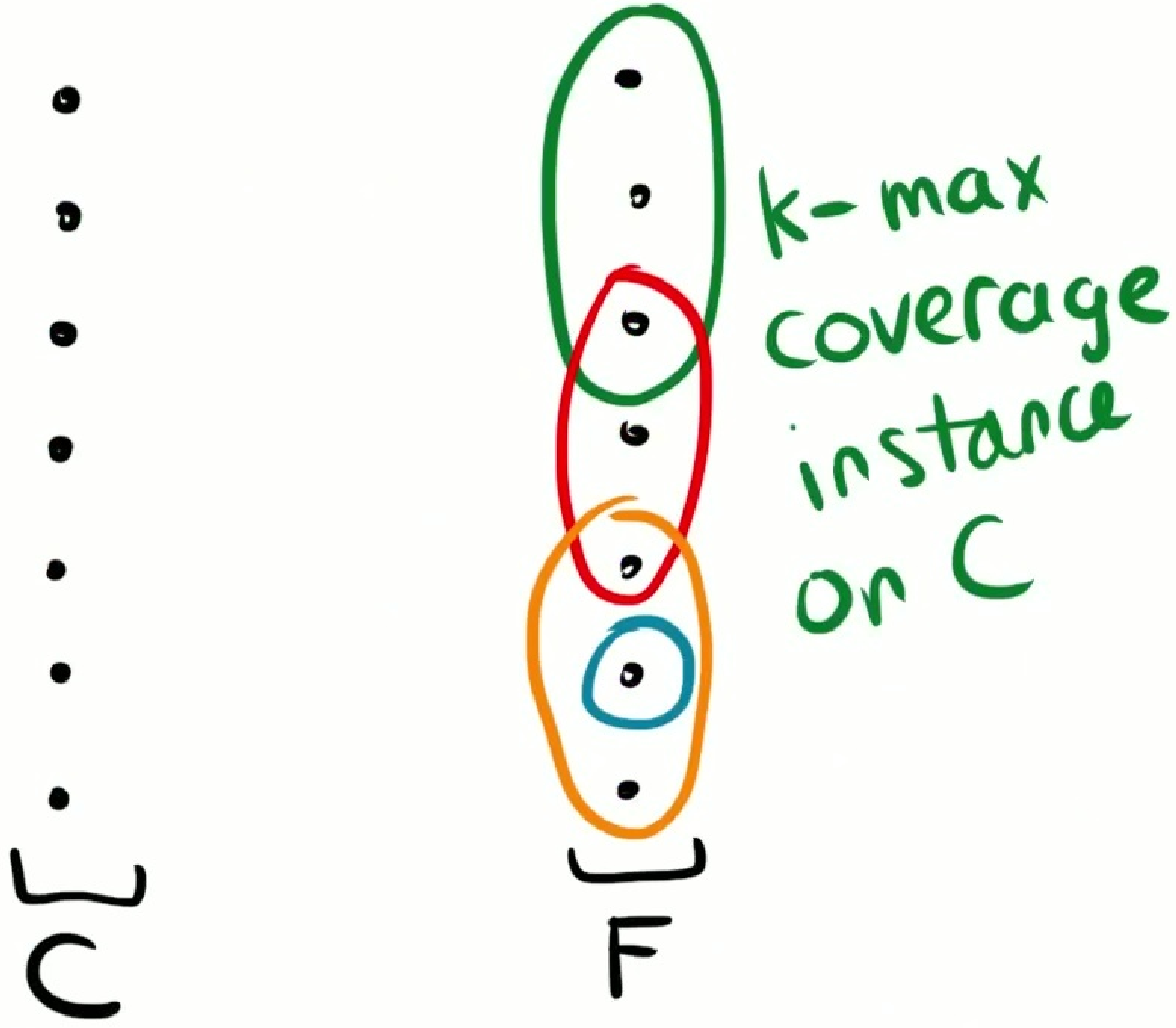

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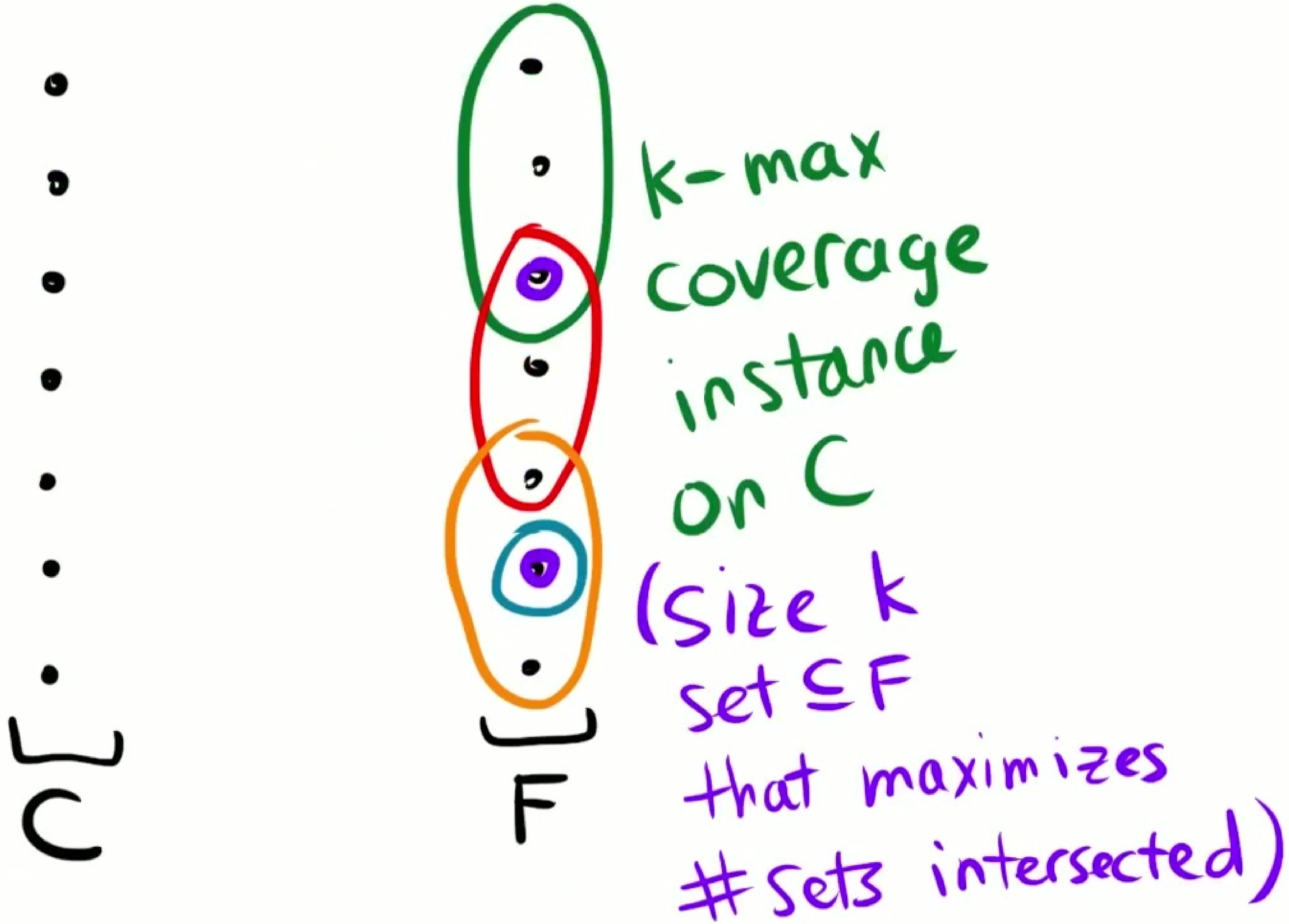

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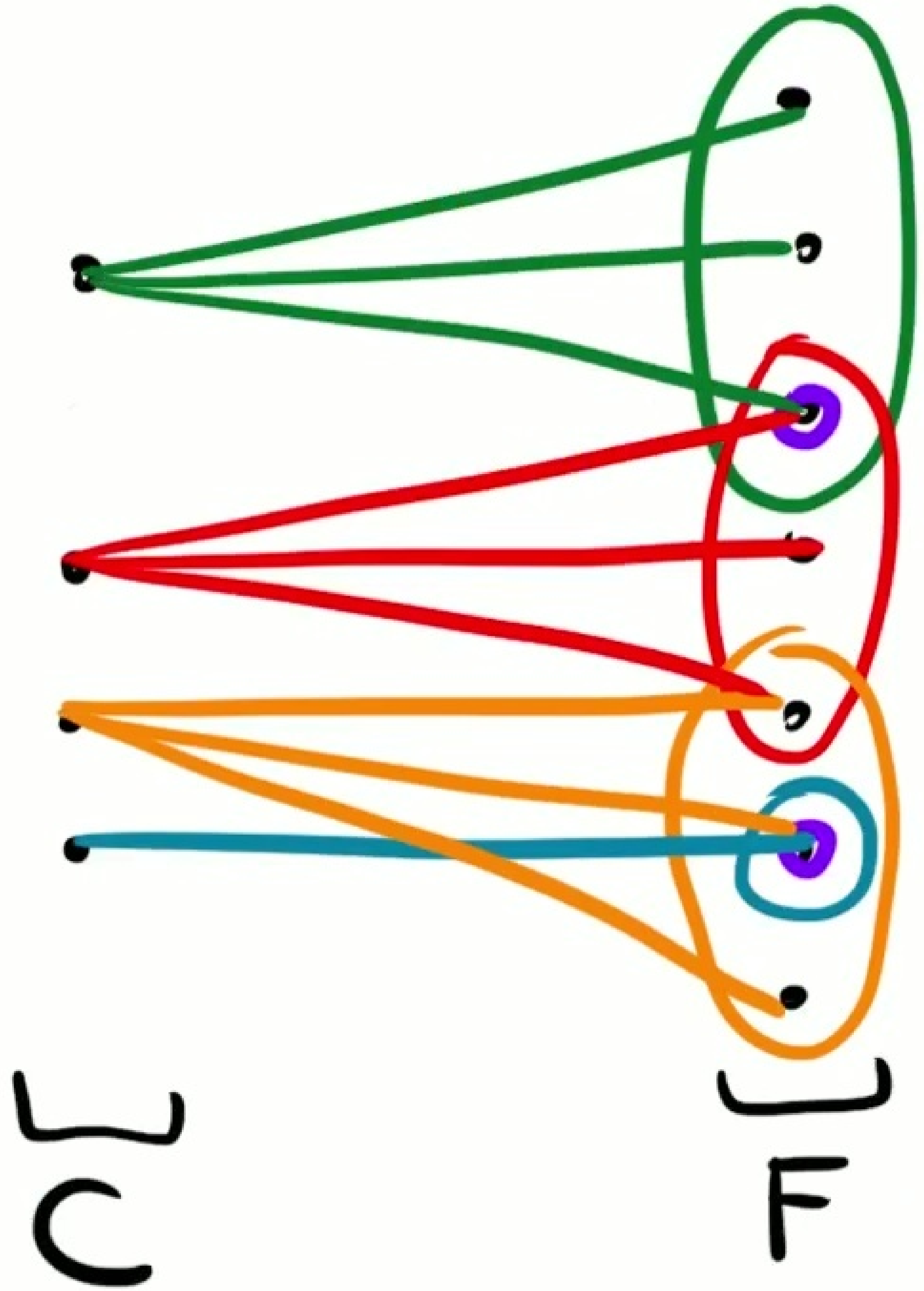

k-max coverage instance on C

(size k set ⊆ F that maximizes #sets intersected)

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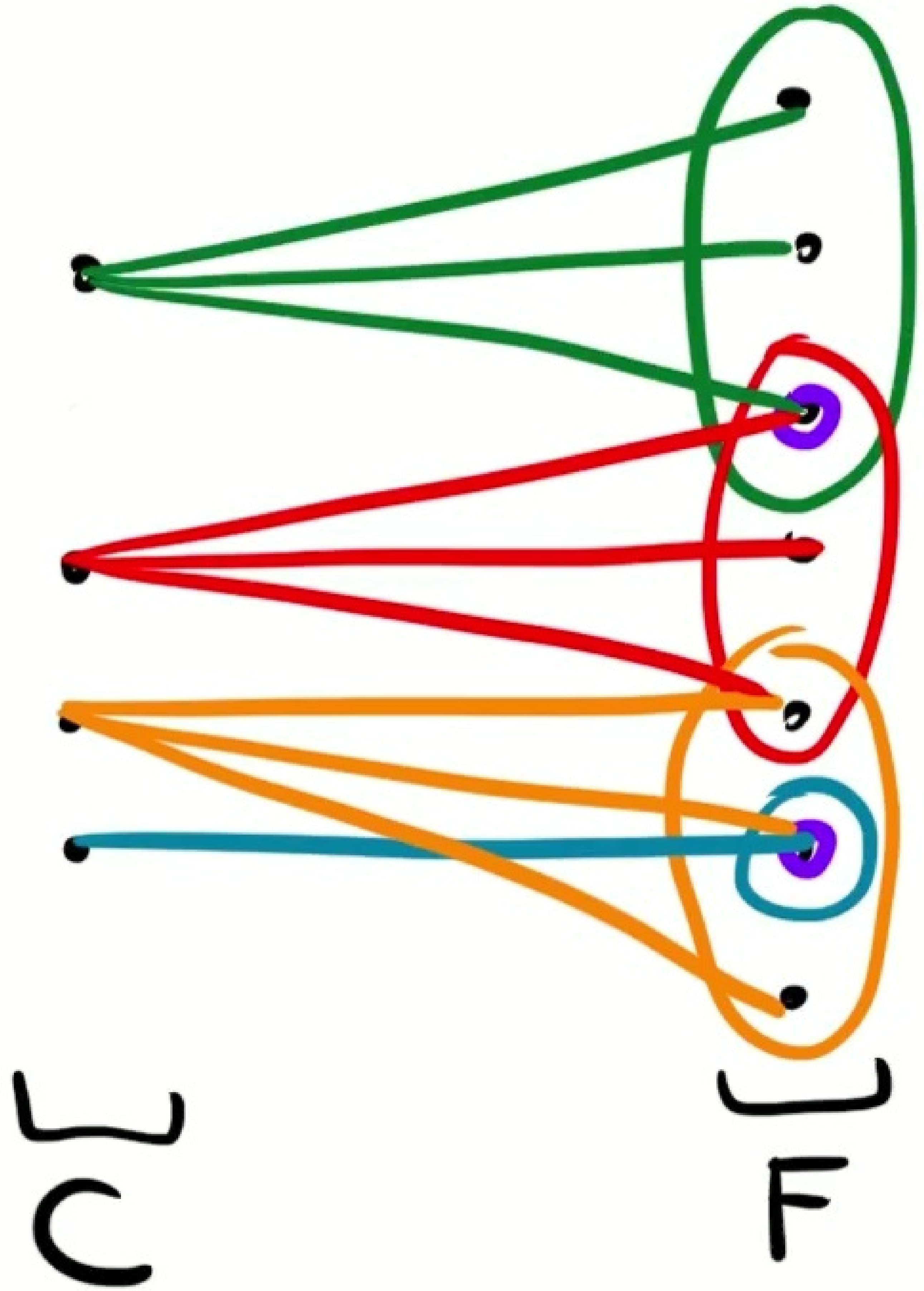

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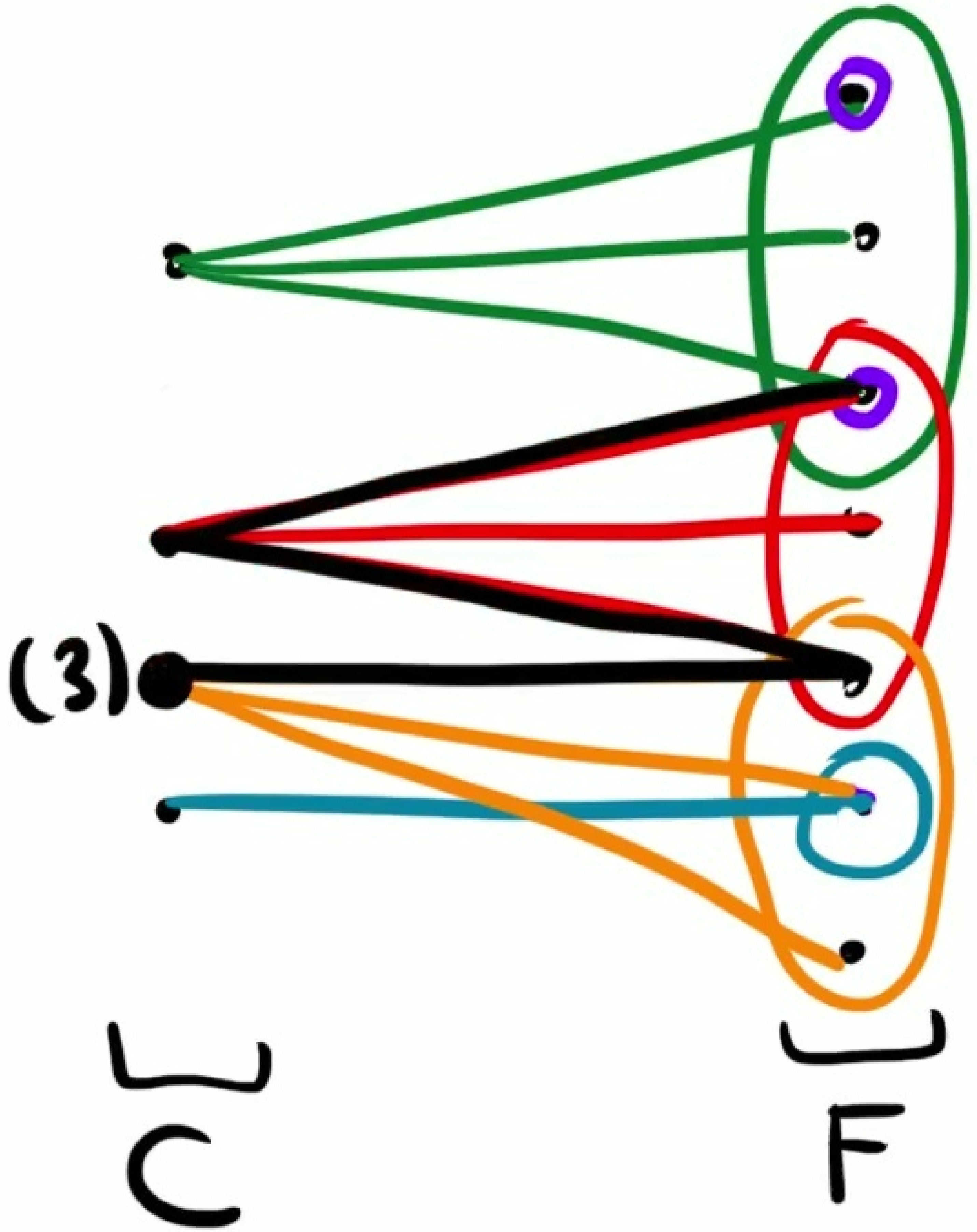

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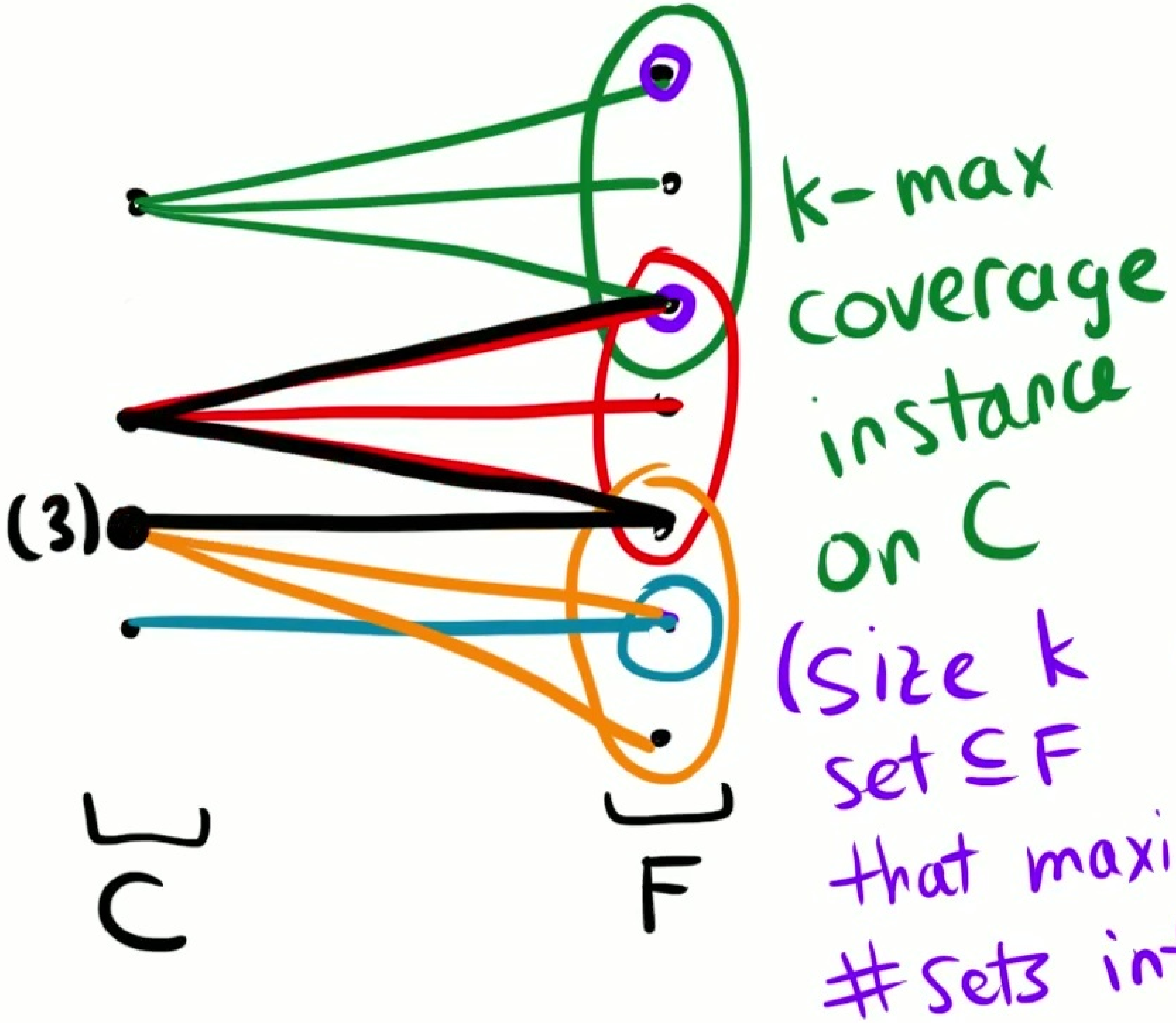

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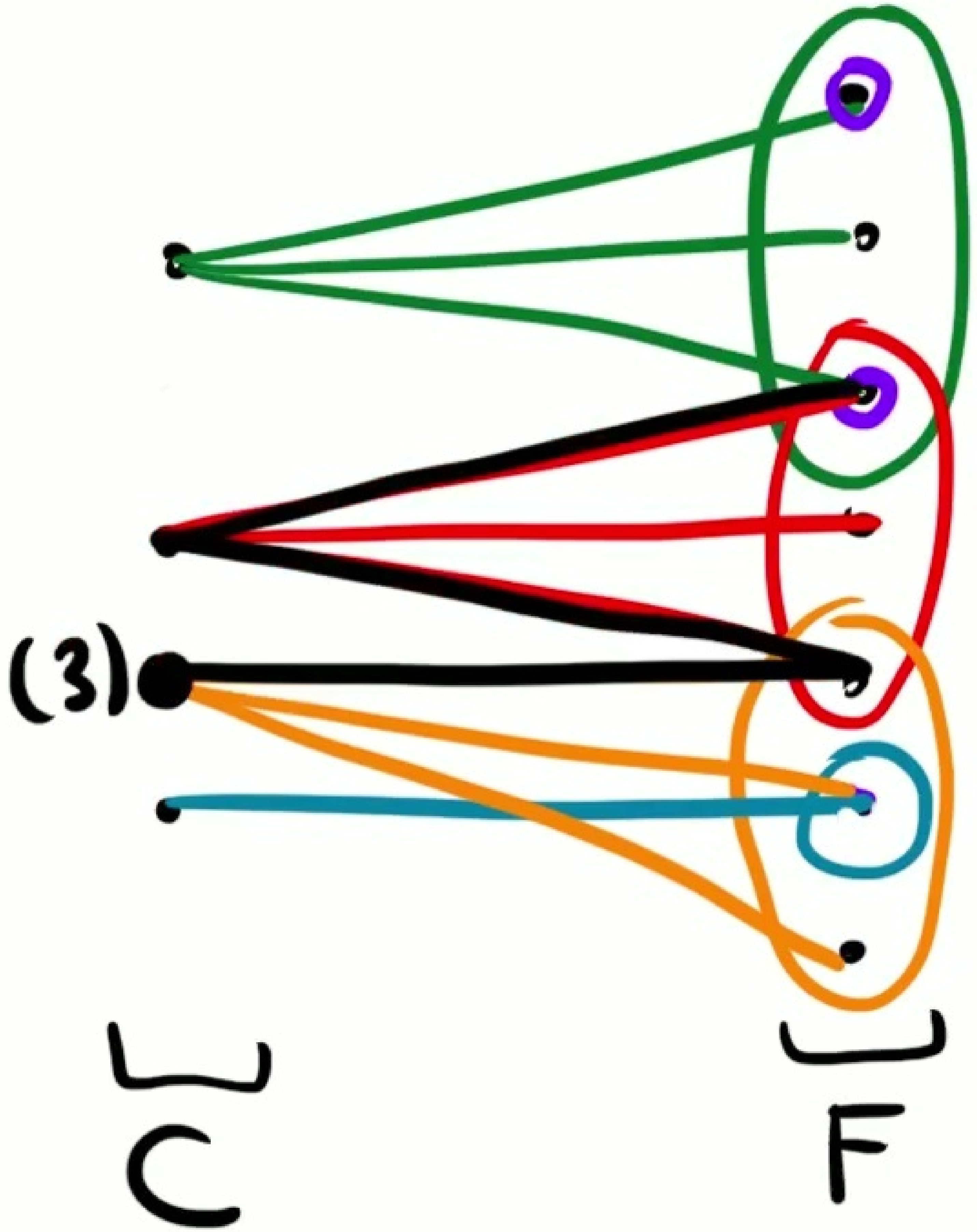

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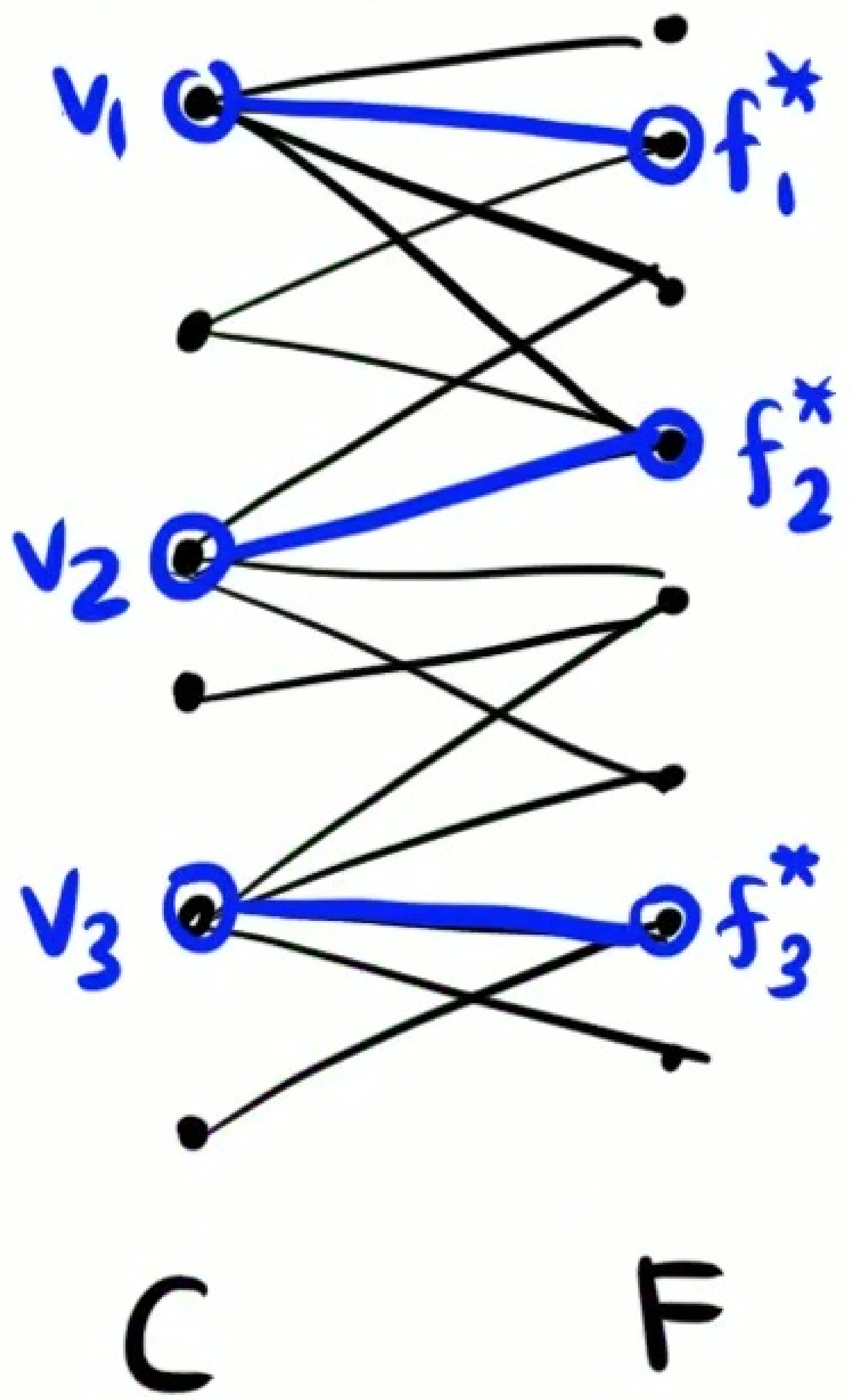
$\Longrightarrow$ k-median $(1-1/e)\cdot 1+\frac{1}{e}\cdot 3$

$=(1+2/e)$-apx hard

Transformation to Algorithm

- Guess k centers $v_1 \dots v_k$
  s.t. $f_i^* \sim v_i \ \forall i$
  $[\mathrm{poly}(\varepsilon^{-1} k \log n)]^k$ time (FPT!)



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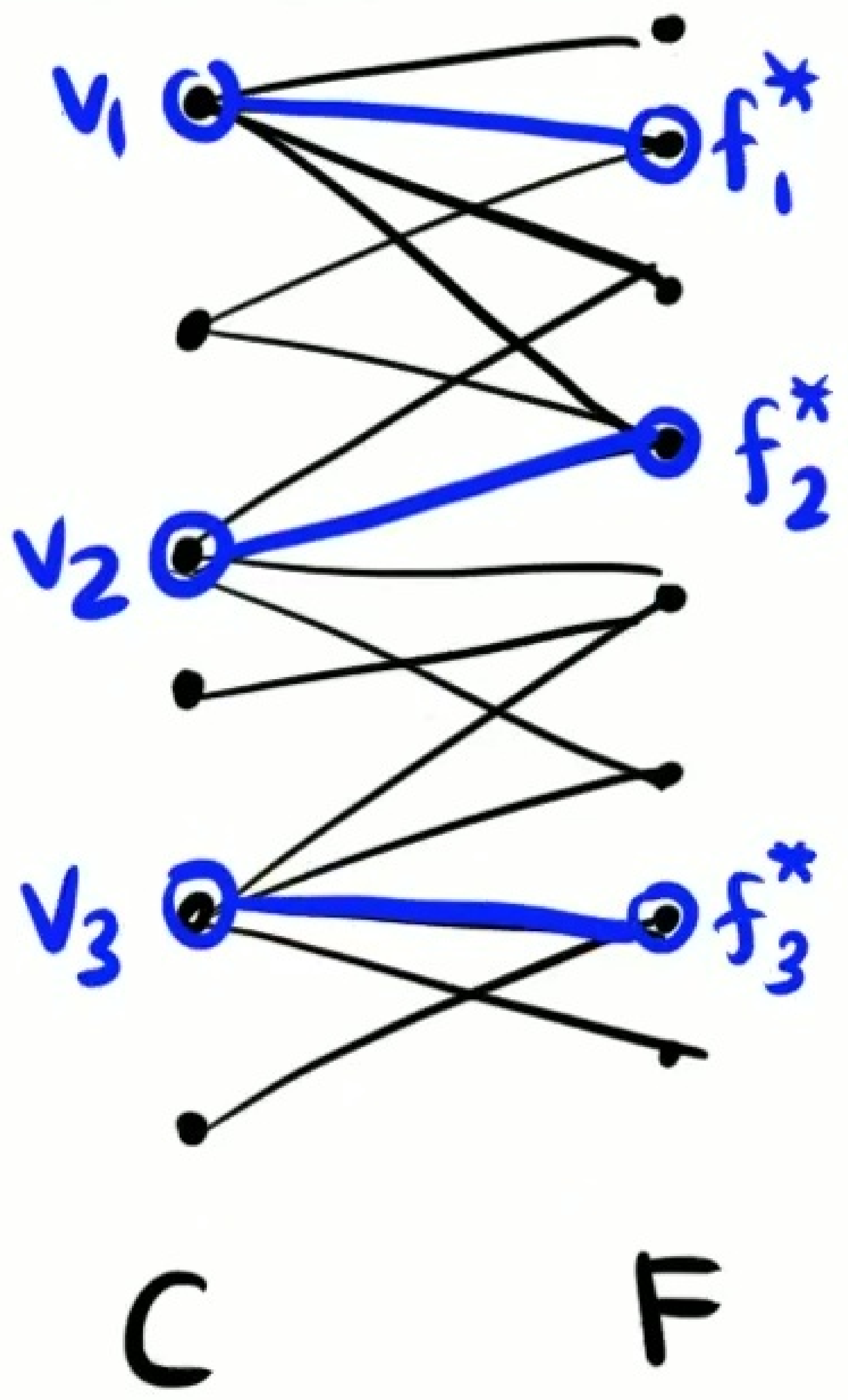

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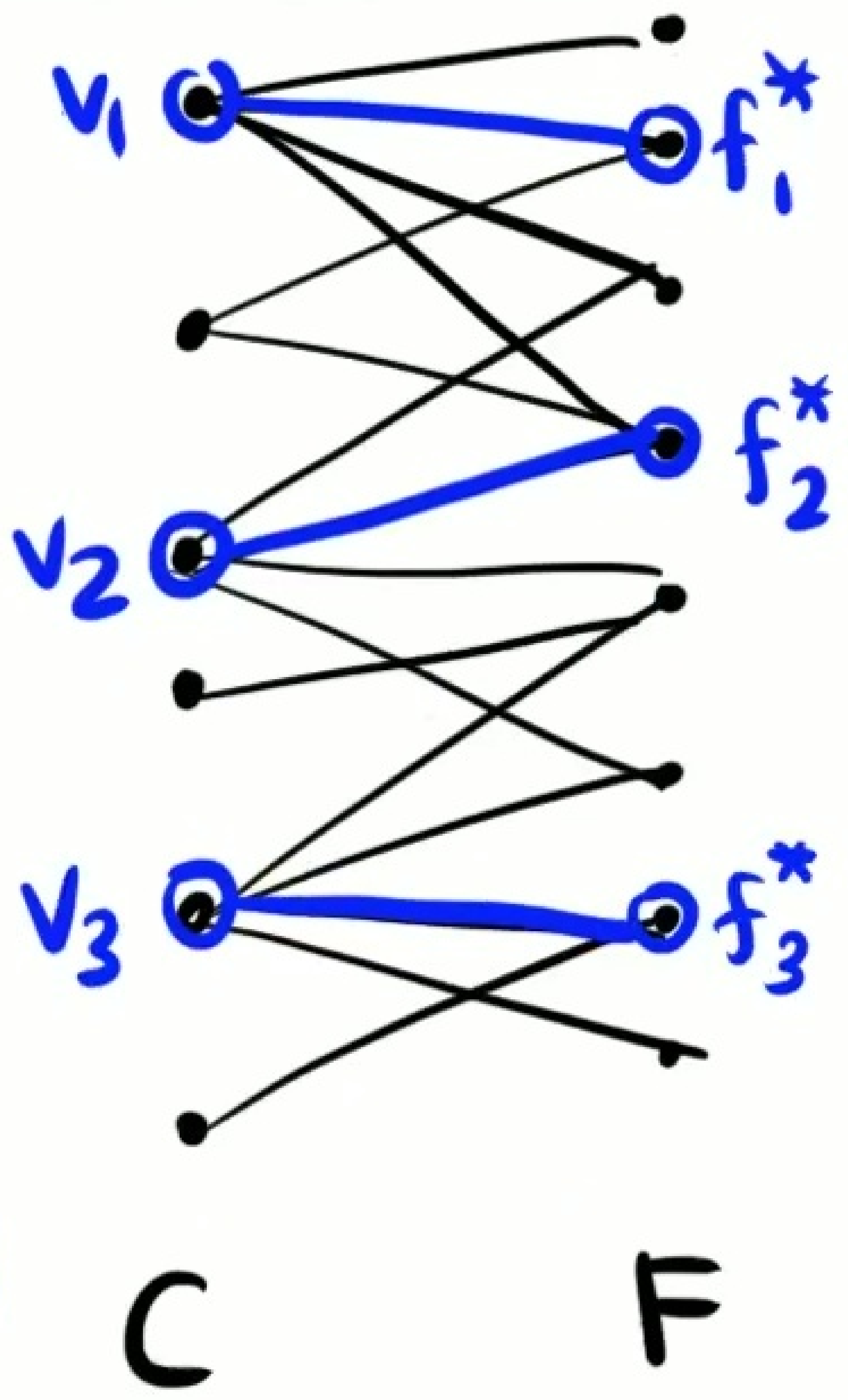

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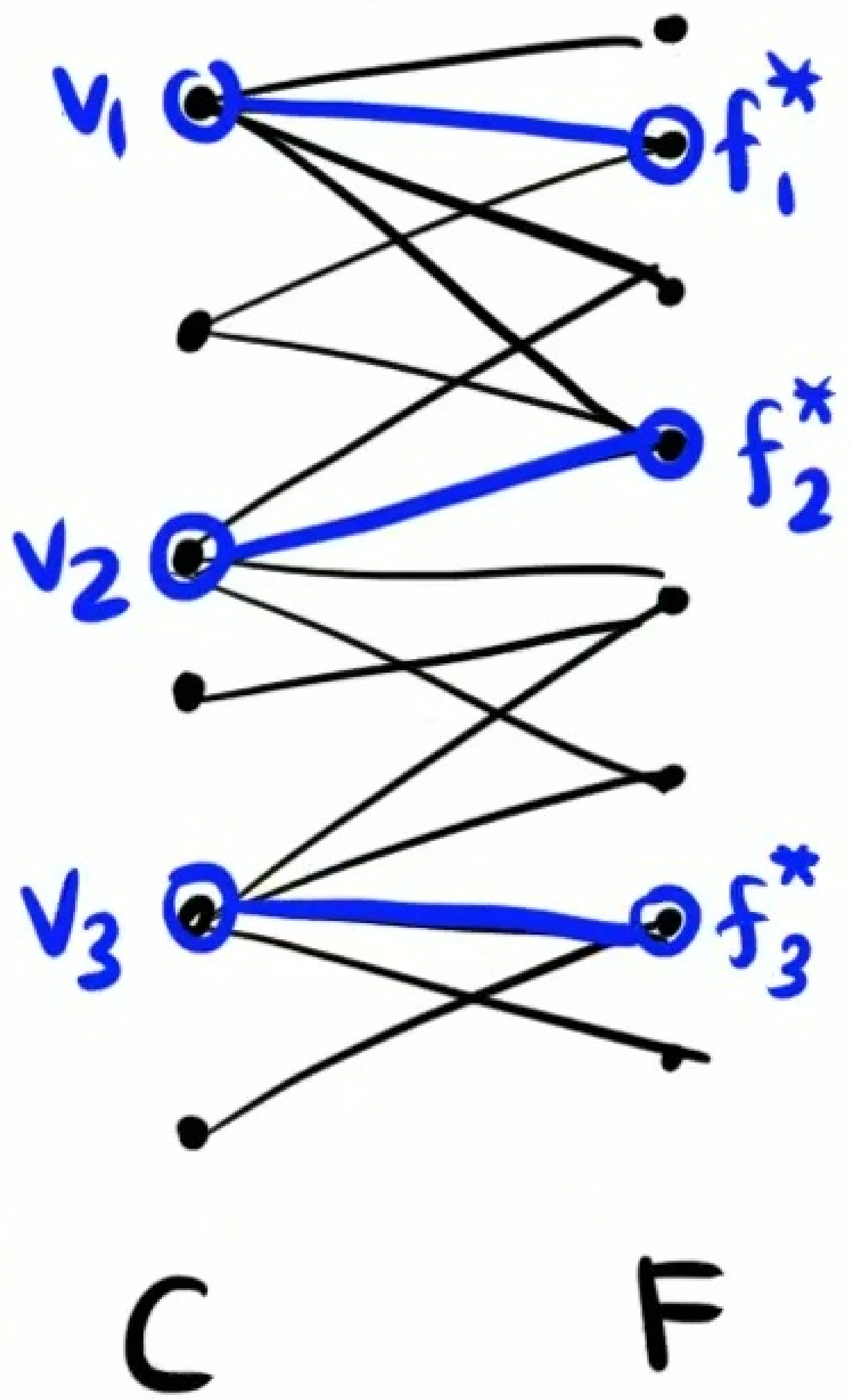

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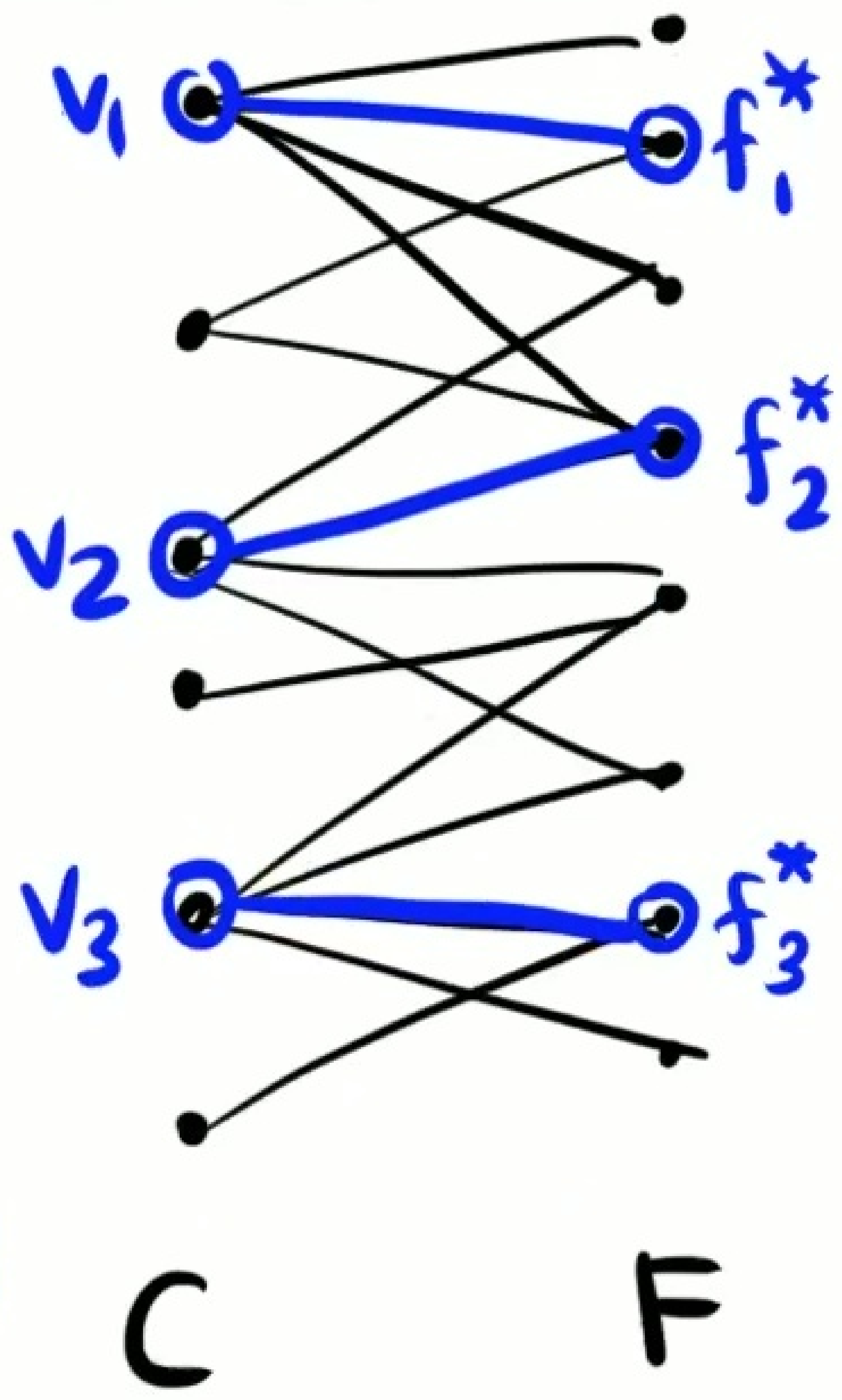

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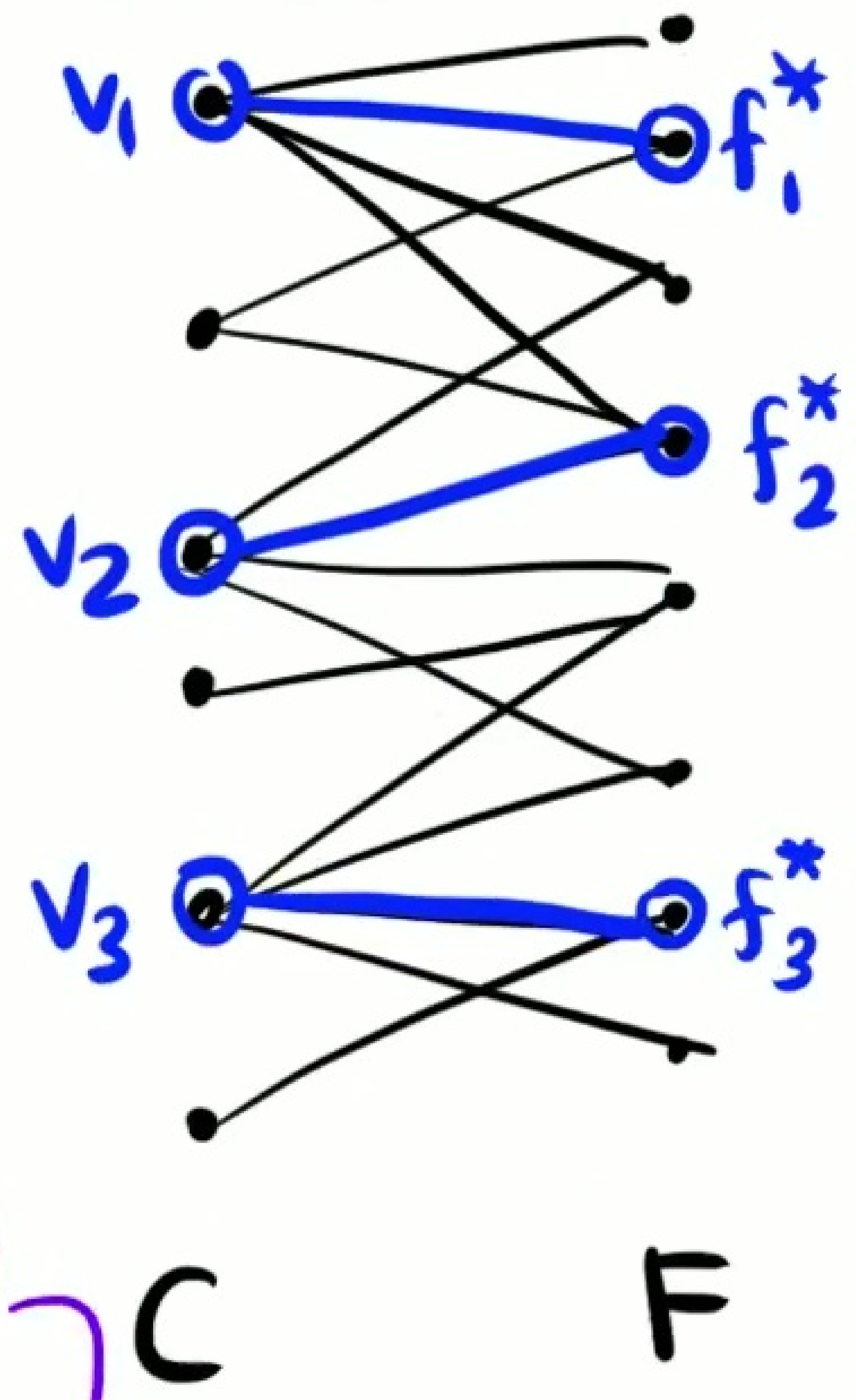

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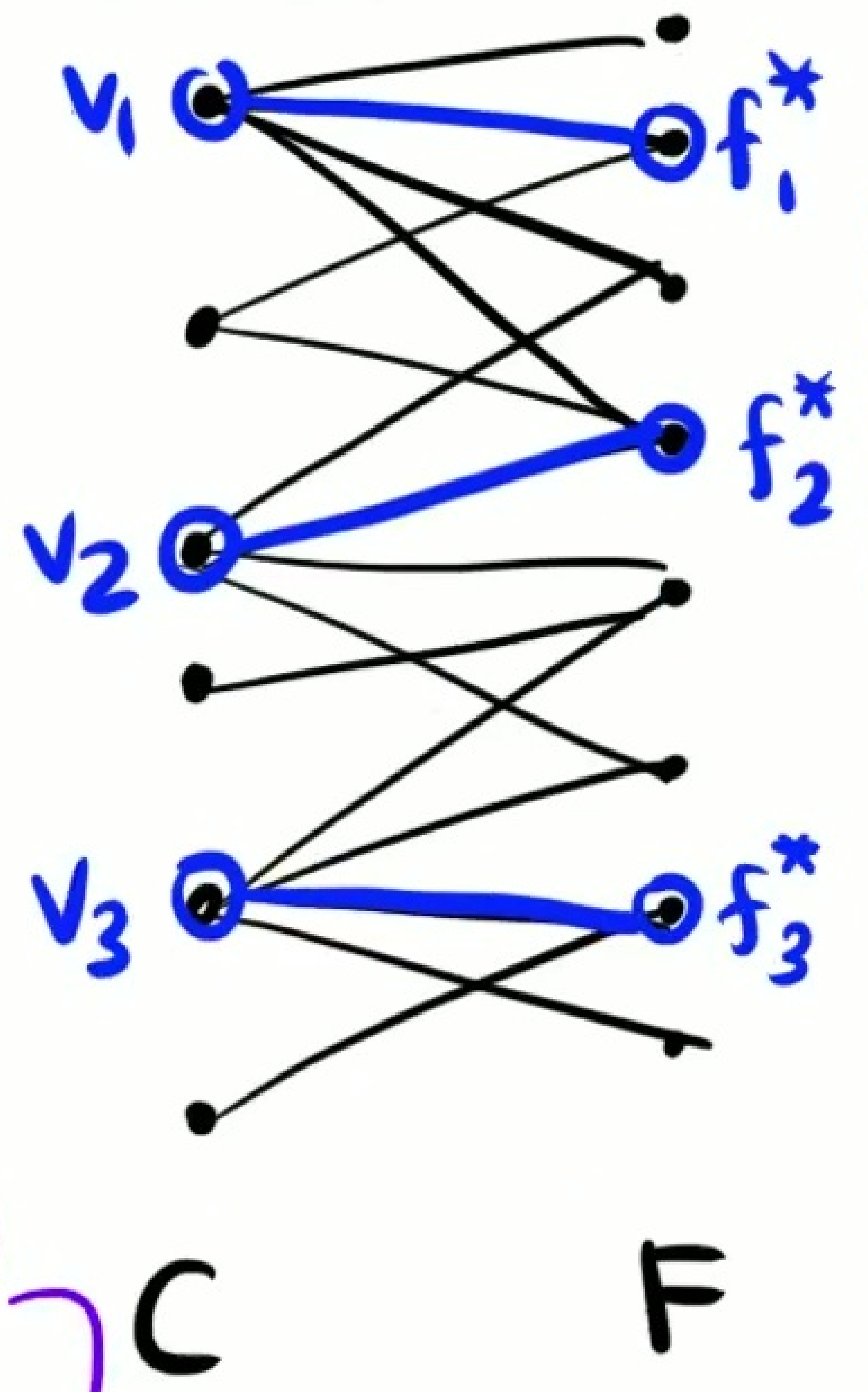

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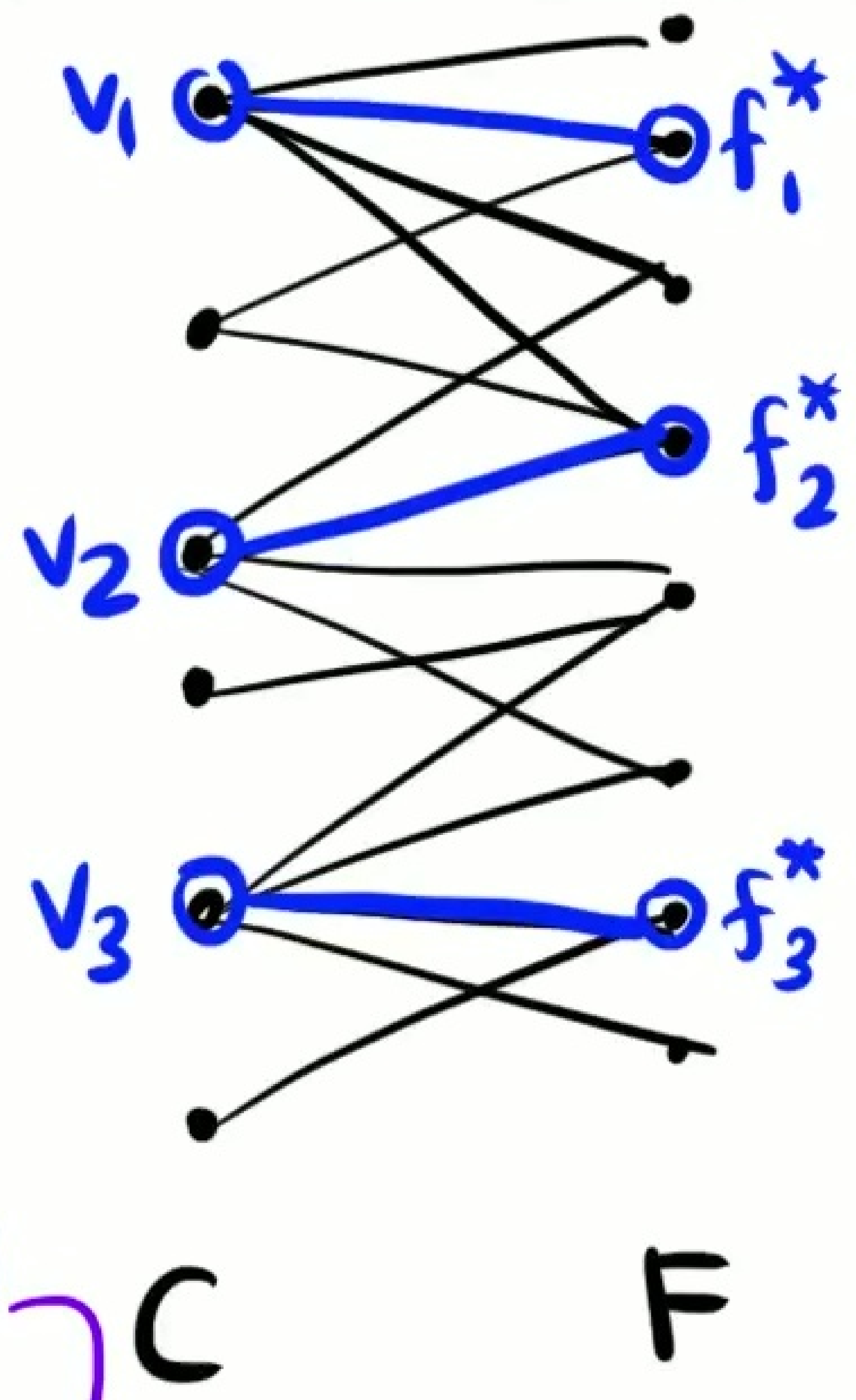

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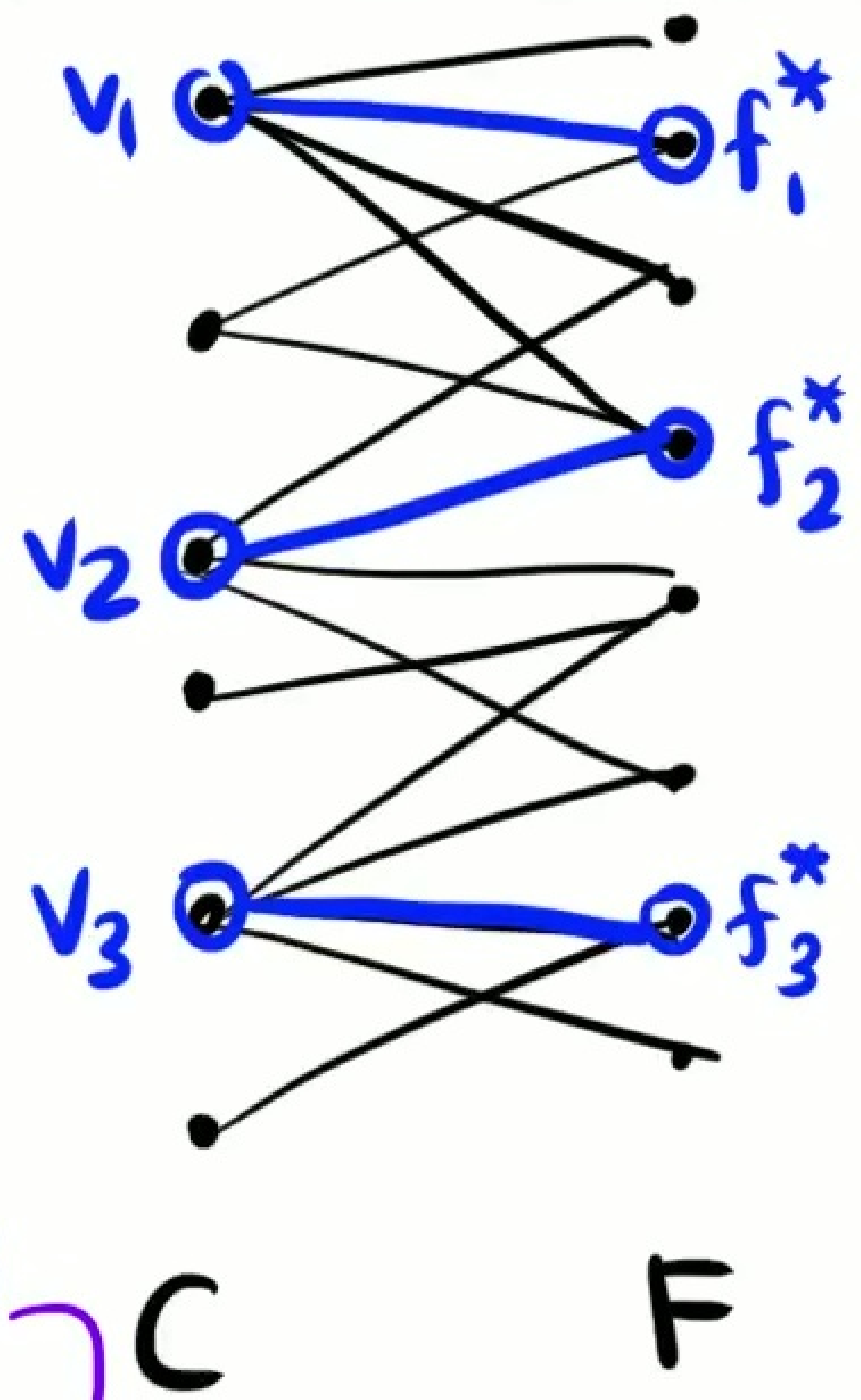

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No such guarantee in poly-time k-med algos!

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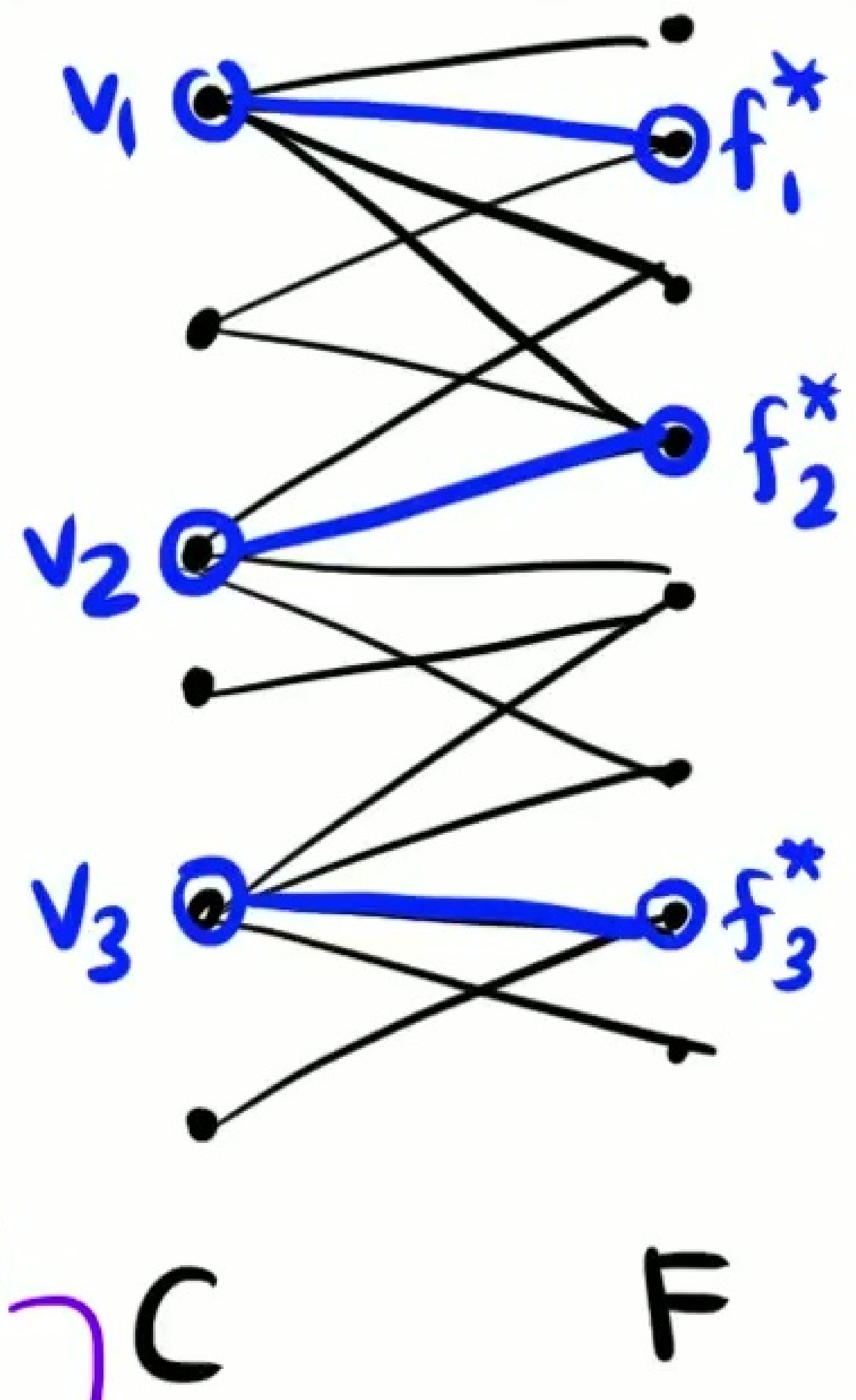

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How to get $(1+2/e)$? don't know $d(v,F^*)$...

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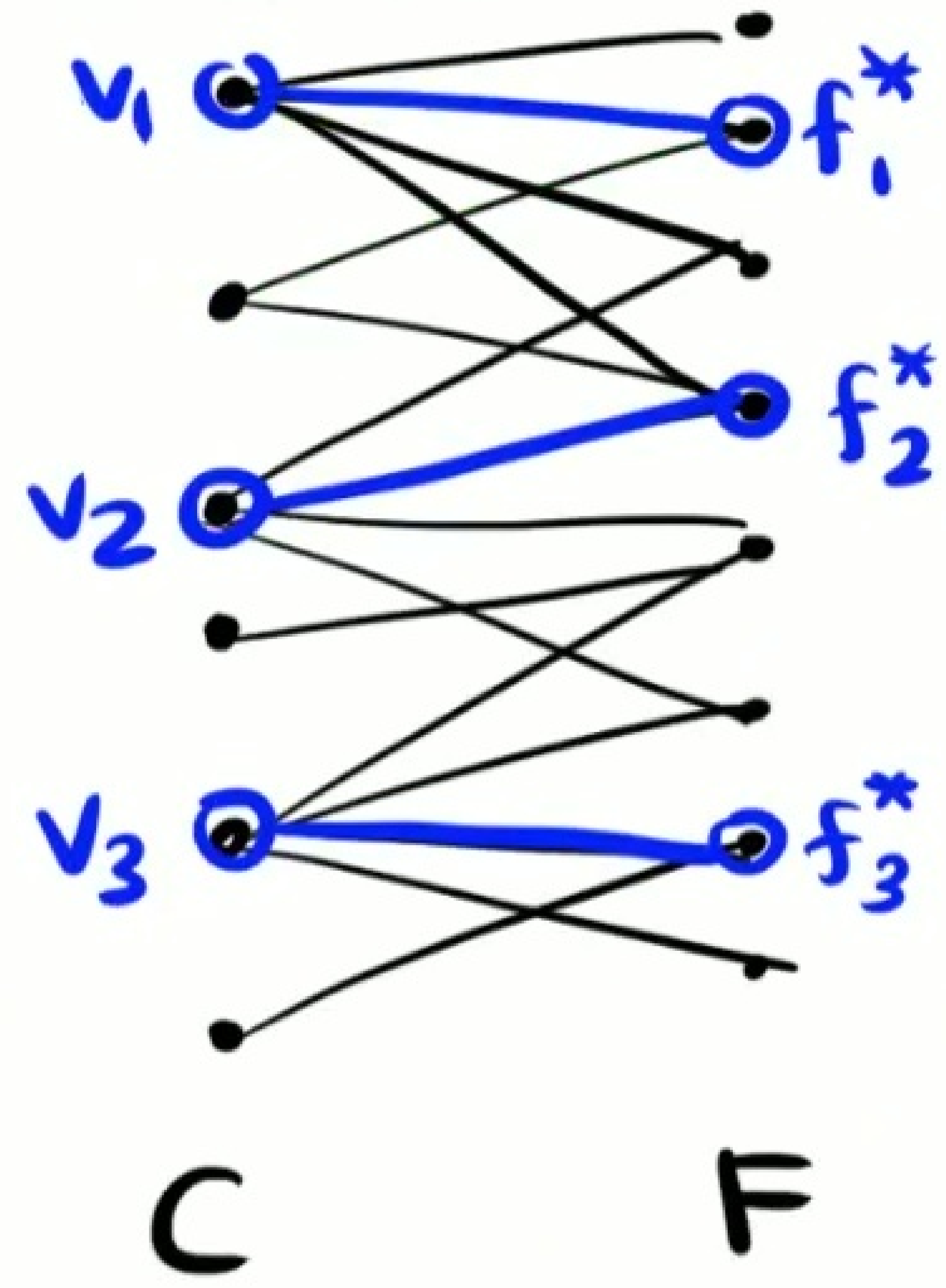
Optimization function



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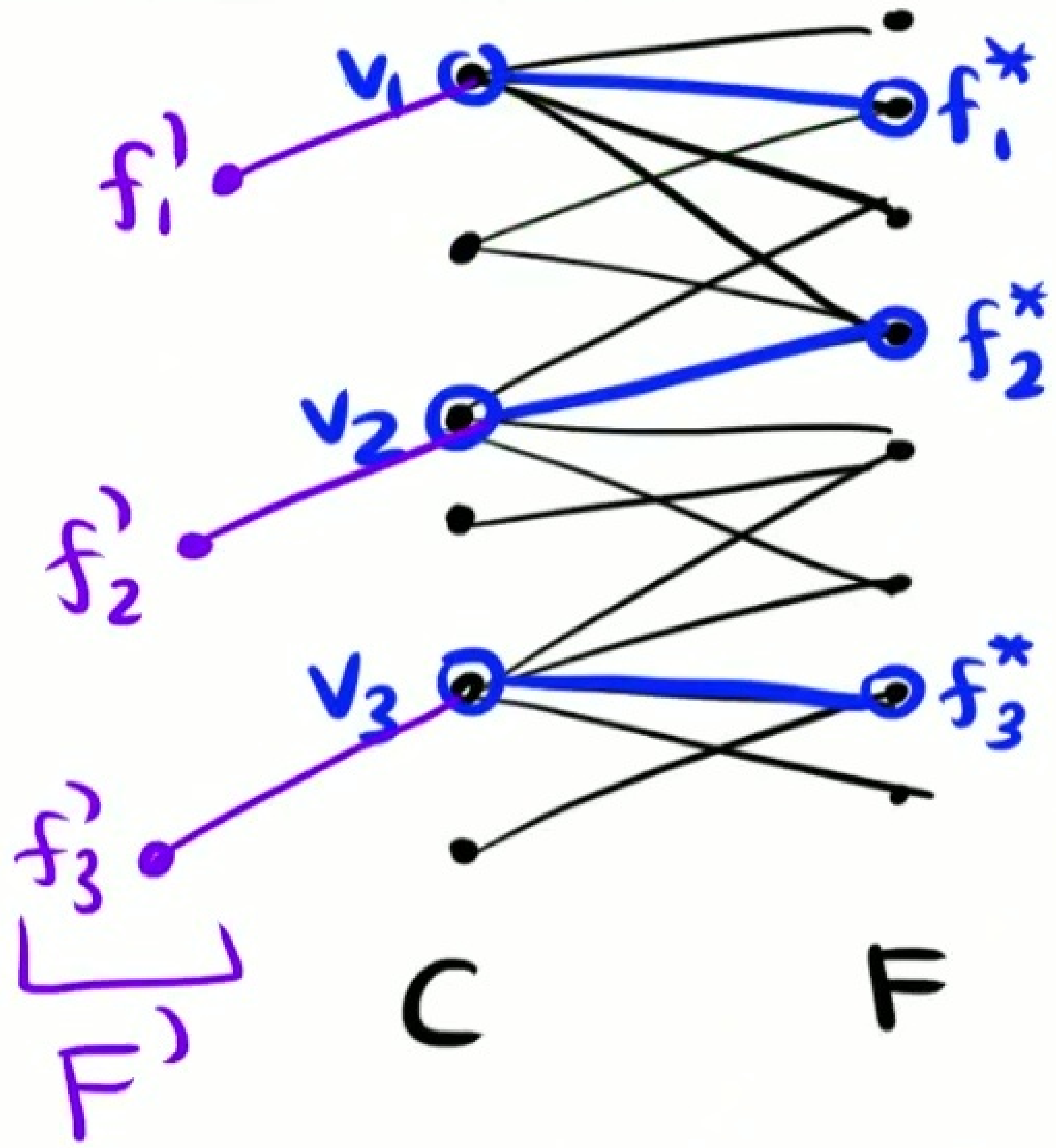
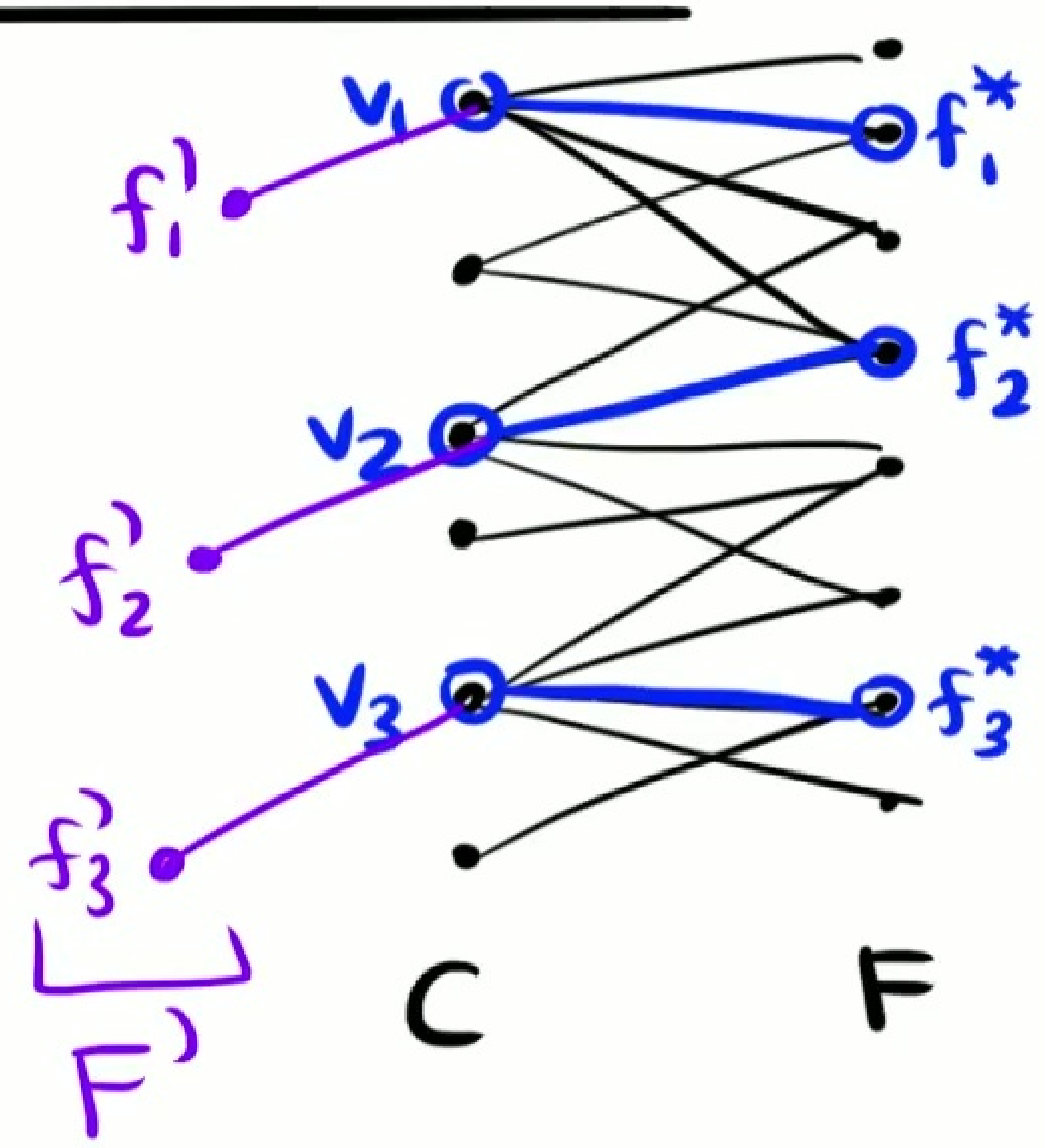


Optimization function

maximize

$$\mathrm{impr}(S) := \sum_{v} w(v)\Big(d(v, F') - d(v, F' \cup S)\Big)$$

over $S \subseteq \mathbb{F}$, $S = \{f_1 \dots f_k,\ f_i \in N(v_i)\}$



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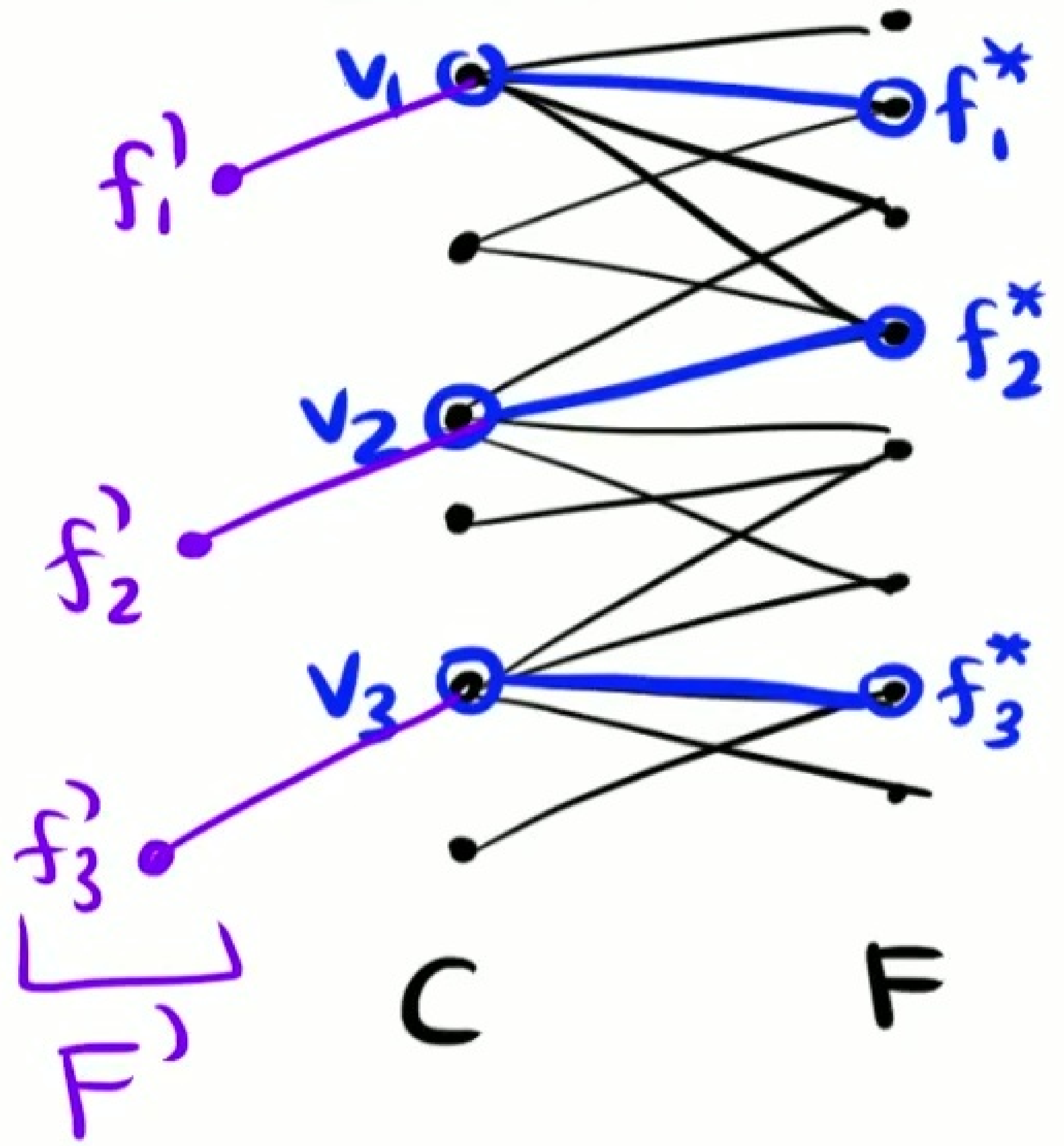

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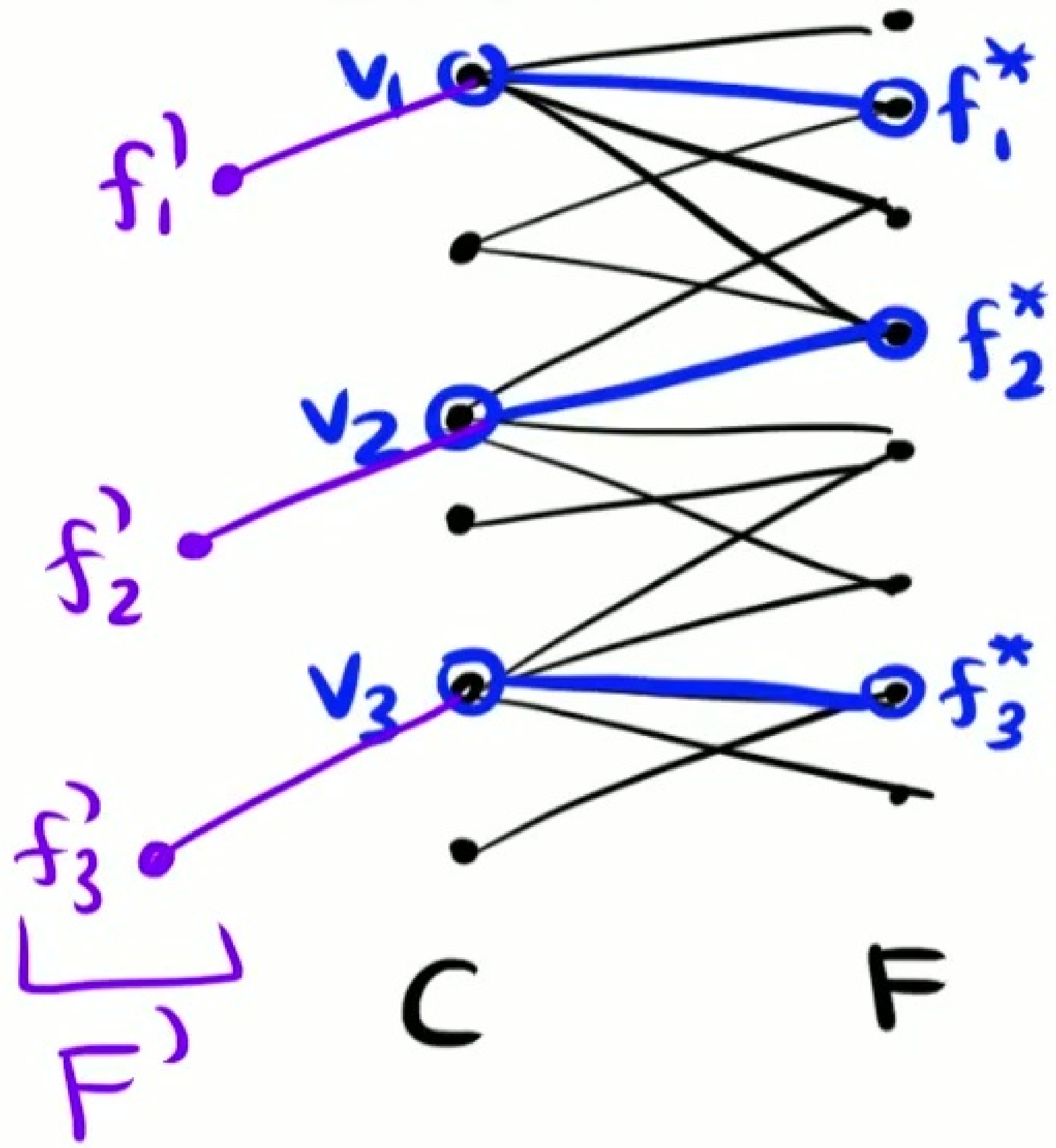
① Adding into S only decreases

$\Rightarrow$ increases impr



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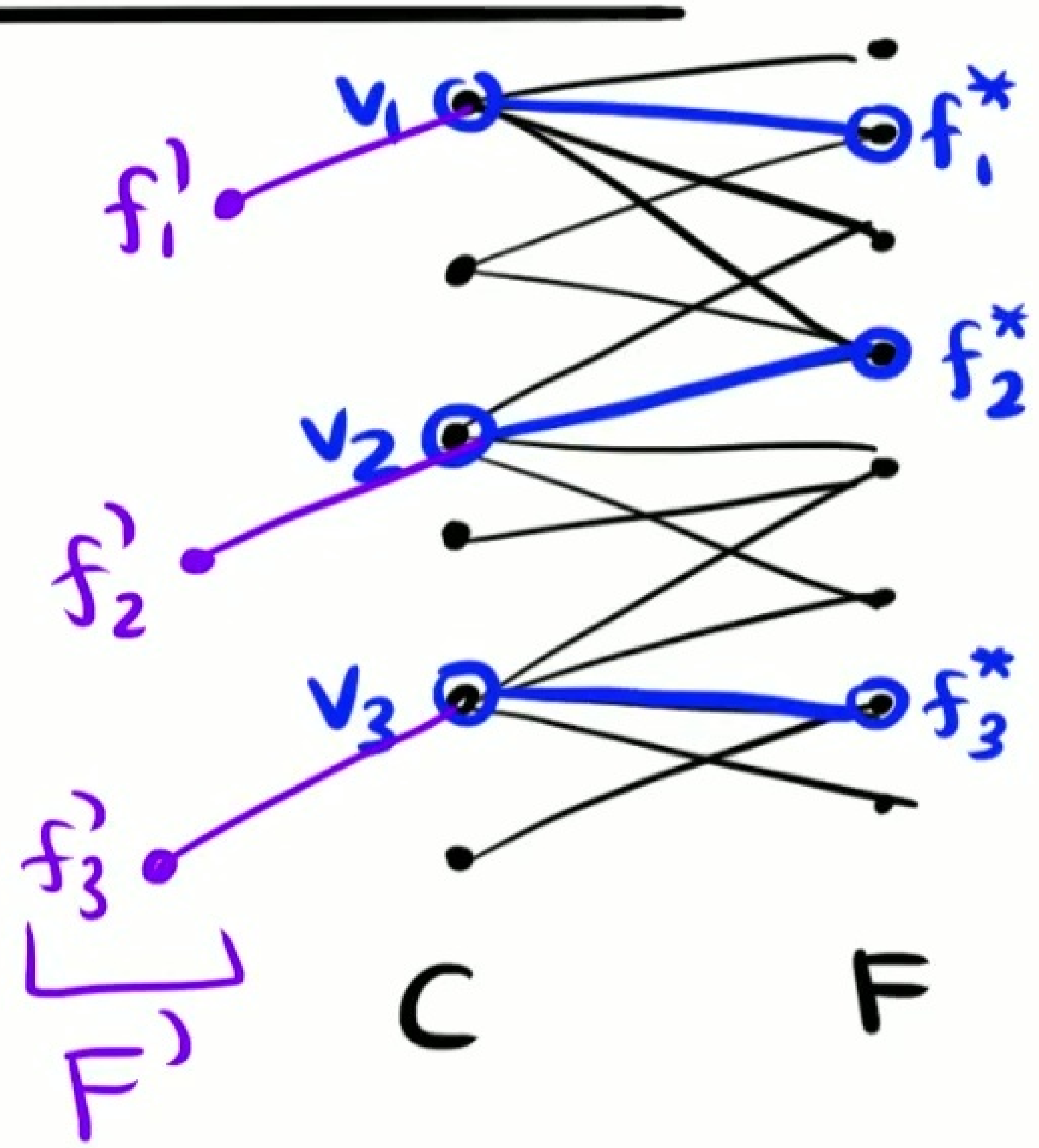

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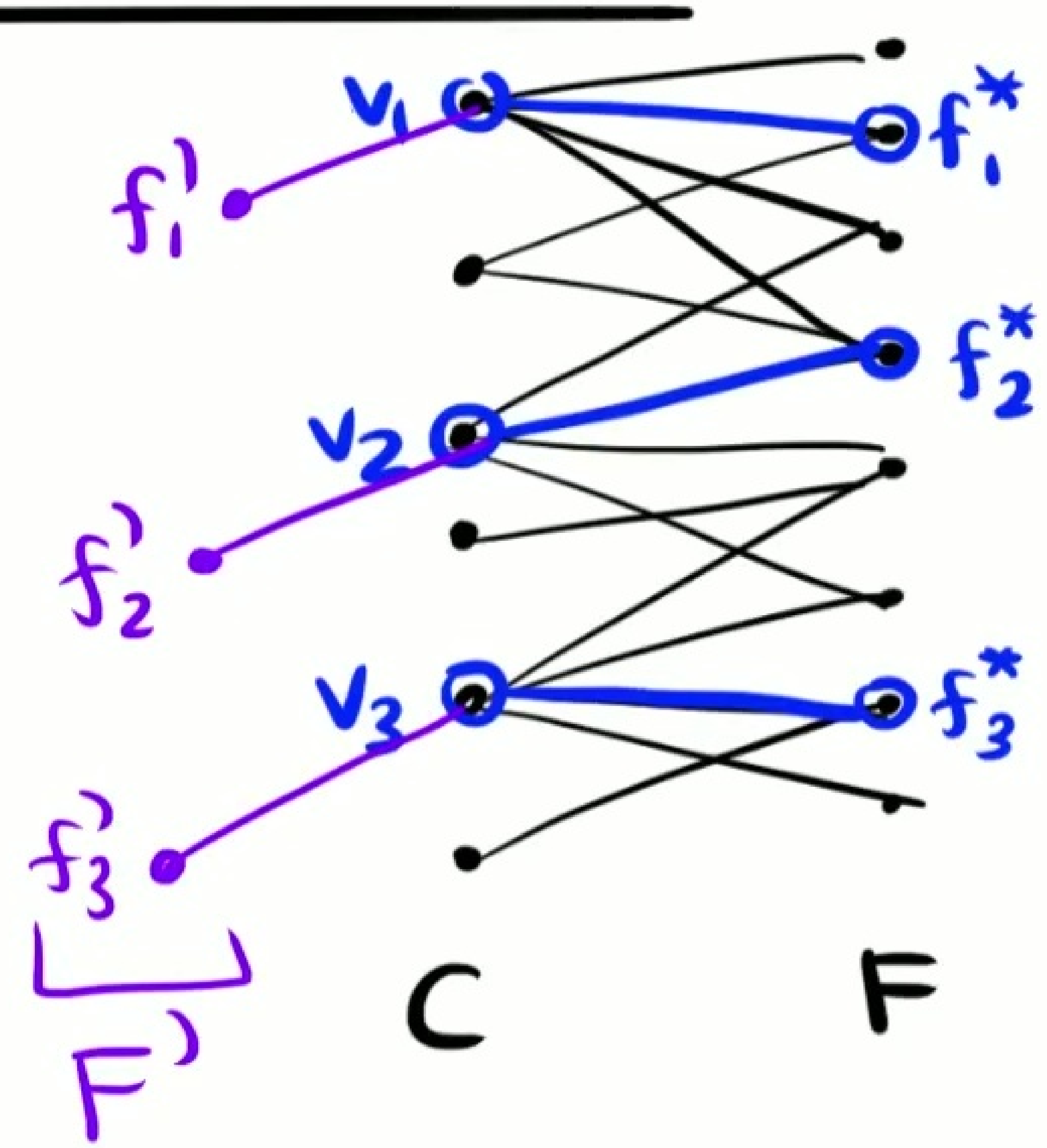
③ Submodular: $\forall S \subseteq T, f$: $\operatorname{impr}(S \cup f) - \operatorname{impr}(S) \geq \operatorname{impr}(T \cup f) - \operatorname{impr}(T)$



Optimization function

maximize

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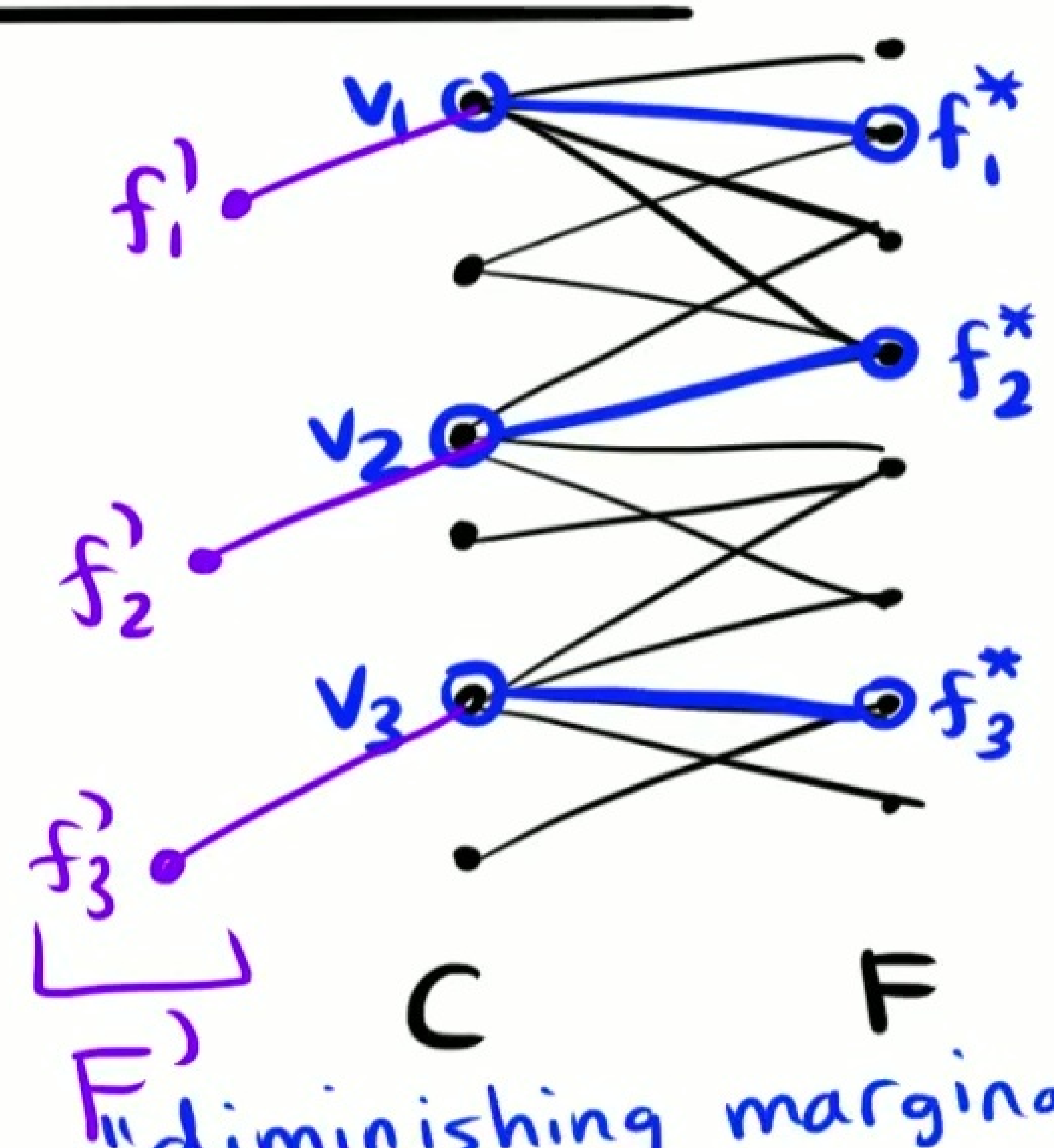

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"diminishing marginal returns"

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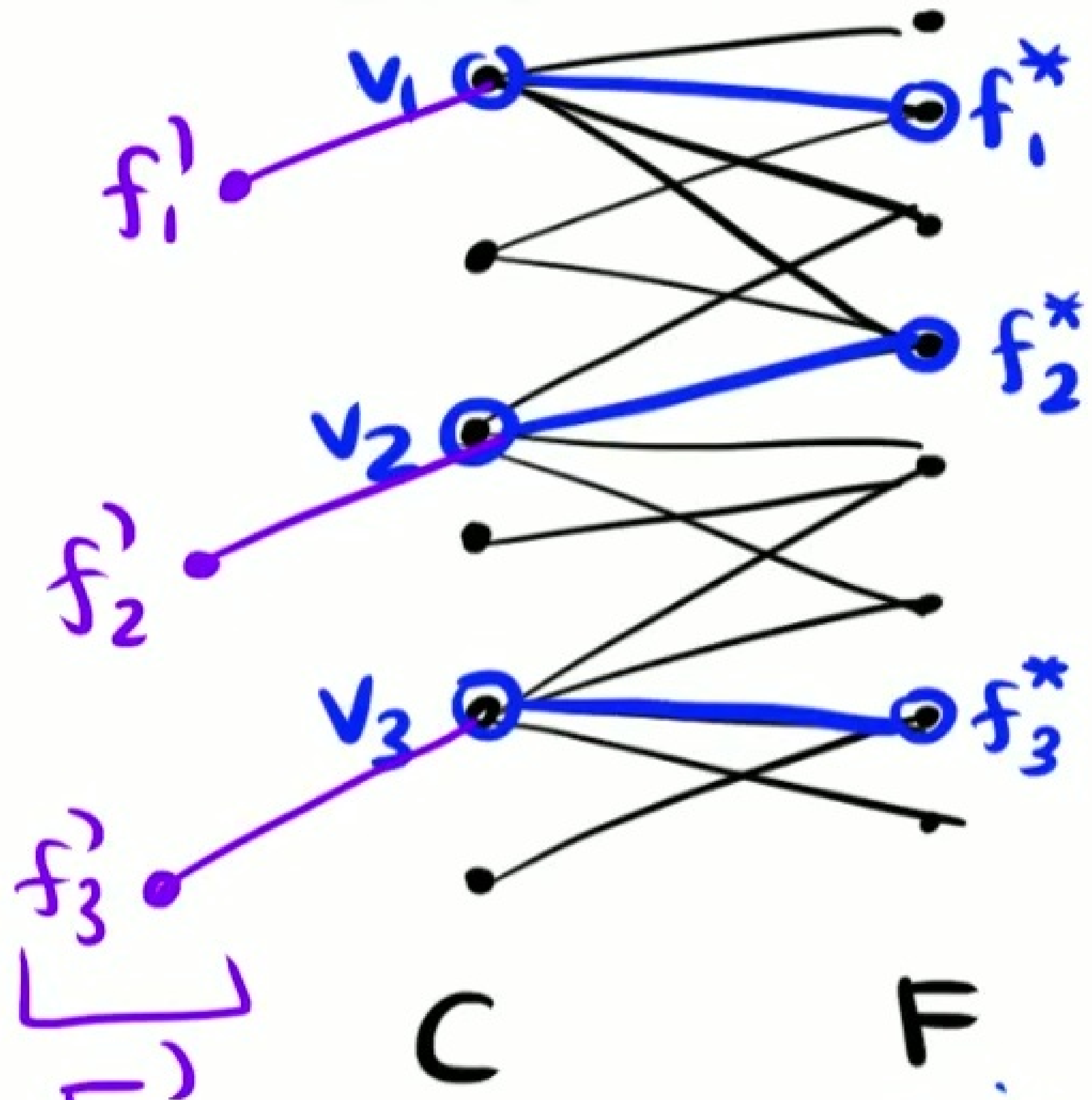

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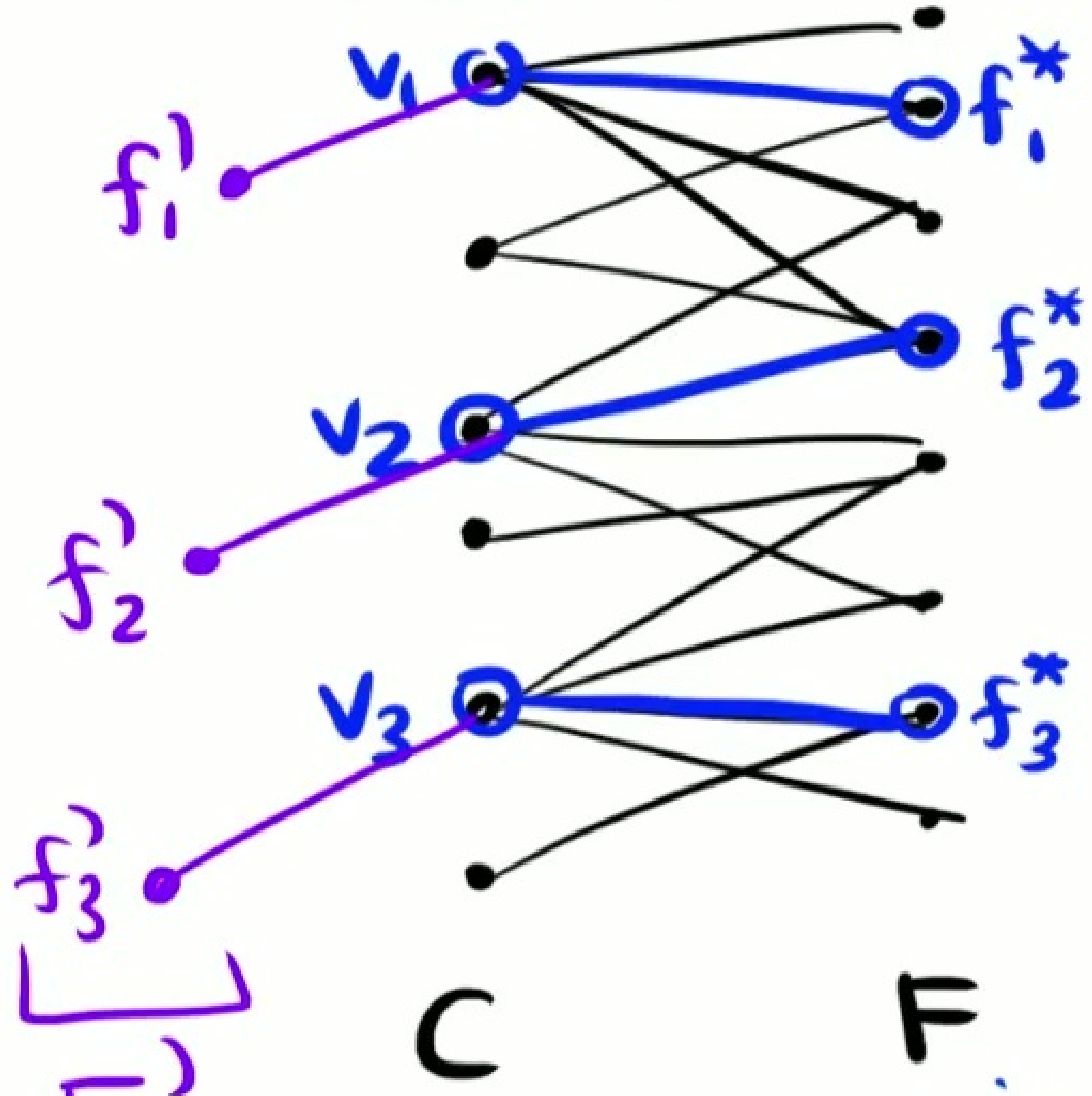

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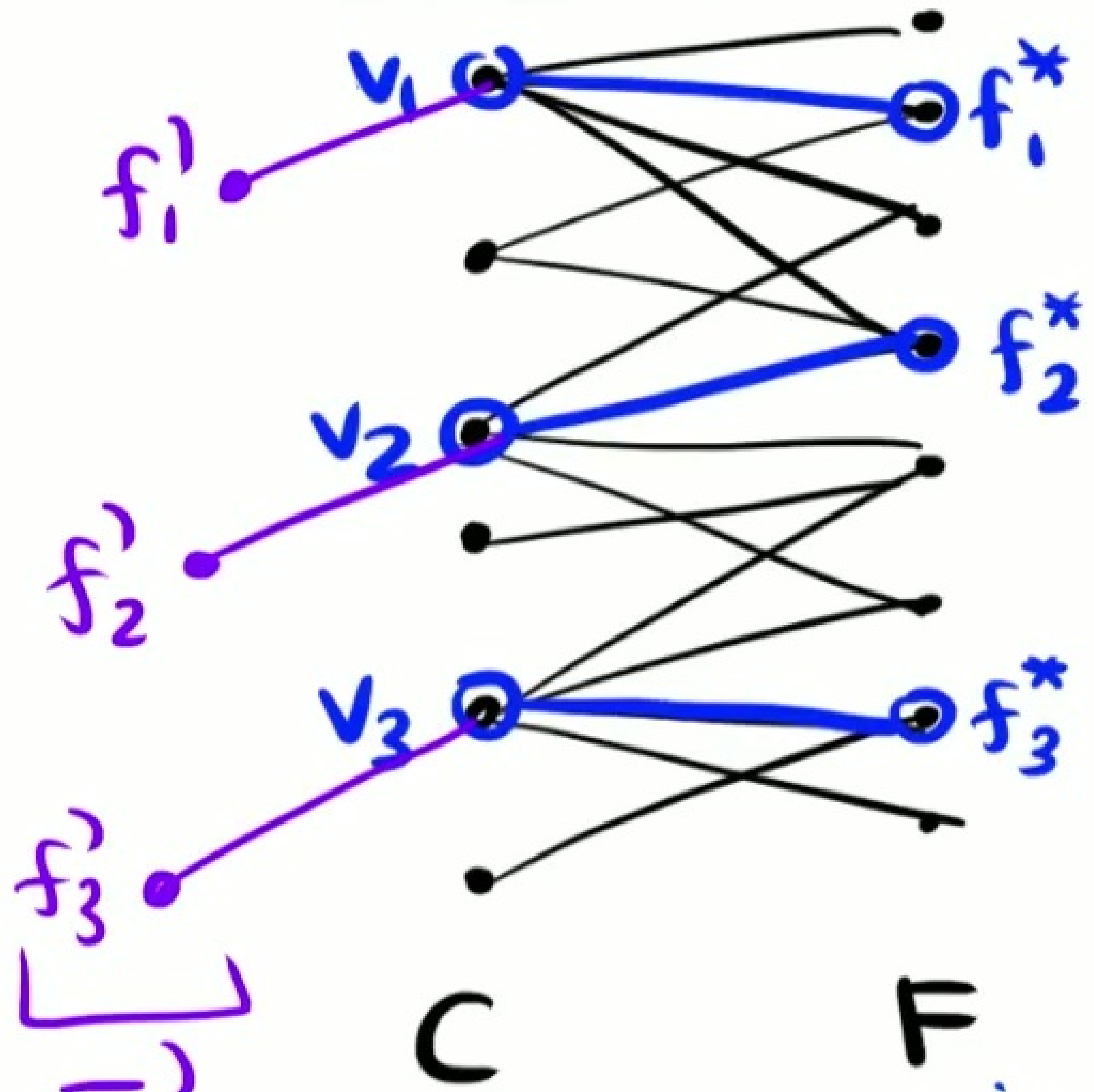

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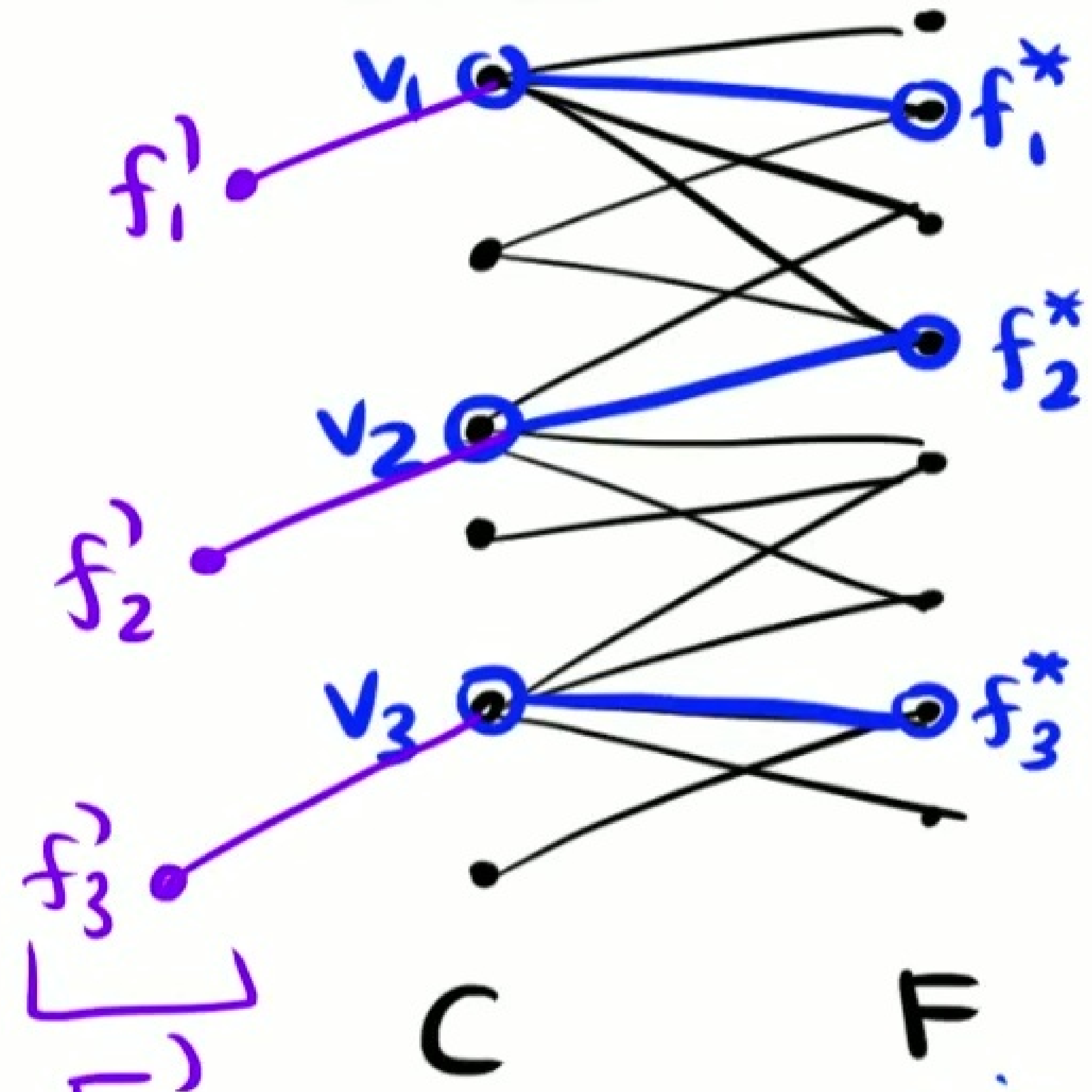
Optimization function



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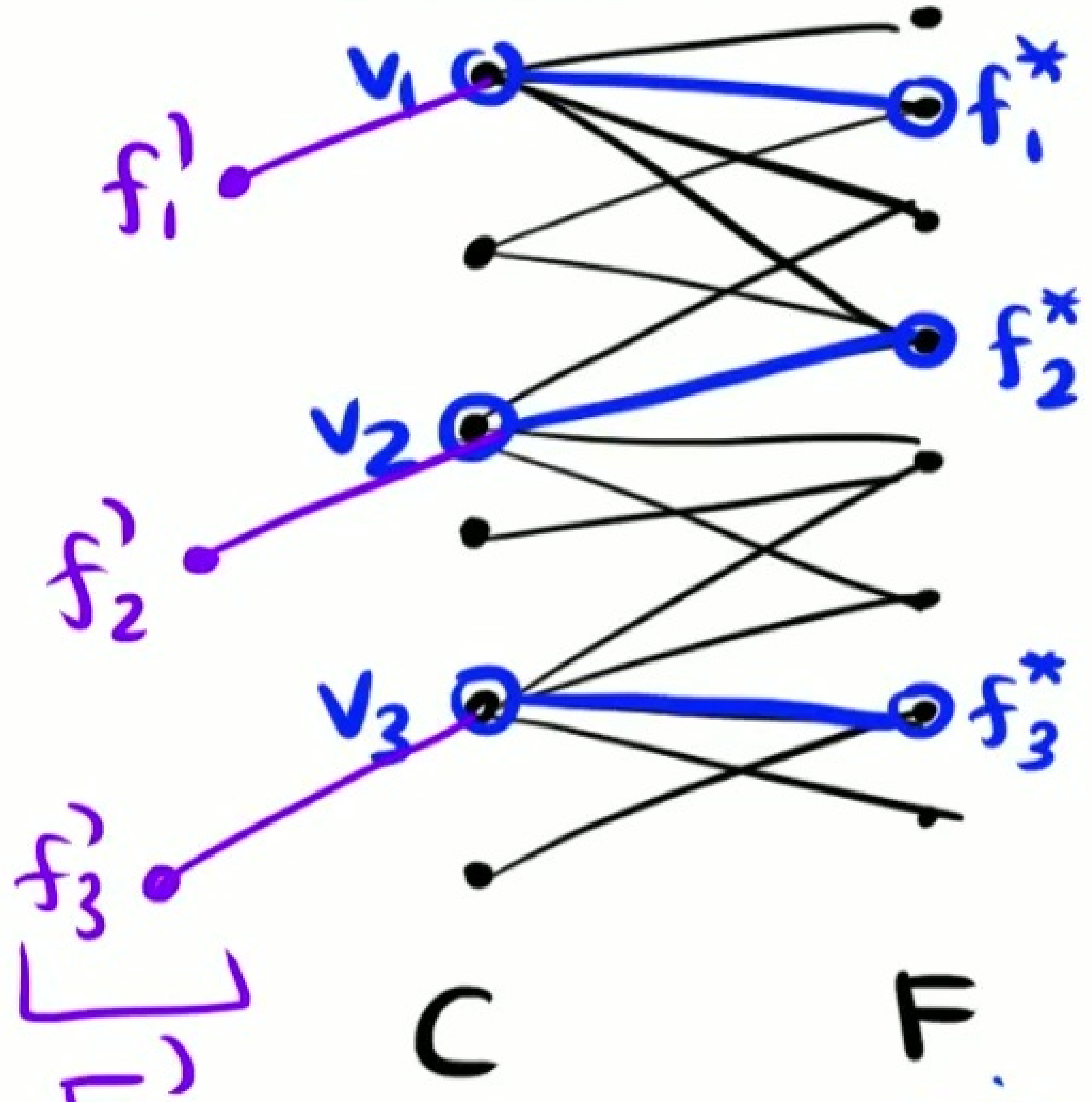

Thm [CCPV] can $(1-1/e)$-approximate monotone submodular.

"diminishing marginal returns"

Optimization function

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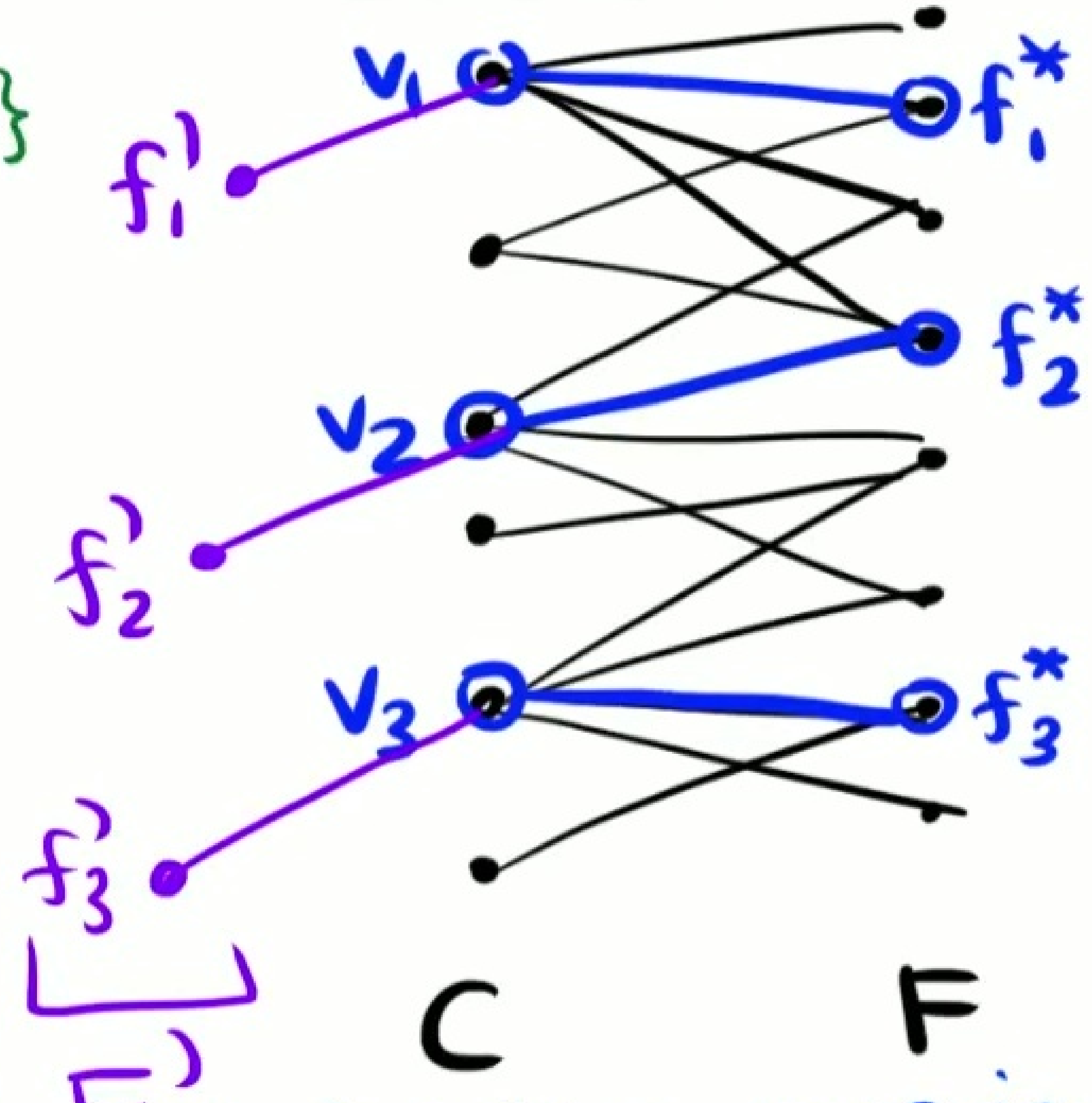

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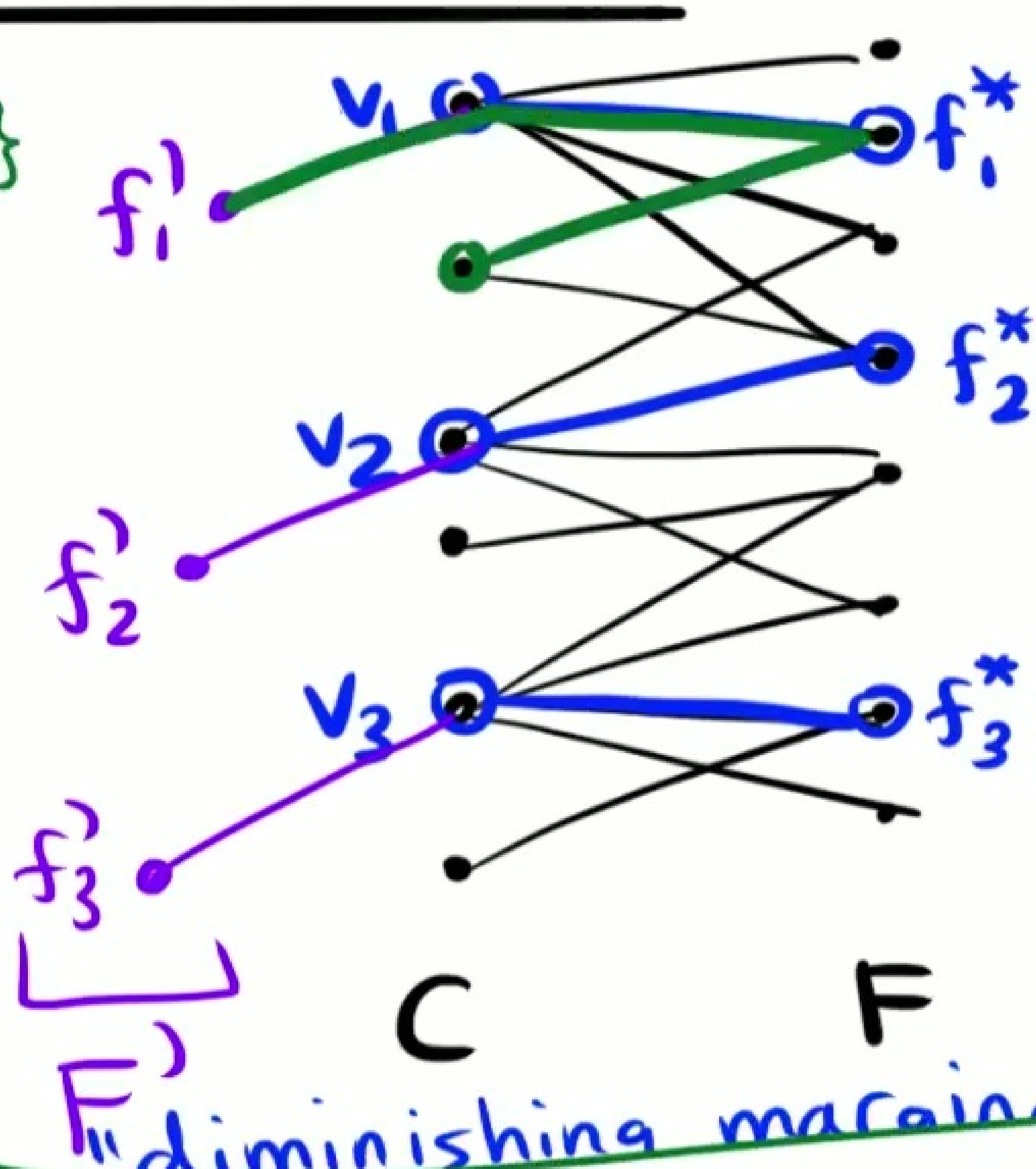

Obs: $\forall v$, $d(v, F') \le 3\, d(v, F^*)$ (same as before)

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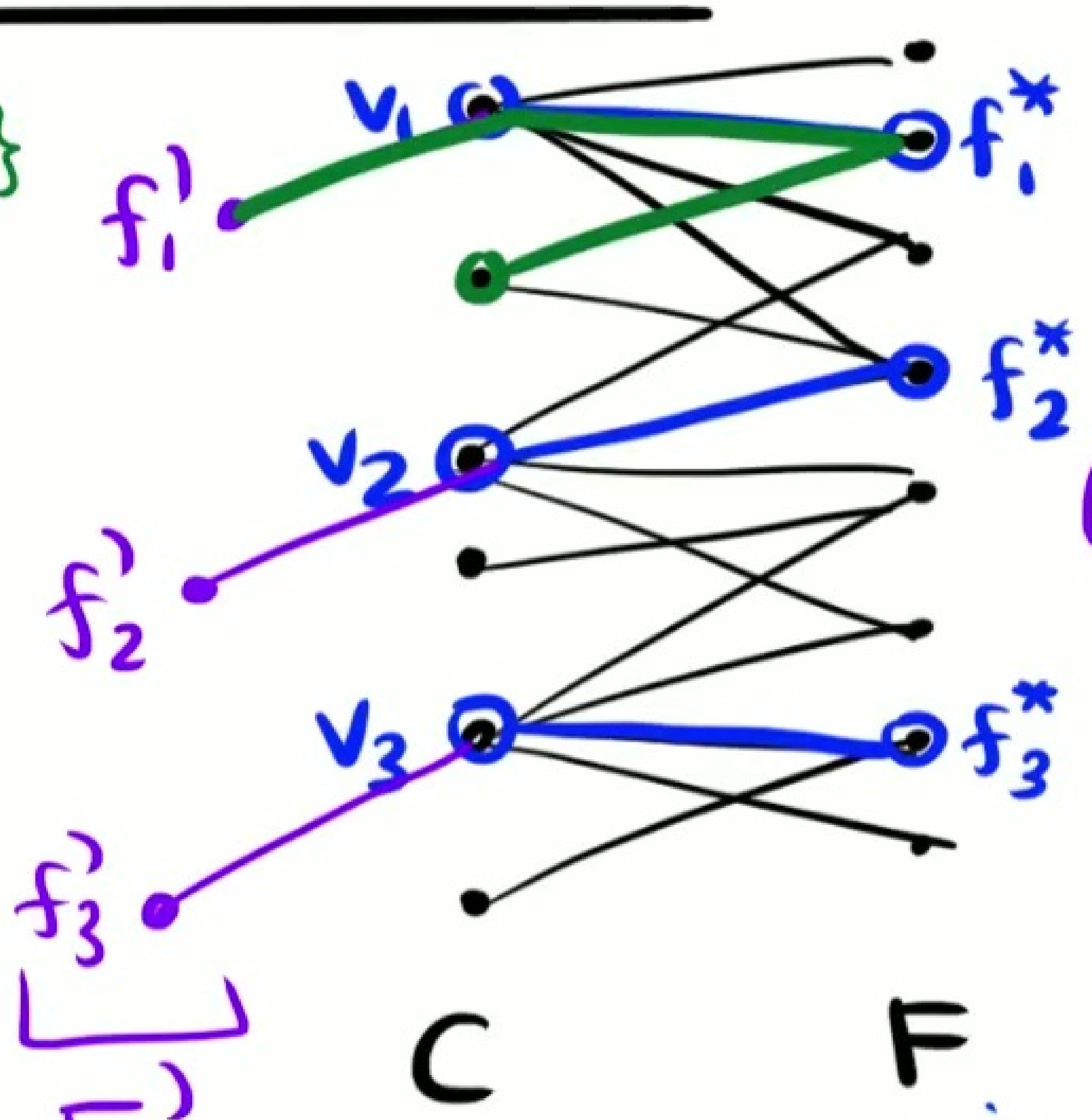

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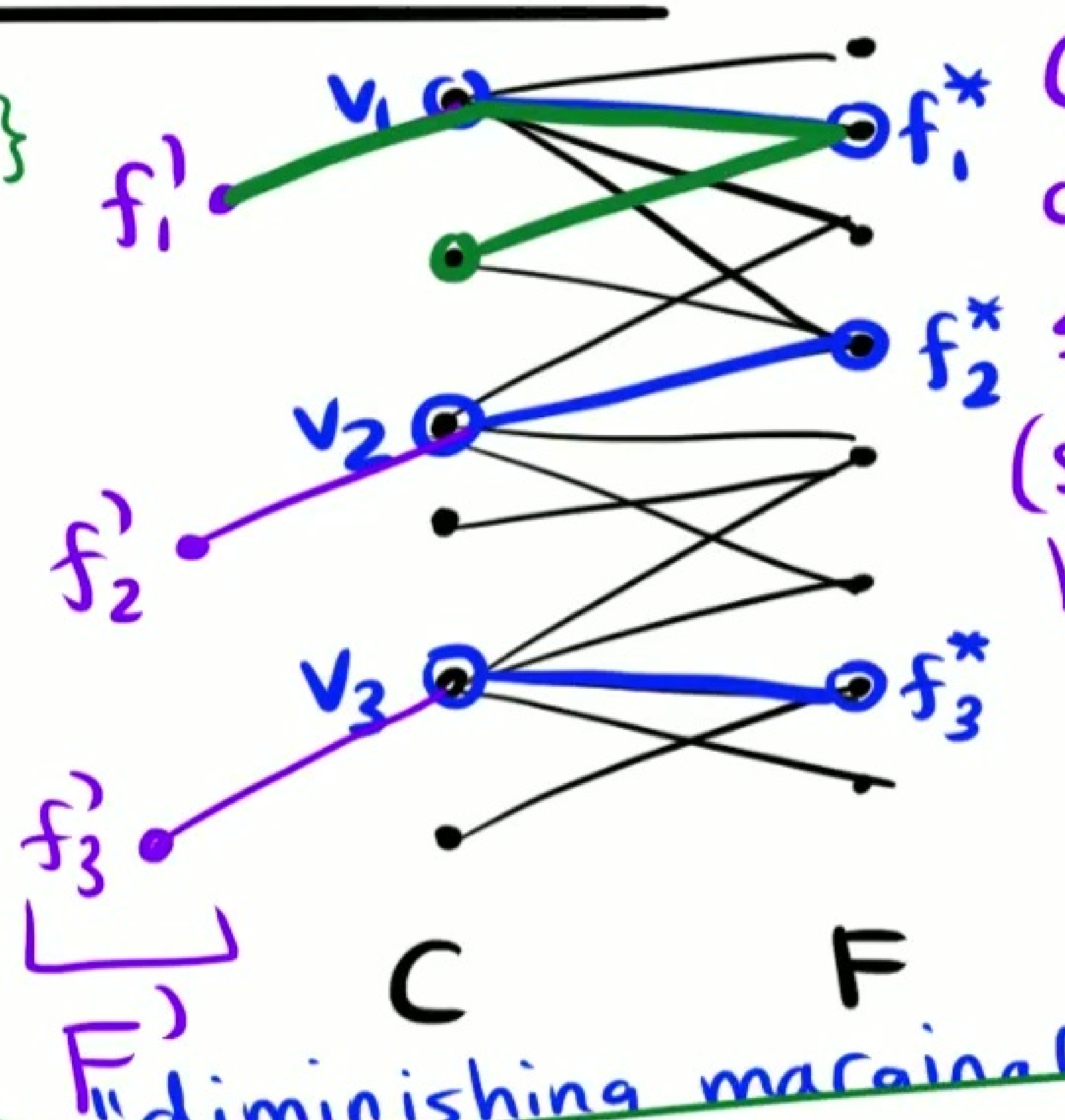

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Optimization function

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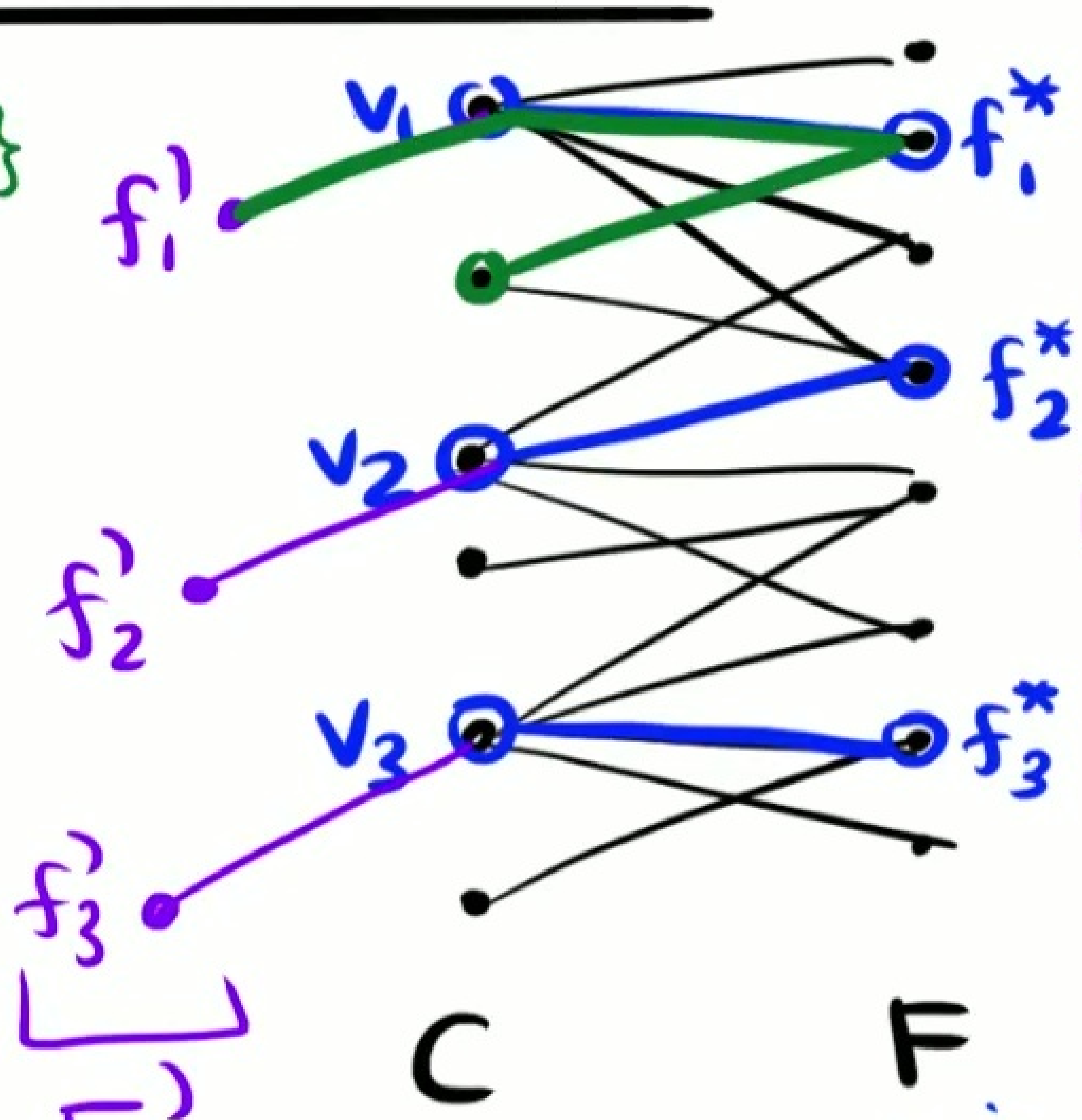

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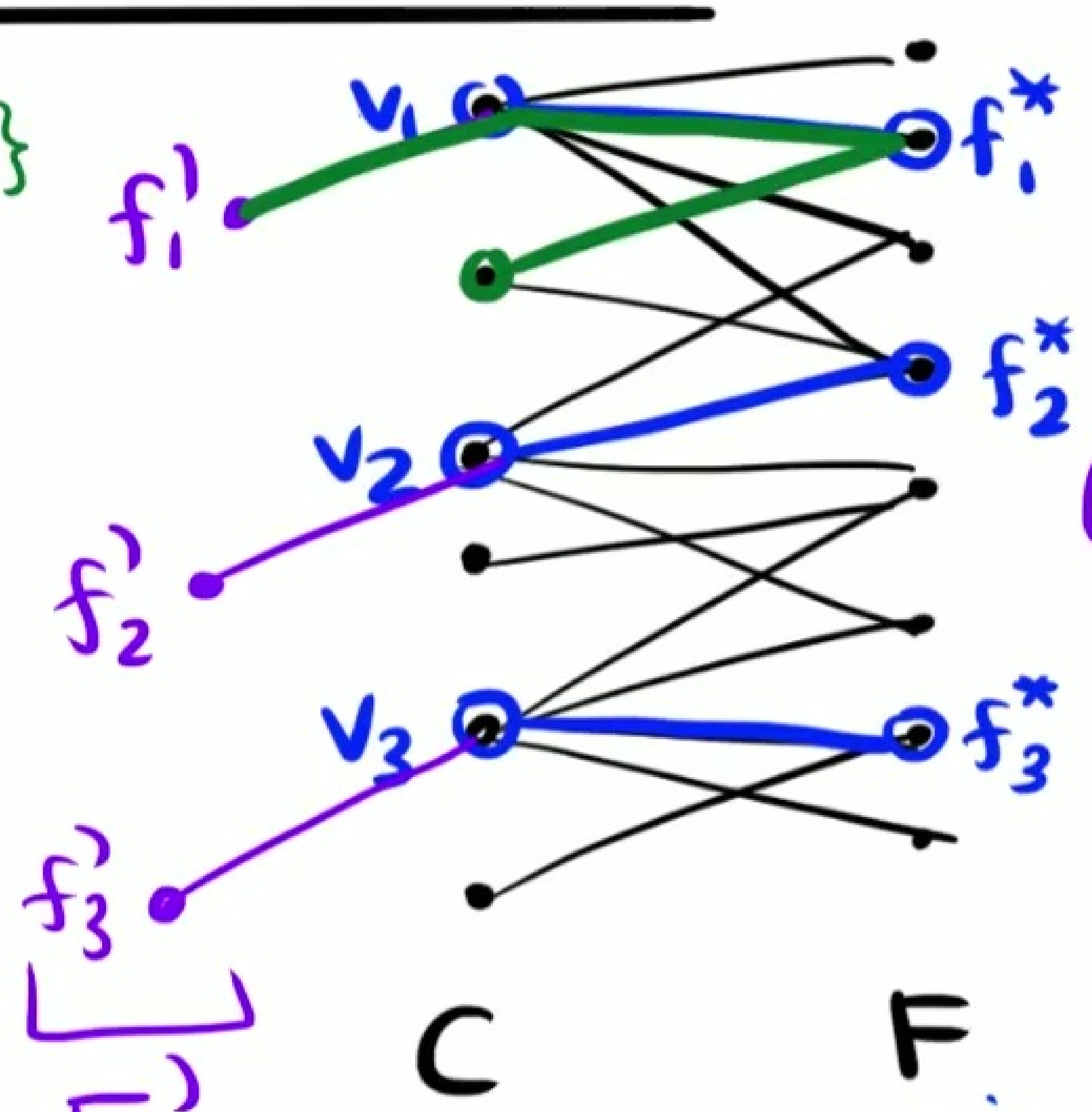

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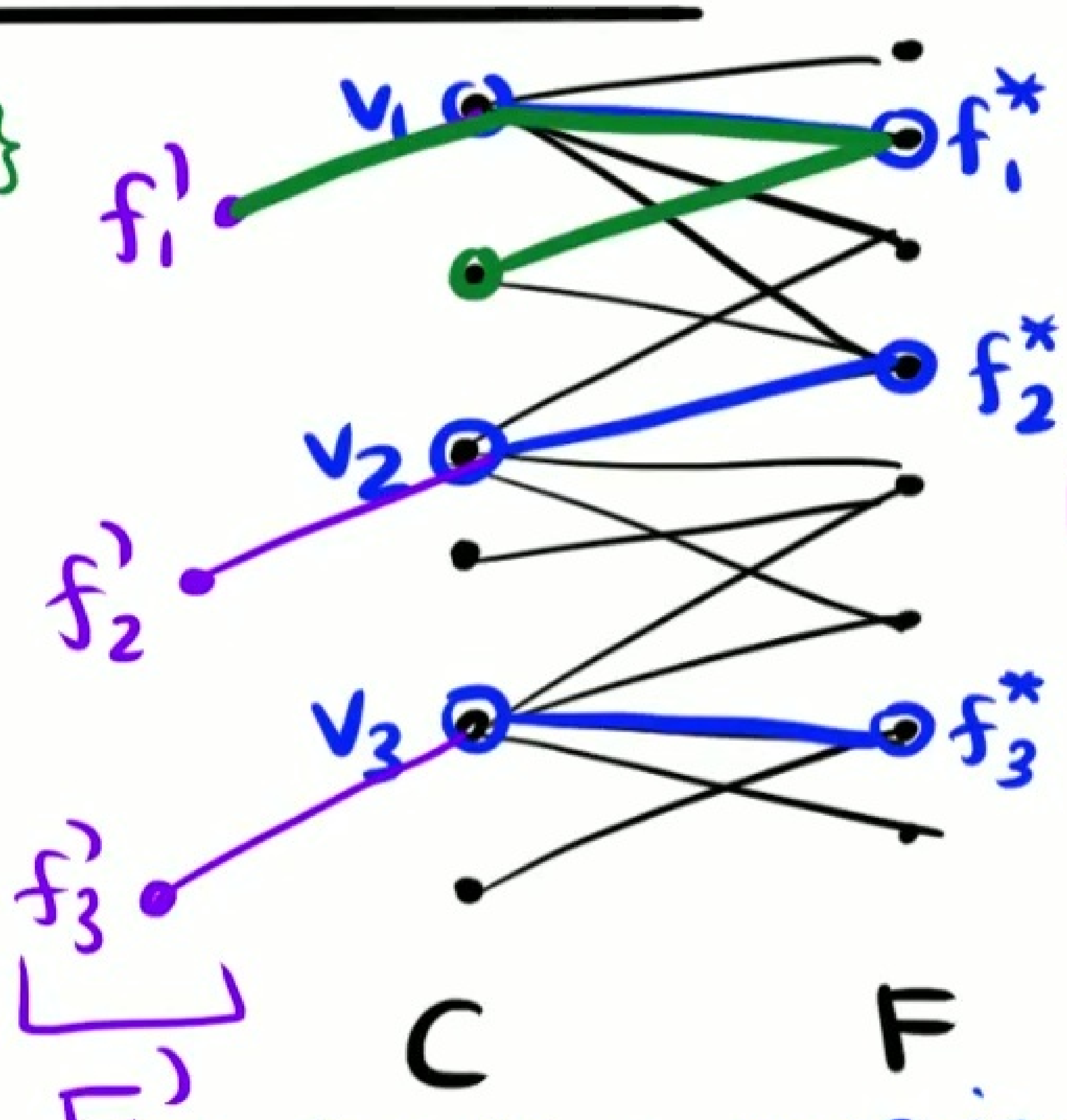

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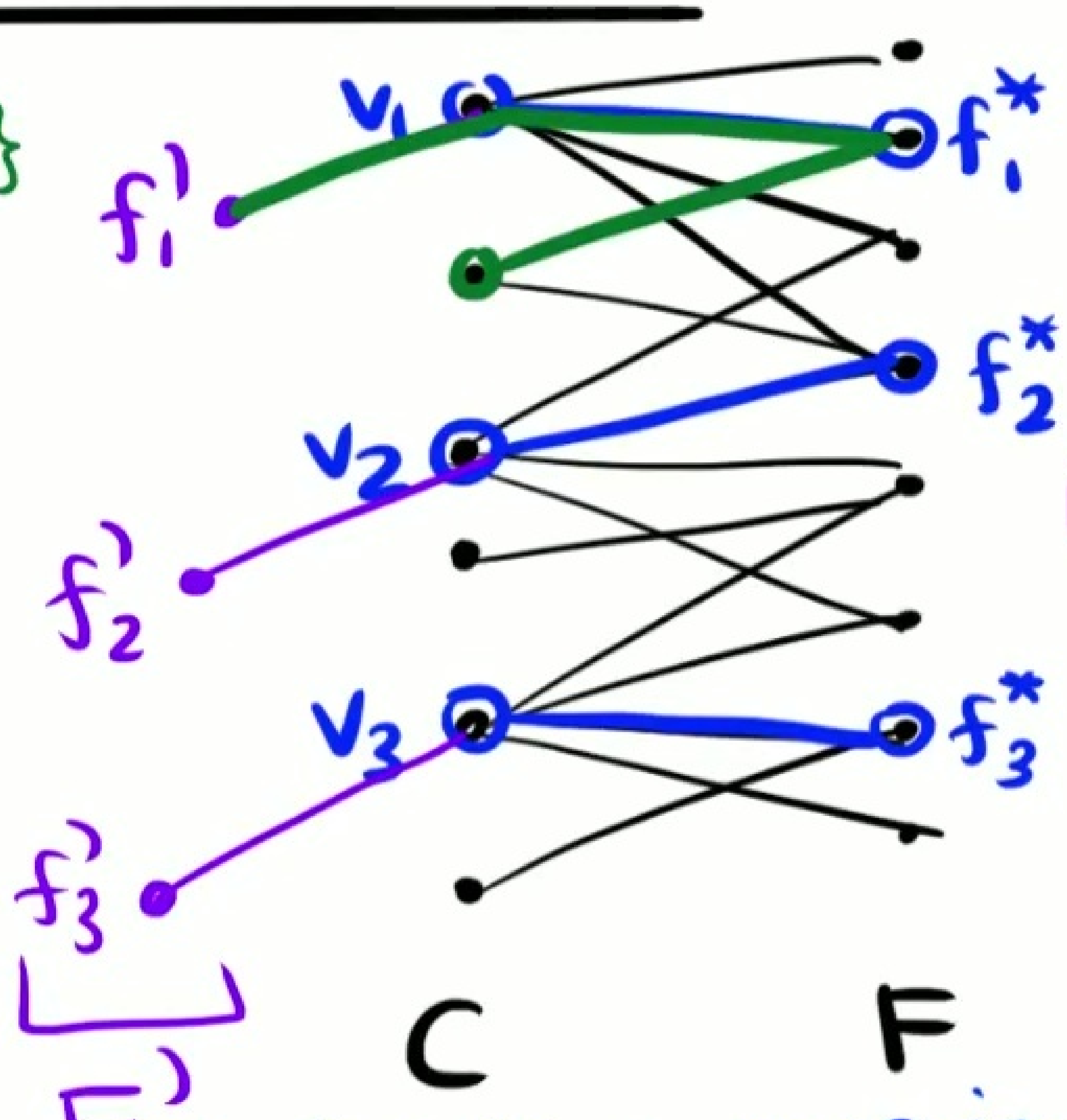

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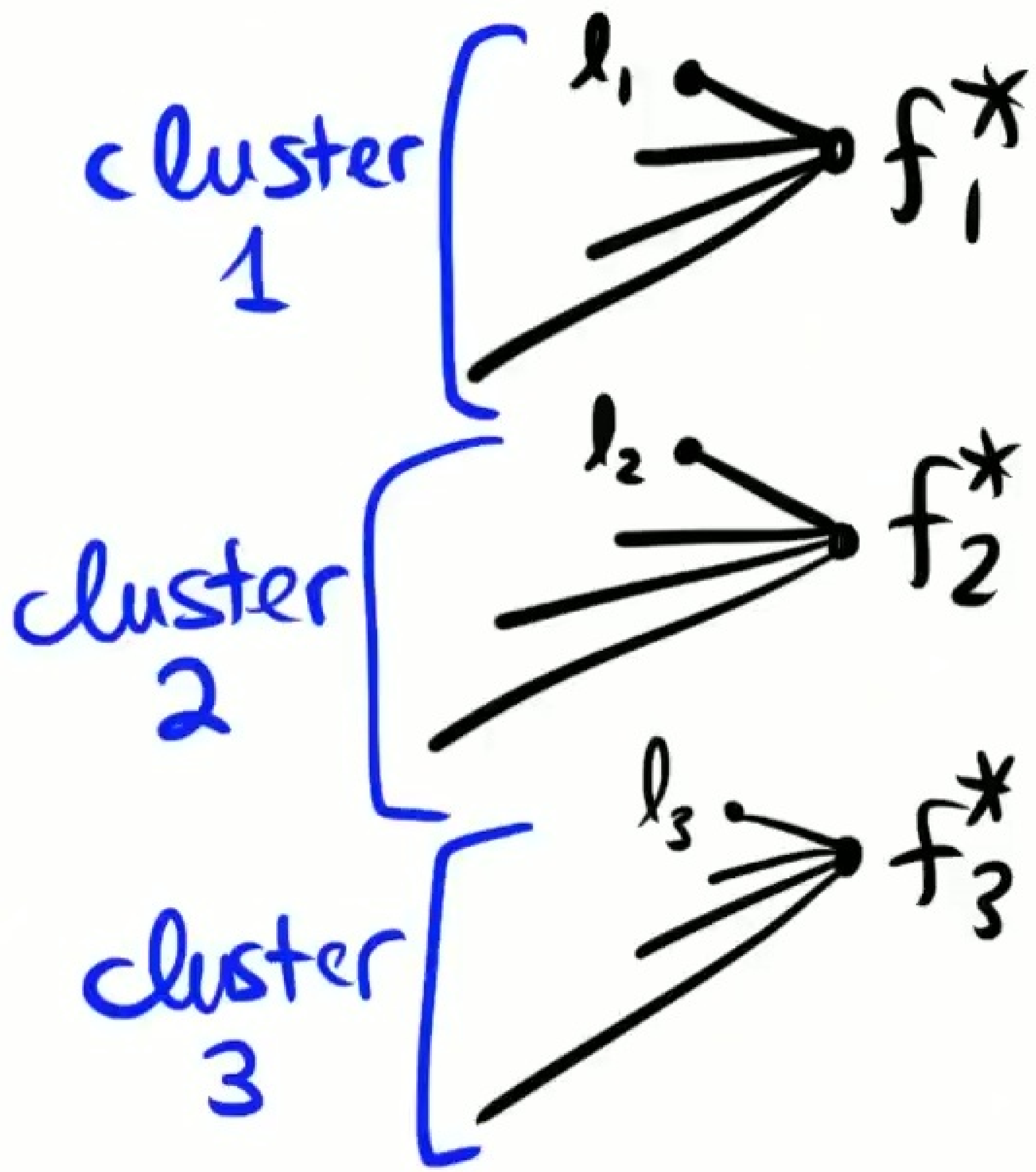
&= \sum_v w(v)\, d(v,F') - \left(1-\tfrac{1}{e}\right)\Big(\sum_v w(v)\big(d(v,F') - d(v,F^*)\big)\Big) \\
&= 1/e \sum w(v) d(v,F') + \left(1-\tfrac{1}{e}\right)\sum w(v)\, d(v,F^*) \\
&\le 3/e \sum w(v) d(v,F^*) + \left(1-\tfrac{1}{e}\right)\sum w(v)\, d(v,F^*)
\end{aligned}$$

General case

- cluster i : all clients assigned to f_i^* in OPT
- l_i := closest client to f_i^* in cluster i



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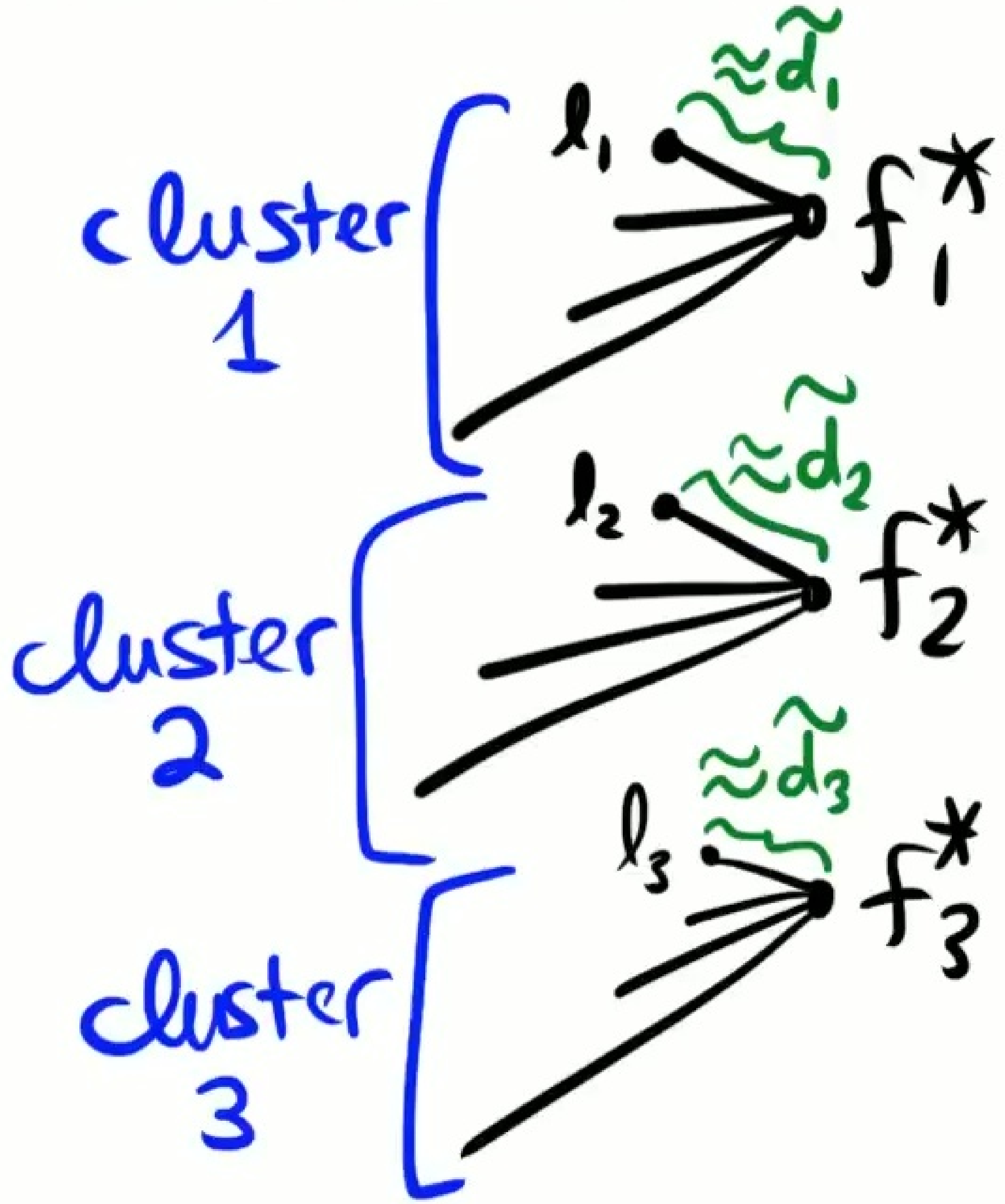
- Guess $l_1 \dots l_K$ ((coreset size)^K choices)
- Guess $\tilde{d}_i \approx d(l_i, f_i^*)$ up to factor $(1+\varepsilon)$



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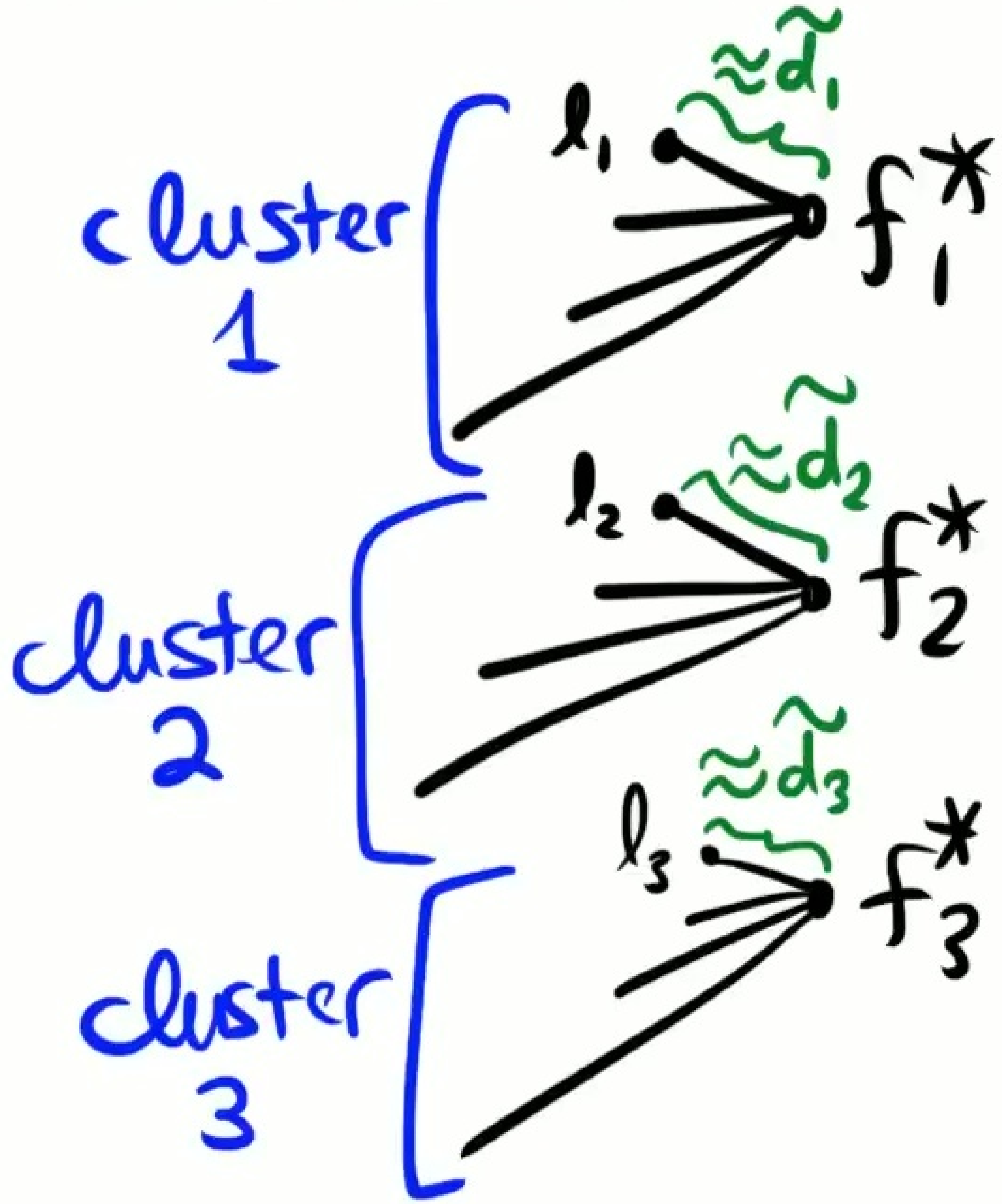

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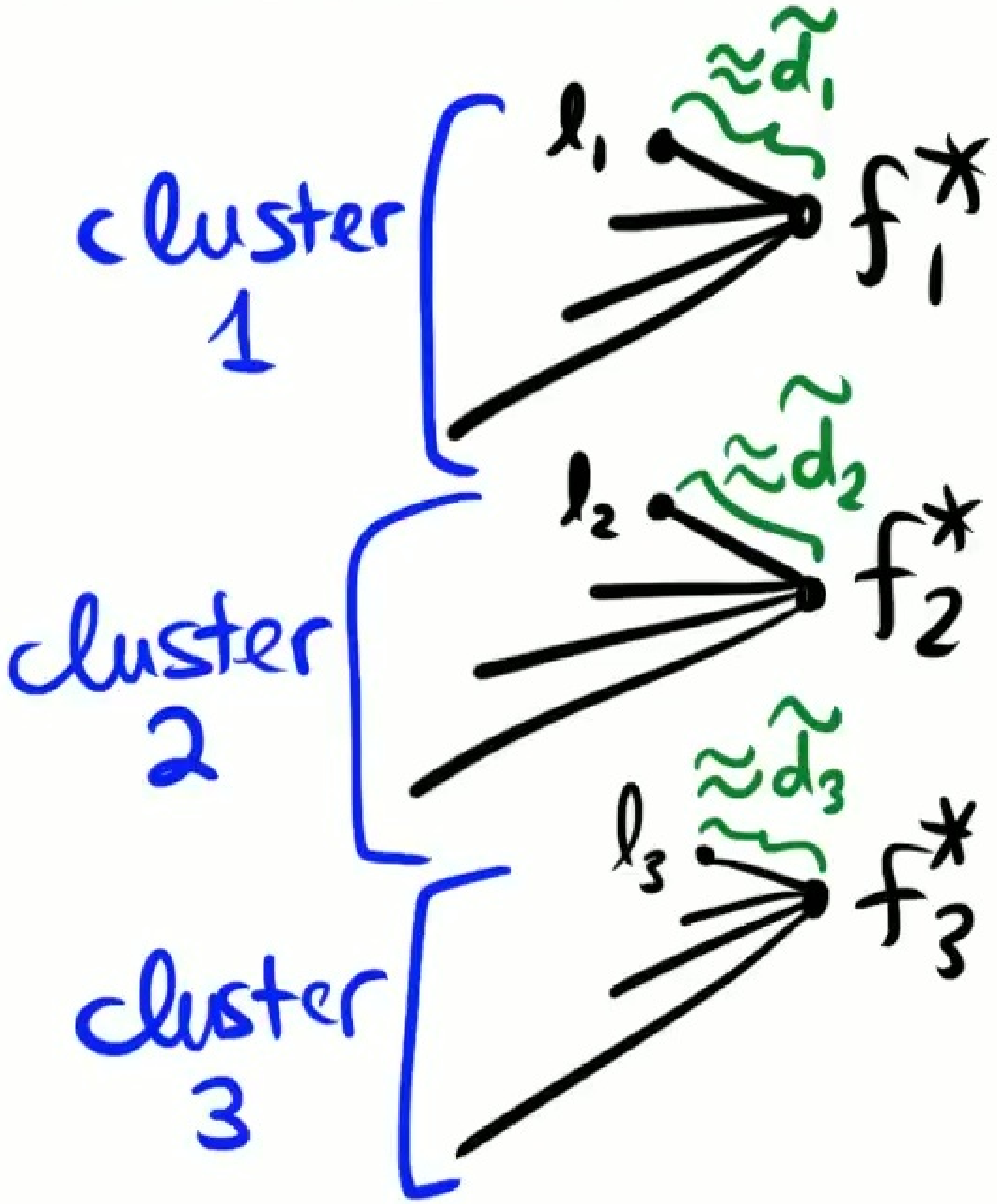

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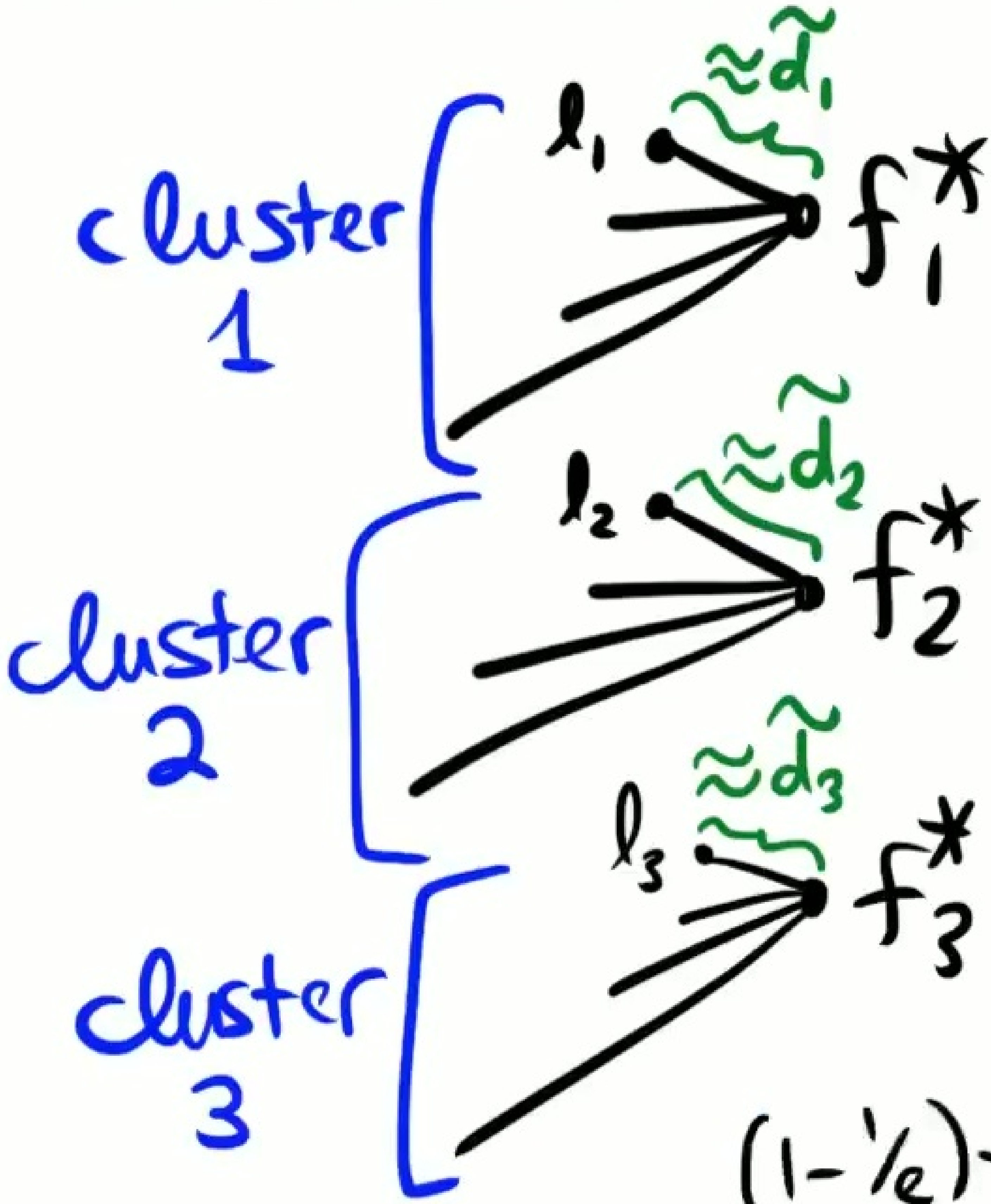

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$(1 - 1/e)$-apx to $\mathrm{impr}(S)$

$\Rightarrow (1 + 2/e + \varepsilon)$-apx to k-median

Open problems

- Can FPT techniques be used for <u>polytime</u> k-median?
  - Fallback of 3-approx, submod opt
- Other variants?
  - Capacitated k-median: upper bound 3+eps
    Lower bound 1+2/e
  - Matroid median: upper bound 2+eps
    Lower bound 1+2/e